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Abstract

We calculate the fragmentation function for a heavy quark to go into quarkonia + X including relativistic and binding corrections. We use a systematic gauge invariant method which starts directly from QCD and which allows for a systematic expansion in quark relative velocity, which is a small natural parameter for heavy quark systems. Inclusion of these $O(v^2)$ corrections modifies the fragmentation functions appreciably.
The formalism developed earlier [1, 2, 5] can be applied to some other processes as well. The present work was initiated by the observation that in all existing treatments of fragmentation, colour gauge invariance is not properly accounted for. Effectively, all authors have implicitly assumed the size of the produced meson to be so small that the gauge-link between the colour sources is a unit operator. This is valid only in the limit of infinitely massive quarks. But certainly this cannot be true for c or b quarks - even for the t quark this would be true only to a few percent.

Fragmentation processes require a variation of the technique developed for the decay processes. We have found that direct introduction of a link operator offers a quicker route to arriving at gauge-invariant matrix elements. While these cannot be calculated ab-initio, nevertheless they can be modeled in a non-relativistic model. Alternatively, they can be extracted from experiment by examining decay rates where large momentum transfers are involved.

Our starting point is the definition of fragmentation function in terms of matrix elements of field operators at light cone separation i.e.,

\[ f(z) = \int \frac{d\lambda}{2\pi} e^{-i\lambda z} Tr \left[ \langle 0 | \psi(0) | P l \rangle \langle P l | \psi(\lambda n) | 0 \rangle \right] \tilde{d} \lambda \]  
(1)

where \( \langle P l | \psi(0) | 0 \rangle \) is the amplitude for a quark to go into a meson + X, shown diagramatically in Fig. 1a.

\[ \langle P l | \tilde{\psi}(0) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \bar{u}(l)(-i\gamma^\nu)M(k)(-i\gamma^\nu)iS_F(P + l)iD_{\mu\nu}(k - \frac{P}{2} + l), \]  
(2)

where \( M(k) \) is non-gauge invariant Bethe-Salpeter amplitude,

\[ M(k) = \int d^4x e^{ik\cdot x} \langle P | \psi(x/2)\tilde{\psi}(-x/2) | 0 \rangle, \]  
(3)

and

\[ D_{\mu\nu}(p) = \left( g_{\mu\nu} + \frac{p_\mu p_\nu + p_\nu p_\mu}{p \cdot n} \right) \frac{1}{p^2}, \]  
(4)

is the gluon propagator. The gauge non-invariance of the B-S amplitude can be fixed by introducing the gauge link operator, i.e., write \( M(k) \) as

\[ M(k) = \int d^4x e^{ik\cdot x} \langle P | \psi(x/2)e^{i\int_{-z}^{-}\xi A(\xi) d\xi} \tilde{\psi}(-x/2) | 0 \rangle. \]  
(5)

Expanding in powers of \( k \), the gluon momentum, and performing the \( k \) integration we get only the covariant derivatives

\[ \langle P l | \tilde{\psi}(0) | 0 \rangle = \bar{u}(l)\gamma^\mu M(0)\gamma^\nu S_F(P + l)D_{\mu\nu} \]
+ \bar{u}(l)\gamma^\mu M^\alpha(0)\gamma^\nu S_F(P + l)D_{\mu\nu,\alpha} \\
+ \bar{u}(l)\gamma^\mu M^\alpha\beta(0)\gamma^\nu S_F(P + l)\frac{1}{2}D_{\mu\nu,\alpha\beta}, \quad (6)

where

\begin{align*}
M(0) & \equiv \langle P, e| \psi \bar{\psi} |0\rangle \\
M^\alpha(0) & \equiv \langle P, e| \psi \mp D^\alpha \bar{\psi} |0\rangle \\
M^\alpha\beta(0) & \equiv \langle P, e| \psi \mp D^\alpha D^\beta \bar{\psi} |0\rangle. \quad (7)
\end{align*}

To proceed, one can perform a Lorentz and CPT invariant decomposition of each of the hadronic matrix elements in the above equation. This is somewhat complicated\[4\] and involves a large number of constants which characterize the hadron. This is discussed in detail in Ref \[5\] and we refer the reader to that reference. The end result is,

where

\begin{align*}
\langle P, e| \psi \bar{\psi} |0\rangle &= \frac{1}{2} M^{1/2} \gamma_5 \left(1 + \frac{\nabla^2}{M^2}\right) \phi \left(1 + \frac{P}{M}\right) - \frac{1}{2} M^{1/2} \gamma_5 \frac{\nabla^2 \phi}{M^2} \left(1 - \frac{P}{M}\right), \\
\langle P, e| \psi \mp D^\alpha \bar{\psi} |0\rangle &= -i M^{1/2} \frac{\nabla^2 \phi}{M^2} \gamma_5 \sigma^\alpha \gamma_\beta, \\
\langle P, e| \psi \mp D^\alpha D^\beta \bar{\psi} |0\rangle &= \frac{M^{5/2}}{6} \frac{\nabla^2 \phi}{M^2} \gamma_5 \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{M^2}\right) \left(1 + \frac{P}{M}\right). \quad (8)
\end{align*}

Using these values of the matrix elements one can readily compute the fragmentation function, including the binding energy correction coming from \( m = M/2 + \epsilon/2 \).

\[
f_{c-\eta}(z) = \frac{64\alpha_s^2}{81\pi} \frac{|R(0)|^2}{M^2} \left[\eta_0 f_0(z) + \eta_B f_B(z) + \eta_W f_W(z)\right], \quad (9)
\]

where

\[
\begin{align*}
f_0(z) &= \frac{z(1-z)^2(48 + 8z^2 - 8z^3 + 3z^4)}{(2-z)^6}, \\
f_B(z) &= \frac{4z^2(1-z)^2(-48 + 48z^2 - 12z^3 + 5z^4)}{(2-z)^8}, \\
\text{and} \\
f_W(z) &= \frac{8z^2(1-z)^2(96 + 144z - 528z^2 + 296z^3 - 102z^4 + 43z^5 - 9z^6)}{3(2-z)^8}, \quad (10)
\end{align*}
\]

and

\[
\phi(0) = R(0)/4\pi. \quad (11)
\]
Note that
\[ \eta_0 = 1, \quad \eta_B = \frac{e_B}{M}, \quad \eta_W = \frac{\nabla^2 \phi}{M^2 \phi}, \]  
(12)

Integrating over z, we obtain the total fragmentation probability:
\[ \int_0^1 dz \, D_{c \to \eta_s}(z, 3m_c) = \frac{64 \alpha_s^2}{27 \pi} \frac{|R(0)|^2}{M^3} (\eta_0 F_0 + \eta_B F_B + \eta_W F_W), \]
(13)

where
\[ F_0 = \frac{773}{30} - 37 \log 2, \]
\[ F_B = -\frac{5639}{105} + \frac{232}{3} \log 2, \]
and
\[ F_W = -\frac{100304}{315} + \frac{4136}{9} \log 2. \]
(14)

The technique developed can be easily extended to calculate the fragmentation of a c (and b) quark to 1−− state. For this we need to calculate the corresponding matrix elements, which turn out to be,
\[ \langle P, \epsilon| \bar{\psi} \gamma^\mu \gamma_5 |0 \rangle = \frac{1}{2} M^{1/2} \left( 1 + \frac{\nabla^2}{M^2} \right) \phi \cdot (1 + \frac{P}{M}) - \frac{1}{2} M^{1/2} \frac{\nabla^2 \phi}{3M^2} \cdot (1 - \frac{P}{M}), \]
\[ \langle P, \epsilon| \bar{\psi} \gamma^\mu D_\alpha \gamma_5 |0 \rangle = -\frac{M^3 \phi}{3} \frac{\nabla^2 \phi}{M^2} \epsilon_\beta \left[ -g^{\alpha\beta} + \frac{i}{M} \epsilon^{\alpha\beta\gamma \delta} P_\gamma \gamma_5 \right], \]
\[ \langle P, \epsilon| \bar{\psi} \gamma^\mu D_\alpha D_\beta \gamma_5 |0 \rangle = \frac{1}{6} M^{5/2} \frac{\nabla^2 \phi}{M^2} \left( g_{\alpha\beta} - \frac{P_\alpha P_\beta}{M^2} \right) \cdot \left( 1 + \frac{P}{M} \right). \]
(15)

With these values of matrix elements the fragmentation function turns out to be
\[ f_{c \to J/\psi}(z) = \frac{64}{27 \pi} \alpha_s(3m_c)^2 \frac{|R(0)|^2}{M^3} |f_0(z) + \eta_B f_B(z) + \eta_W f_W(z)|, \]
(16)

where
\[ f_0(z) = \frac{z(1 - z)^2(16 - 32z + 72z^2 - 32z^3 + 5z^4)}{(2 - z)^6}, \]
\[ f_B(z) = \frac{4z^2(1 - z)^2(48 - 144z + 152z^2 - 28z^3 + 13z^4 - 2z^5)}{3(z - 2)^8}, \]
and
\[ f_W(z) = \frac{4z^2(1 - z)^2(144 + 128z - 304z^2 - 256z^3 + 261z^4 - 50z^5)}{9(2 - z)^8}. \]
(17)

Integrating over z, we obtain the total fragmentation probability:
\[ \int_0^1 dz \, D_{c \to \eta_s}(z, 3m_c) = \frac{64 \alpha_s^2}{27 \pi} \frac{|R(0)|^2}{M^3} (\eta_0 F_0 + \eta_B F_B + \eta_W F_W), \]
(18)
where

\[
\begin{align*}
F_0 & = \frac{1189}{30} - 57 \log 2, \\
F_B & = \frac{-2327}{35} + 96 \log 2, \\
\text{and } F_W & = \frac{-655558}{945} + \frac{9008}{9} \log 2.
\end{align*}
\]

We then use the Altarelli-Parisi equations to evolve this up to the scale \( Q^2 = (M_Z/2)^2 \) appropriate for \( Z_0 \) decay. The fragmentation functions including the \( O(R \nabla^2 R) \) corrections are plotted in the Fig. 2 and 3, at the scale \( Q^2 = (M_Z/2)^2 \).

For \( \eta_c \) we have used \( \alpha_S = 0.19, M_{\eta_c} = 2.98 \) and \( m_c = 1.43 \). The value of the wavefunction at the origin is taken to be \( |\phi_{\eta_c}|^2 = (0.077 \text{ GeV})^2 \) whereas \( \nabla^2 \phi/\phi = -.7 \text{ GeV}^2 \). For a \( J/\psi \) we use the same values except that now \( M_{J/\psi} = 3.096 \) and \( |\phi_{J/\psi}|^2 = (0.078 \text{ GeV})^2 \).

1 Applications

Once the fragmentation functions are calculated, they can be used to predict the cross sections and decay widths of a large number of processes. For example the decay widths of \( Z^0 \) via fragmentation or the production rate of heavy quarkonium states in \( W^\pm \), top quark, and Higgs decays.
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References


Figure Captions

1. a) Fragmentation of a quark into a $J/\Psi + X$.
   b) Fragmentation function for $c \rightarrow \eta_c$.

2. The dotted line shows the zeroth order fragmentation function for $c \rightarrow \eta_c$ at $Q^2 = (M_Z/2)^2$. The solid line includes the binding and wavefunction corrections.

3. Same as above for $c \rightarrow J/\psi$