A Generalization of Coupled Integrable, Dispersionless System

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Abstract

A generalized inverse scheme of the integrable, dispersionless system is proposed from the group theoretical point of view.

Key Words

inverse problem, integrable equation, dispersionless equation, group theory, Lie algebra

The authors¹⁾ solved the following coupled, integrable dispersionless equations:

$$\begin{aligned} q_{xt} + (rs)_x &= 0, \\ r_{xt} - 2q_x r &= 0, \\ s_{xt} - 2q_x s &= 0. \end{aligned} \tag{1}$$

by the inverse scattering problem with the boundary conditions:

$$\left. \begin{aligned} q_x &\rightarrow q_0 \\ r &\rightarrow 0 \\ s &\rightarrow 0 \end{aligned} \right\} \quad \text{for } |x| \rightarrow \infty, \tag{2}$$

where q_0 is constant. Eq.(1) has an important conserved quantity

$$q_x^2 + r_x s_x = q_0^2. \tag{3}$$

If $r = s^2$, (3) is rewritten as

$$q_x^2 + r_x^2 = q_0^2, \tag{4}$$

then (4) is related to the Euclidian group in two dimensions E_2 . Eq.(1) was found to be connected to the sine-Gordon equation.³⁾⁴⁾ If $r = s^{*5)}$, (3) is rewritten as

$$q_x^2 + \text{Re}(r_x)^2 + \text{Im}(r_x)^2 = q_0^2, \tag{5}$$

then (5) is related to the symmetry of $O(3) \sim SU(2)$. Eq.(1) was shown to be equivalent to the Pohlmeyer-Lund-Regge equation.⁶⁾ By changing variables $r = \rho + \sigma$ and $s = \rho - \sigma$, (3) yields

$$q_x^2 + \rho_x^2 - \sigma_x^2 = q_0^2, \tag{6}$$

(6) is related to the symmetry of $O(2,1) \sim SL(2,R)$. The inverse scheme of (1) is given as

$$\begin{aligned} V_x &= UV, \\ V_t &= WV, \end{aligned} \tag{7}$$

where

$$\begin{aligned} U &= -i\lambda \begin{pmatrix} q_x & r_x \\ s_x & -q_x \end{pmatrix}, \\ W &= \begin{pmatrix} 0 & -r \\ s & 0 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{8}$$

Here we shall extend our scheme to a more general one based on the group theoretical consideration.

Let us introduce the following Lagrangian as

$$L = \text{Tr} \left(\frac{1}{2} S_x S_t - \frac{1}{3} G[S, [S_x, S]] \right). \quad (9)$$

Here S is defined by

$$S = \phi_a T^a = \eta_{ab} \phi^a T^b, \quad (10)$$

where ϕ is a vector field with components $(\phi^1, \phi^2, \dots)$, T^a is a generator of the Lie algebra satisfying the commutation relation with the structure constant f^{ab}_c :

$$[T^a, T^b] = i f^{ab}_c T^c, \quad (11)$$

and the tensor η_{ab} is the inverse of the metric tensor η^{ab} defined by

$$\eta^{ab} = \text{Tr} (T^a T^b). \quad (12)$$

A constant matrix G is defined with a constant vector κ with components $(\kappa^1, \kappa^2, \dots)$ as

$$G = \eta_{ab} \kappa^a T^b. \quad (13)$$

The Lagrangian is invariant under the following transformation as

$$\begin{aligned} S' &= \Omega S \Omega^{-1}, \\ G' &= \Omega G \Omega^{-1} \end{aligned} \quad (14)$$

where

$$\Omega = e^{i\theta_a T^a} \quad (15)$$

with a constant vector θ_a .

Equation of motion is given by

$$S_{xt} + [S_x, [S, G]] = 0. \quad (16)$$

We can immediately obtain conserved quantities for integer n as

$$\text{Tr} (S_x^n). \quad (17)$$

For $n = 2$, (17) reduces the relations (4),(5) and (6) for E_2 , $SU(2)$ and $SL(2,R)$, respectively.

We write the inverse scheme of (16) and its compatibility condition as follows

$$\begin{aligned} V_x &= UV, \\ V_t &= WV, \end{aligned} \tag{18}$$

and

$$U_t - W_x + [U, W] = 0. \tag{19}$$

Assume U and W with an eigenvalue λ such as

$$\begin{aligned} U &= \lambda S_x, \\ W &= W_0 + \frac{1}{\lambda} W_1, \end{aligned} \tag{20}$$

where

$$\begin{aligned} W_0 &= [S, G], \\ W_1 &= G. \end{aligned} \tag{21}$$

Then the compatibility condition yields the equation of motion (16), that is,

$$S_{xt} + [S_x, W_0] = 0. \tag{22}$$

Our scheme is very general and we show some examples. Consider $SU(2)$ group. Without loss of generality we take ϕ , κ and T^a as

$$\phi = (\text{Re}(r), \text{Im}(r), q), \tag{23}$$

$$\kappa = (0, 0, 1), \tag{24}$$

and

$$T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{25}$$

where $\eta^{ab} = \delta_{ab}$. Then we can reproduce the equation of motion and the inverse scheme discussed in Konno⁵⁾ and Kotlyarov⁶⁾. For $SU(3)$ group, ϕ , κ and T^a are chosen as

$$\phi = (\text{Re}(r_1), \text{Im}(r_1), \phi_3, \text{Re}(r_2), \text{Im}(r_2), \text{Re}(r_3), \text{Im}(r_3), \phi_8), \tag{26}$$

$$\kappa = (0, 0, 2, 0, 0, 0, 0, 0), \tag{27}$$

and

$$\begin{aligned}
T^1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T^4 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T^5 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & T^6 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
T^7 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & T^8 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{28}$$

The metric tensor is obtained as

$$\eta^{ab} = \delta_{ab}. \tag{29}$$

U and W are given as

$$\begin{aligned}
U &= \frac{\lambda}{\sqrt{2}} \begin{pmatrix} \phi_{3x} + \frac{\phi_{8x}}{\sqrt{3}} & r_{1x} & r_{2x} \\ r_{1x}^* & -\phi_{3x} + \frac{\phi_{8x}}{\sqrt{3}} & r_{3x} \\ r_{2x}^* & r_{3x}^* & -\frac{2\phi_{8x}}{\sqrt{3}} \end{pmatrix}, \\
W &= \sqrt{2} \begin{pmatrix} 0 & -2r_1 & -r_2 \\ 2r_1^* & 0 & r_3 \\ r_2^* & -r_3^* & 0 \end{pmatrix} + \frac{2}{\lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{30}$$

The equations of motion are given as

$$\begin{aligned}
\phi_{3xt} + 2|r_1|_x^2 + \frac{1}{2}|r_2|_x^2 + \frac{1}{2}|r_3|_x^2 &= 0, \\
\phi_{8xt} + \frac{\sqrt{3}}{2}|r_2|_x^2 - \frac{\sqrt{3}}{2}|r_3|_x^2 &= 0, \\
r_{1xt} - 4\phi_{3x}r_1 - r_{2x}r_3^* + r_2r_{3x}^* &= 0, \\
r_{2xt} - (\phi_{3x} + \sqrt{3}\phi_{8x})r_2 + r_{1x}r_3 + 2r_1r_{3x} &= 0, \\
r_{3xt} - (\phi_{3x} - \sqrt{3}\phi_{8x})r_3 - r_{1x}^*r_2 - 2r_1^*r_{2x} &= 0,
\end{aligned} \tag{31}$$

where we see that $\text{Tr}(S_x^2)$

$$\left[(\phi_{3x})^2 + (\phi_{8x})^2 + |r_{1x}|^2 + |r_{2x}|^2 + |r_{3x}|^2 \right]_t = 0 \quad (32)$$

is satisfied.

According to the method found by Wadati, Sanuki and Konno⁷⁾, the inverse scheme (18) may be rewritten in a form of conservation law as

$$\frac{\partial}{\partial t} \left(\sum_j U_{ij} \frac{V_j}{V_i} \right) = \frac{\partial}{\partial x} \left(\sum_j W_{ij} \frac{V_j}{V_i} \right). \quad (33)$$

Expanding V_j/V_i ($j \neq i$) in the power series of $1/\lambda$ and equating the terms of the same power of $1/\lambda$, we can formally obtain an infinite number of conserved quantities.

Application to other compact and non-compact groups will be presented in separate papers.

References

- 1) K.Konno and H.Kakuhata, J. Phys. Soc. Japan **64** (1995) 2707.
- 2) K.Konno and H.Oono, J. Phys. Soc. Japan **63** (1994) 377.
- 3) R.Hirota and S. Tsujimoto, J. Phys. Soc. Japan **63** (1994) 3533.
- 4) K.Konno and H.Oono, J. Phys. Soc. Japan **63** (1994) 3534.
- 5) K.Konno, Applicable Analysis **57** (1995) 209.
- 6) V.P.Kotlyarov, J. Phys. Soc. Japan **63** (1994) 3535.
- 7) M. Wadati, H.Sanuki and K.Konno, Prog. Theor. Phys. **53** (1975) 419.