A Generalization of Coupled Integrable, Dispersionless System

Hiroshi Kakuhata and Kimiaki Konno*

Tsuruga Women's Junior College, Tsuruga 914.
and
Atomic Energy Research Institute, Nihon University, Tokyo 101

*Department of Physics, College of Science and Technology, Nihon University, Tokyo 101.

Abstract

A generalized inverse scheme of the integrable, dispersionless system is proposed from the group theoretical point of view.

Key Words

inverse problem, integrable equation, dispersionless equation, group theory, Lie algebra
The authors\textsuperscript{1) }solved the following coupled, integrable dispersionless equations:

\begin{align*}
q_{xt} + (rs)_x &= 0, \\
r_{xt} - 2q_x r &= 0, \\
s_{xt} - 2q_x s &= 0. \\
\end{align*}

by the inverse scattering problem with the boundary conditions:

\begin{equation}
\begin{array}{c}
q_x \to q_0 \\
r \to 0 \\
s \to 0
\end{array}
\right\}
\text{for } |x| \to \infty,
\end{equation}

where $q_0$ is constant. Eq.(1) has an important conserved quantity

\begin{equation}
q_x^2 + r_x s_x = q_0^2.
\end{equation}

If $r = s^2$, (3) is rewritten as

\begin{equation}
q_x^2 + r_x^2 = q_0^2,
\end{equation}

then (4) is related to the Euclidian group in two dimensions $E_2$. Eq.(1) was found to be connected to the sine-Gordon equation.\textsuperscript{3,4)} If $r = s^5$, (3) is rewritten as

\begin{equation}
q_x^2 + \text{Re}(r_x)^2 + \text{Im}(r_x)^2 = q_0^2,
\end{equation}

then (5) is related to the symmetry of $O(3) \sim SU(2)$. Eq.(1) was shown to be equivalent to the Pohlmeyer-Lund-Regge equation.\textsuperscript{6)} By changing variables $r = \rho + \sigma$ and $s = \rho - \sigma$, (3) yields

\begin{equation}
q_x^2 + \rho_x^2 - \sigma_x^2 = q_0^2,
\end{equation}

(6) is related to the symmetry of $O(2,1) \sim SL(2,R).$ The inverse scheme of (1) is given as

\begin{equation}
\begin{align*}
V_x &= UV, \\
V_t &= WV,
\end{align*}
\end{equation}

where

\begin{equation}
\begin{align*}
U &= -i\lambda \begin{pmatrix} q_x & r_x \\ s_x & -q_x \end{pmatrix}, \\
W &= \begin{pmatrix} 0 & -r \\ s & 0 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\end{equation}
Here we shall extend our scheme to a more general one based on the group theoretical consideration.

Let us introduce the following Lagrangian as

\[ L = \text{Tr} \left( \frac{1}{2} S_x S_t - \frac{1}{3} G[S, [S_x, S]] \right). \]  

Here \( S \) is defined by

\[ S = \phi_a T^a = \eta_{ab} \phi^a T^b, \]

where \( \phi \) is a vector field with components \( (\phi^1, \phi^2, \cdots) \), \( T^a \) is a generator of the Lie algebra satisfying the commutation relation with the structure constant \( f^{ab}_{\phantom{ab}c} \):

\[ [T^a, T^b] = i f^{ab}_{\phantom{ab}c} T^c, \]

and the tensor \( \eta_{ab} \) is the inverse of the metric tensor \( \eta^{ab} \) defined by

\[ \eta^{ab} = \text{Tr} \left( T^a T^b \right). \]

A constant matrix \( G \) is defined with a constant vector \( \kappa \) with components \( (\kappa^1, \kappa^2, \cdots) \) as

\[ G = \eta_{ab} \kappa^a T^b. \]

The Lagrangian is invariant under the following transformation as

\[ S' = \Omega S \Omega^{-1}, \]
\[ G' = \Omega G \Omega^{-1} \]

where

\[ \Omega = e^{i \theta_a T^a} \]

with a constant vector \( \theta_a \).

Equation of motion is given by

\[ S_{xt} + [S_x, [S, G]] = 0. \]

We can immediately obtain conserved quantities for integer \( n \) as

\[ \text{Tr} (S^n_x). \]
For \( n = 2 \), (17) reduces the relations (4),(5) and (6) for \( E_2 \), \( SU(2) \) and \( SL(2,R) \), respectively.

We write the inverse scheme of (16) and its compatibility condition as follows

\[
V_x = U V, \\
V_i = W V,
\]

and

\[
U_x - W_x + [U, W] = 0.
\]

Assume \( U \) and \( W \) with an eigenvalue \( \lambda \) such as

\[
U = \lambda S_x, \\
W = W_0 + \frac{1}{\lambda} W_1,
\]

where

\[
W_0 = [S_x, G], \\
W_1 = G.
\]

Then the compatibility condition yields the equation of motion (16), that is,

\[
S_{2x} + [S_x, W_0] = 0.
\]

Our scheme is very general and we show some examples. Consider \( SU(2) \) group. Without loss of generality we take \( \phi, \kappa \) and \( T^a \) as

\[
\phi = (\text{Re}(r), \text{Im}(r), q), \\
\kappa = (0, 0, 1).
\]

and

\[
T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
T^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \eta^{ab} = \delta_{ab} \). Then we can reproduce the equation of motion and the inverse scheme discussed in Konno\textsuperscript{5}) and Kotlyarov\textsuperscript{6}). For \( SU(3) \) group, \( \phi, \kappa \) and \( T^a \) are chosen as

\[
\phi = (\text{Re}(r_1), \text{Im}(r_1), \phi_3, \text{Re}(r_2), \text{Im}(r_2), \text{Re}(r_3), \text{Im}(r_3), \phi_8), \\
\kappa = (0, 0, 2, 0, 0, 0, 0, 0).
\]
and
\[
T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
T^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (28)
\]
\[
T^7 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

The metric tensor is obtained as
\[
\eta^{ab} = \delta_{ab}. \quad (29)
\]

\(U\) and \(W\) are given as
\[
U = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} \phi_{3z} + \frac{\phi_{8z}}{\sqrt{3}} & r_{1z} & r_{2z} \\ r_{1z}^* & -\phi_{3z} + \frac{\phi_{8z}}{\sqrt{3}} & r_{3z} \\ r_{2z}^* & r_{3z}^* & 2\phi_{8z}/\sqrt{3} \end{pmatrix},
\]
\[
W = \sqrt{2} \begin{pmatrix} 0 & -2r_1 & -r_2 \\ 2r_1^* & 0 & r_3 \\ r_2^* & -r_3^* & 0 \end{pmatrix} + \frac{2}{\lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The equations of motion are given as
\[
\phi_{3z} + 2|r_{1z}|^2 + \frac{1}{2}|r_{2z}|^2 + \frac{1}{2}|r_{3z}|^2 = 0,
\]
\[
\phi_{8z} + \frac{\sqrt{3}}{2}|r_{2z}|^2 - \frac{\sqrt{3}}{2}|r_{3z}|^2 = 0,
\]
\[
4\phi_{3z}r_1 - r_{2z}r_3^* + r_{2z}r_3 = 0,
\]
\[
2\phi_{3z} + \sqrt{3}\phi_{8z} r_2 + r_{1z}r_3 + 2r_1r_3 = 0,
\]
\[
\phi_{3z} - \sqrt{3}\phi_{8z} r_3 - r_{1z}r_2 - 2r_1r_2 = 0,
\]

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where we see that \( \text{Tr}(S_x^2) \)

\[
\left[ (\phi_{2x})^2 + (\phi_{2x})^2 + |r_{1x}|^2 + |r_{2x}|^2 + |r_{3x}|^2 \right]_t = 0
\]  

(32)

is satisfied.

According to the method found by Wadati, Sanuki and Konno\(^7\), the inverse scheme (18) may be rewritten in a form of conservation law as

\[
\frac{\partial}{\partial t} \left( \sum_j U_{ij} \frac{V_j}{V_i} \right) = \frac{\partial}{\partial x} \left( \sum_j W_{ij} \frac{V_j}{V_i} \right).
\]  

(33)

Expanding \( V_j/V_i \ (j \neq i) \) in the power series of \( 1/\lambda \) and equating the terms of the same power of \( 1/\lambda \), we can formally obtain an infinite number of conserved quantities.

Application to other compact and non-compact groups will be presented in separate papers.
References