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**CUMULANTS IN PERTURBATION EXPANSIONS FOR
NON-EQUILIBRIUM FIELD THEORY**

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Cumulants in perturbation expansions for non-equilibrium field theory*

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Abstract

The formulation of perturbation expansions for a quantum field theory of strongly interacting systems in a general non-equilibrium state is discussed. Non-vanishing initial correlations are included in the formulation of the perturbation expansion in terms of cumulants. The cumulants are shown to be the suitable candidate for summing up the perturbation expansion. Also a linked-cluster theorem for the perturbation series with cumulants is presented. Finally a generating functional of the perturbation series with initial correlations is studied. We apply the methods to a simple model of a fermion-boson system.

Keywords: Perturbation theory, non-equilibrium field theory, cumulant expansion, Wick theorem, initial correlations, linked-cluster theorem
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1 Introduction

The treatment and description of systems in non-equilibrium is a problem of great current interest in many branches of physics, as e.g. in nuclear heavy ion collisions [1,2], in plasma physics or solid state physics. Since in such strongly interacting systems exact methods are not available, a common and powerful method consists in perturbation expansions with partial

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summations of diagrams. While these methods are well developed for systems in equilibrium [3] or with special types of density matrices [1], there is a growing interest to extend these also to non-equilibrium systems.

A special feature that distinguishes non-equilibrium from equilibrium systems is the appearance of initial correlations, which are present already in the initial state of a system. The necessity to include initial correlations in the perturbation series was already stated a long time ago in the literature [4-7] but it was first carried out in a consistent way in Ref. [6]. In a previous paper the present authors [7] reformulated the argument and generalized it to arbitrary systems of interacting fermions and bosons.

Formally initial correlations arise in non-equilibrium systems because the expectation value of normal ordered products does not vanish for an arbitrary statistical matrix. In Ref. [7] the initial correlations were formulated in terms of correlation functions introduced in [6] which were derived from whiteman functions. Here we show that the use of these correlation functions, which have the property to appear at most once in a given diagram of perturbation theory, leads to at least two types of difficulties: the interpretation of the simplest correlation function, namely the expectation value of the boson field, as an external fields becomes problematic, and the summation of diagrams into self-energies to be used in a Dyson equation becomes cumbersome.

In this paper we propose to circumvent these difficulties by expressing the initial correlations in terms of cumulants. These are irreducible correlations from which all correlations of lower order have been subtracted out. These are well known in textbooks of statistical mathematics [8] and in the literature of statistical physics [9-11]. Furthermore it is known for equilibrium systems that the perturbation expansion of cumulants contains only connected diagrams [12]. We show that cumulants may appear arbitrarily many times in a diagram and thus one obtains the usual interpretation and the summation techniques of perturbation theory. Thus we assert that cumulants are the appropriate tool for the perturbation expansion of non-equilibrium systems.

This paper is organized in the following way: In Section 2 the perturbation expansion in non-equilibrium within the Schwinger-Keldysh formalism and the results for the non-equilibrium Wick theorem with correlation functions of Ref. [7] are reviewed. In Section 3 the mathematical properties of n -variate cumulants are reviewed and specialized for our purposes. In Section 4 these are applied to a reformulation of the non-equilibrium Wick theorem. These results are demonstrated in Section 5 for the fermion Green

function of a simple model of interacting fermions and bosons, which we use for demonstrations. The summation of the perturbation series of the Green functions in form of Dyson equations and a linked-cluster theorem are presented in Section 6. Finally in Section 7 we discuss the formulation of generating functionals for the free and the exact Green functions of the perturbation series with cumulants, and we end with a summary and an outlook.

2 Perturbation theory with correlation functions

In this section we first review very briefly the results of Ref. [7] in order to define the notation and the context. We then demonstrate that the perturbation series using correlation functions as defined in Refs. [6] and [7] is not yet satisfactory. This will be the motivation to introduce cumulants.

To have an illustration for the considerations on the basis of a certain model, we refer, as in Ref. [7], all explanations to the model of a fermion Dirac field interacting by minimal coupling with a scalar boson field. This model can be easily extended to a theory of a Dirac field interacting with gauge fields or to effective meson-nucleon field theories of nuclear matter which form the starting point of, e.g., the Walecka theory.

$$L(x) = L_0(x) + L'(x) \quad (1)$$

$$L_0(x) = \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) + \frac{1}{2}(\partial_\mu\phi(x)\partial^\mu\phi(x) - m_s^2\phi^2(x)), \quad (2)$$

$$L'(x) = g_s\phi(x)\bar{\psi}(x)\psi(x) = -h'(x), \quad (3)$$

$$H'(t_x) = \int d^3x h'(t_x, \mathbf{x}), \quad (4)$$

where L_0 represents the free Lagrangian, L' the interaction Lagrangian and L the total Lagrangian. ψ is a fermion field with mass m and ϕ a scalar field with mass m_s .

The non-equilibrium situation is represented by a statistical operator in the Heisenberg picture denoted by H

$$\rho_H(t) = \rho_H(t_0) = \rho_H. \quad (5)$$

Furthermore the non-equilibrium situation is taken into account by the closed-time-path (Schwinger-Keldysh) contour C which runs along the real

time axis from $-\infty$ to $+\infty$ and back to $-\infty$ [1]. On this contour in analogy to the time-ordering a contour-ordering operator O_C is defined which puts field operators with time arguments lying later on the contour to the left. With this contour-ordering operator a contour ordered Green function is defined as follows

$$iG(x, y) := \text{tr} \left[\rho_H O_C \left(\psi_H(x) \bar{\psi}_H(y) \right) \right], \quad (6)$$

in which the usual chronologically ordered Green function G_c , the anti-chronologically ordered Green function G_a , as well as the 1-particle correlations $G_>$ and $G_<$ are included [1].

In the perturbation expansion the full Green function G is expressed as a functional of the free Green function

$$iG^0(x, y) := \text{tr} \left[\rho_H O_C \left(\psi_I(x) \bar{\psi}_I(y) \right) \right] \quad (7)$$

by writing the field operators in the Heisenberg picture in terms of those in the interaction picture [1]

$$iG(x, y) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_C dt_1 \dots \int_C dt_n \times \text{tr} \left[\rho_H O_C \left(H'_I(t_1) \dots H'_I(t_n) \psi_I(x) \bar{\psi}_I(y) \right) \right]. \quad (8)$$

$\int_C$ indicates the integration along the Schwinger-Keldysh time contour C .

Ref. [7] and the present paper are concerned with the factorization of the right hand side of Eq. (8) for a general non-equilibrium situation. In Ref. [7] some convenient abbreviations were introduced to make the formulae more compact and transparent. All field operators are combined into a single symbol $\chi := \{\phi, \psi, \bar{\psi}\}$. The following functions are defined from these operators: the Wightman function as the expectation value of a product of field operators, $W_n := \langle \chi^n \rangle$, the normal ordered product, $N_n := : \chi^n :$, and the contour ordered product, $O_n := O_C(\chi^n)$ where we used $\langle . \rangle$ for $\text{tr}(\rho_H .)$. Since all operators are in the interaction picture $\langle O_2 \rangle$ is equal to the free Green function (Eq. (7)).

Also two complex functions, the pairing and the contour ordered contraction, are introduced

$$\underbrace{\chi\chi} := \chi\chi - : \chi\chi := W_2 - \langle N_2 \rangle, \quad \underbrace{\chi\chi}_C := O_C(\chi\chi) - : \chi\chi := \langle O_2 \rangle - \langle N_2 \rangle. \quad (9)$$

The contraction on the contour includes the pairing. The pairing and the contraction are complex functions since they are equal to the commutator of the field operators unless they vanish. E.g., the pairing and the contraction are zero for the pairs $\phi\psi$, $\phi\bar{\psi}$, $\psi\phi$, $\bar{\psi}\phi$, $\psi\bar{\psi}$ and $\bar{\psi}\bar{\psi}$.

Furthermore, an n -particle correlation function defined in Ref. [6] is introduced

$$C_n := \sum_{a.p.} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^P (-1)^k W_{n-2k} W_2^k, \quad (10)$$

where $\lfloor \frac{n}{2} \rfloor$ is largest integer less or equal $\frac{n}{2}$. The abbreviation $a.p.$ denotes the sum over the set of all possible pairs of the field operators but not including the pairs for which the contraction and the pairing are zero.

The expectation value of the Wick theorem for a contour ordered product [3] of field operators then has the following form in the above notations

$$\langle O_n \rangle = \sum_{a.p.} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^P \langle N_{n-2k} \rangle (\langle O_2 \rangle - \langle N_2 \rangle)^k, \quad (11)$$

and an analogous expression is valid for the Wightman function W_n with $\langle O_2 \rangle$ replaced by W_2 on the r.h.s.

The essential difference relative to the equilibrium situation is that the expectation values of normal ordered products do not vanish because it is not possible to define field operators that annihilate the non-equilibrium state at all times. As shown in [7] the functions $\langle N_n \rangle$ in (11) can be substituted by the functions W_n from the analogous equation for W_n . We arrive at

$$\langle O_n \rangle = \sum_{a.p.} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^P W_{n-2k} (\langle O_2 \rangle - W_2)^k. \quad (12)$$

Finally by using the definition of the correlation functions, Eq. (10), one can substitute the Wightman function W_n by correlation functions C_n to arrive at a Wick theorem for a non-equilibrium system including initial correlations

$$\langle O_n \rangle = \sum_{a.p.p.} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^P C_{n-2k} \langle O_2 \rangle^k. \quad (13)$$

This result was obtained in Ref. [6] for a scalar field theory and in Ref. [7] extended for interacting fermion and boson fields.

To illustrate this expression we write down the fermion Green function up to first order in the coupling constant for the above model (Eqs. (1)-(3)), keeping only connected diagrams

$$\begin{aligned}
iG(x, y) = & iG^0(x, y) + ig_s \int_C d^4 1 \left[i^2 G^0(x, 1) \langle \phi(1) \rangle^0 G^0(1, y) \right. \\
& + iG^0(x, 1) C_{\phi\psi\bar{\psi}}^0(1, 1, y) - iG^0(1, 1) C_{\phi\psi\bar{\psi}}^0(1, x, y) \\
& \left. - iG^0(1, y) C_{\phi\bar{\psi}\psi}^0(1, 1, x) + C_{\phi\bar{\psi}\psi\bar{\psi}}^0(1, 1, 1, x, y) \right]. \quad (14)
\end{aligned}$$

We remark that only terms containing at most one correlation function appear, and this is also true for higher orders as is seen from Eq. (13). A term containing two correlation functions as e.g. $\langle \phi(1) \rangle^0 C_{\psi\psi\bar{\psi}\bar{\psi}}^0(1, 1, x, y)$ is not allowed.

A first difficulty with the interpretation of Eq. (13) or (14) arises when we consider $\langle \phi(1) \rangle^0$. We can interpret the first term of first order in Eq. (14) as an interaction of the fermion with an external field. However, this interpretation becomes problematic in higher order since we would expect in a perturbation theory that the fermion interacts arbitrarily many times with the external field.

Additional difficulties appear when we consider the summation of diagrams with the help of irreducible self-energies. An irreducible self-energy is an insertion in the fermion line which cannot be disconnected by cutting any internal line. In an usual field theory the irreducible self-energy is used to sum up the perturbation expansion by an integral equation for the exact Green function

$$G(x, y) = G^0(x, y) + \int d^4 1 d^4 2 G^0(x, 1) \Sigma(1, 2) G(2, y). \quad (15)$$

In our situation we would have to distinguish between irreducible self-energies with and without a correlation function which are denoted by Σ_C and Σ , respectively. Since the correlation functions appear only once in an entire diagram, irreducible self-energy terms containing a correlation function appear only once in contrast to irreducible self-energy terms without correlation functions which appear arbitrarily many times in a diagram of the perturbation expansion (see Fig. 1). If we neglect for a moment correlation functions which are connected with one or both of the external points x and y of the exact Green function $G(x, y)$, we can write the following system of equations

$$G(x, y) = G^0(x, y) + \int d^4 1 d^4 2 \bar{G}(x, 1) \Sigma_C(1, 2) \bar{G}(2, y), \quad (16)$$

$$\bar{G}(x, y) = G^0(x, y) + \int d^4 1 d^4 2 G^0(x, 1) \Sigma(1, 2) \bar{G}(2, y), \quad (17)$$

which is also illustrated in Fig. 1. If we consider diagrams with correlation functions at the external points, there would be two more equations. It is clear that the irreducible self-energy with a correlation function cannot be treated in the same manner as one without correlation functions. The consequence is a more involved and unsymmetric treatment in the summation of the perturbation expansion.

The main task of this paper is to show that we can overcome this unsymmetric treatment of irreducible self-energies with and without correlation functions by using a perturbation expansion by cumulants instead of the correlation functions defined above. This will be seen in the subsequent sections.

3 Cumulants

At the beginning of this section we briefly want to recall the mathematical definition of cumulants. The cumulants are also known as irreducible correlations, since all correlations of lower order are separated off. Therefore they are of great importance in treating statistical problems in, e.g., multi-particle measurements [9,11]. We start the review with the univariate case, i.e. with one observable, and then treat the multivariate or n -variate case. The basic statistical as well as physical quantities are the moments of the observable(s). If in the univariate case the observable is noted by x , the first moment of the observable x is its expectation value $\langle x \rangle$. The expectation value of the n -th power of x is the n -th moment $\langle x^n \rangle$.

The moments can be derived from a moment generating function Φ which is often called the characteristic function [8]. In contrast to the literature we use a real generating function

$$\Phi(t) = \langle e^{tx} \rangle, \quad \langle x^n \rangle = \frac{d^n}{dt^n} \Phi(t) \Big|_{t=0}. \quad (18)$$

Let κ_n denote the cumulant of order n in x . The cumulants and the moments are related to each other in the following way

$$\begin{aligned} \Phi(t) &= 1 + \langle x \rangle t + \frac{\langle x^2 \rangle t^2}{2!} + \frac{\langle x^3 \rangle t^3}{3!} + \dots \\ &= \exp \left(\kappa_1 t + \frac{\kappa_2 t^2}{2!} + \frac{\kappa_3 t^3}{3!} + \dots \right). \end{aligned} \quad (19)$$

The inverse relation is given by the logarithm of Eq. (19) and one sees that $\ln \Phi$ is the generating function of the cumulants

$$\kappa_n = \frac{d^n}{dt^n} \ln \Phi(t) \Big|_{t=0}. \quad (20)$$

The generalization of the above relations to n -variate cumulants and moments is straightforward. We consider the moments of the observables $x_1, \dots, x_n$ for orders $k_1, \dots, k_n$: $\langle x_1^{k_1} \dots x_n^{k_n} \rangle$. The generating function of the general n -variate moment uses a parameter t_i for each observable x_i

$$\Phi(t_1, \dots, t_n) := \left\langle \exp \left(\sum_i x_i t_i \right) \right\rangle. \quad (21)$$

$$\langle x_1^{k_1} \dots x_n^{k_n} \rangle = \frac{\partial^{k_1}}{\partial t_1^{k_1}} \dots \frac{\partial^{k_n}}{\partial t_n^{k_n}} \Phi(t_1, \dots, t_n) \Big|_{t_1=\dots=t_n=0}. \quad (22)$$

The n -variate cumulant is specified by integers $i_1, \dots, i_n$ for the order of the cumulant in each observable $x_1, \dots, x_n$: $\kappa(x_1^{i_1}, \dots, x_n^{i_n})$. The relation between the n -variate cumulants and moments extends Eq. (19) in an obvious way

$$\sum_{k_1, \dots, k_n=0}^{\infty} \langle x_1^{k_1} \dots x_n^{k_n} \rangle \frac{t_1^{k_1} \dots t_n^{k_n}}{k_1! \dots k_n!} = \exp \left(\sum_{i_1, \dots, i_n=0}^{\infty} \kappa(x_1^{i_1}, \dots, x_n^{i_n}) \frac{t_1^{i_1} \dots t_n^{i_n}}{i_1! \dots i_n!} \right), \quad (23)$$

and we obtain the inverse relation by the logarithm of Eq. (23).

In the context of the Wick theorem we are only interested in cumulants and moments in which each observable appears at most in first order. The relation between moments and cumulants of this kind can be obtained from Eq. (23) and its inverse by differentiation with respect $t_1, \dots, t_n$ and then setting $t_1, \dots, t_n$ equal to zero. We calculate the moments as functional of the cumulants

$$\langle x_1 \dots x_n \rangle = \frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_n} \exp \left(\sum_{i_1, \dots, i_n=0}^{\infty} \kappa(x_1^{i_1}, \dots, x_n^{i_n}) \frac{t_1^{i_1} \dots t_n^{i_n}}{i_1! \dots i_n!} \right) \Big|_{t_1=\dots=t_n=0} \quad (24)$$

$$= \sum_{a.p.} \sum_{r=1}^n \kappa(x_1, \dots, x_n)_r. \quad (25)$$

In the last line $\kappa(x_1, \dots, x_n)_r$ denotes a product of r cumulants whose arguments are a partition of the n arguments $x_1, \dots, x_n$ into r disjunct sets. $\sum_{a.p.}$

is the sum over all possible partitions. In Eq. (25) two factors $r!$ cancel: a factor $\frac{1}{r!}$ accounts for the fact that a product of r cumulants appears at the r -th power in the power series of the exponential and a factor $r!$ counts the number how often we can get the same partition in the exponential series.

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As an example we factorize $\langle x_1 x_2 x_3 x_4 \rangle$ into cumulants

$$\begin{aligned} \langle x_1 x_2 x_3 x_4 \rangle = & \kappa(1, 2, 3, 4) + \kappa(1) \kappa(2, 3, 4) + \dots + \kappa(1, 2) \kappa(3, 4) + \dots \\ & + \kappa(1) \kappa(2) \kappa(3, 4) + \dots + \kappa(1) \kappa(2) \kappa(3) \kappa(4). \end{aligned} \quad (26)$$

We used the abbreviation $\kappa(1, 3, 4)$ for a cumulant of three observables with the arguments 1, 3 and 4 and so on. The inverse relation is found to be

$$\kappa(x_1, \dots, x_n) = \sum_{a.p.} \sum_{r=1}^n (-1)^{(r-1)} (r-1)! \langle x_1, \dots, x_n \rangle_r, \quad (27)$$

where $\langle x_1, \dots, x_n \rangle_r$ denotes a product of r moments whose arguments are a partition of the n arguments $x_1, \dots, x_n$ into r disjunct sets. As an example the cumulant $\kappa(x_1, x_2, x_3, x_4)$ is written as

$$\begin{aligned} \kappa(1, 2, 3, 4) = & \langle x_1 x_2 x_3 x_4 \rangle - \langle x_1 \rangle \langle x_2 x_3 x_4 \rangle - \dots - \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle - \dots \\ & + 2! \langle x_1 \rangle \langle x_2 \rangle \langle x_3 x_4 \rangle + \dots - 3! \langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle \langle x_4 \rangle. \end{aligned} \quad (28)$$

Until now x was a real observable. To describe fermions besides bosons x also has to be a Grassmann variable. Each commutation of two Grassmann variables which is necessary to obtain the correct ordering generates a factor (-1) . Then the cumulant expansion of fermion variables has a factor $(-1)^P$ on the r.h.s. of Eq. (25) from the permutations necessary for the reordering of the fermion variables.

The cumulants which we use in the following, should be complex functions also for fermion field operators. Therefore contributions with moments from an unequal number of fermion field operators must be excluded.

4 Non-equilibrium Wick theorem with cumulants

We now use the cumulants introduced above in the perturbation expansion of non-equilibrium systems taking as the observables x the field operators χ . We will see that using cumulants one can avoid the difficulty of the unsymmetrical appearance of self-energies with and without correlation functions

in the perturbation expansion discussed at the end of section 2. We start from Eq. (12), where the $(n - 2k)$ -point Wightman function is just the moment of the corresponding field operators in the above terminology. We note that all field operators can be distinguished in our formulation, i.e. we have the case outlined above in which the different observables appear only up to the first order. Using Eq. (25) one writes W_n as a functional of cumulants

$$W_n = \sum_{r=1}^n \sum_{a.p.} (-1)^P \kappa\{n\}_r, \quad (29)$$

where $\kappa\{n\}_r$ is the abbreviation for the product of r cumulants whose arguments are a partition of the n arguments of W_n into r sets and $\sum_{a.p.}$ is the sum over all possible partitions. The number of fermion field operators in each set is even. The factor $(-1)^P$ takes into account the necessary permutations of fermion field operators to get the correct ordering.

However, since in Eq. (12) also W_2 appears, which contains κ_2 , it is necessary to isolate factors of κ_2 in Eq. (29). Thus we write

$$W_n = \sum_{r=1}^n \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{a.p.} (-1)^P \kappa\{n - 2i\}'_r \kappa_2^i. \quad (30)$$

Here $\kappa\{n\}'_r$, as $\kappa\{n\}_r$, denotes a product of r cumulants whose arguments are a partition of the n arguments into r distinct sets but with the constraint that no 2-point-cumulant $\kappa\{2\}_1$ appears in this product. This expansion is inserted in Eq. (12) and rearranged to order the 2-point correlations. The explicit steps are executed in the Appendix. The result is

$$\langle O_n \rangle = \sum_{a.p.} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{n-2s} (-1)^P \kappa\{n - 2s\}'_r \overline{\langle O_2 \rangle}^s. \quad (31)$$

Here $\overline{\langle O_2 \rangle}$ is the 2-point Green function, where in the case of bosons the expectation values of the field amplitudes are subtracted

$$\overline{\langle O_2 \rangle} = \begin{cases} \langle O_2 \rangle - \langle \phi \rangle^0 \langle \phi \rangle^0 & \text{for bosons,} \\ \langle O_2 \rangle & \text{for fermions.} \end{cases} \quad (32)$$

Eq. (31) is the non-equilibrium Wick-theorem with cumulants. We note that any cumulant except for the cumulant κ_2 can appear any number of times in a diagram depending on the number of vertices. This is in contrast

to Eq. (13), where each correlation function could appear only once. When Eq. (31) is used in Eq. (8), the perturbation series becomes a sum over products of free Green functions and cumulants.

Let us make a few comments on the Wick theorem obtained. Comparing the cumulant expansion of the Wightman function W_n in Eq. (30) with the corresponding expansion of $\langle O_n \rangle$ in Eq. (31) one sees that κ_2 is just replaced by the free Green function $\overline{\langle O_2 \rangle}$. Thus the Wick theorem is a special cumulant expansion for contour ordered products. On the other hand one could have written down a cumulant expansion for $\langle O_n \rangle$ in analogy to Eq. (29) right away as

$$\begin{aligned} \langle O_n \rangle &= \sum_{a.p.} \sum_{r=1}^n (-1)^P \kappa_C \{n\}_r \\ &= \sum_{a.p.} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{n-2s} (-1)^P \kappa_C \{n-2s\}_r \overline{\langle O_2 \rangle}^s. \end{aligned} \quad (33)$$

The cumulants $\kappa \{n\}_r$ are replaced by contour ordered cumulants $\kappa_C \{n\}_r$ and the contour ordered 2-point cumulant $\kappa_C \{2\}_1$ is the free Green function $\overline{\langle O_2 \rangle}$. In contrast to Eq. (33) the cumulants in the Wick theorem, Eq. (31), are not contour ordered. Thus, the Wick theorem is a simplification for the cumulant expansion of the expectation value of a contour ordered product, since the causal structure of the factorization is simpler.

5 Perturbation expansion of the fermion Green function

To get familiar to the Wick theorem with cumulants we apply it, as at the end of section 2, to the perturbation series of the fermion Green function but now up to second order in the coupling constant. We use the same notation as there, but now imply the use of cumulants. The simplest cumulant is the expectation value of the meson field $C_\phi^0(1) = \langle \phi(1) \rangle^0$. An example of a higher order cumulants is: $C_{\phi\bar{\psi}\psi}^0(1,2,3) = \kappa(\phi(1)\bar{\psi}(2)\psi(3))$. The perturbation series is then

$$\begin{aligned} iG(x,y) &= \langle O_C \left(\psi(x)\bar{\psi}(y) \right) \rangle + ig_s \int_C d^4 1 \langle O_C \left(\phi(1)\bar{\psi}(1)\psi(1)\psi(x)\bar{\psi}(y) \right) \rangle \\ &\quad + \frac{(ig_s)^2}{2!} \int_C d^4 1 d^4 2 \langle O_C \left(\phi(1)\bar{\psi}(1)\psi(1) \right. \end{aligned}$$

$$\begin{aligned}
& \times \phi(2) \bar{\psi}(2) \psi(2) \psi(x) \bar{\psi}(y) \rangle \rangle \quad (34) \\
= & iG^0(x, y) + (ig_s) \int_C d^4 1 \left[i^2 G^0(x, 1) \langle \phi(1) \rangle^0 G^0(1, y) \right. \\
& + i [C_{\phi \bar{\psi}}^0(1, 1, y) G^0(x, 1) - C_{\phi \bar{\psi}}^0(1, x, y) G^0(1, 1) \\
& \quad - C_{\phi \bar{\psi}}^0(1, 1, x) G^0(1, y)] \\
& + [C_{\phi \bar{\psi} \psi \bar{\psi}}^0(1, 1, 1, x, y) + \langle \phi(1) \rangle^0 C_{\bar{\psi} \psi \bar{\psi}}^0(1, 1, x, y)] \Big] \\
& + (ig_s)^2 \int_C d^4 1 d^4 2 \left[i^4 [G^0(x, 1) \bar{D}^0(1, 2) G^0(1, 2) G^0(2, y) \right. \\
& \quad - G^0(x, 1) \bar{D}^0(1, 2) G^0(2, 2) G^0(1, y)] \\
& + i^3 [G^0(x, 1) \langle \phi(1) \rangle^0 G^0(1, 2) \langle \phi(2) \rangle^0 G^0(2, y)] \\
& \left. + i^2 [G^0(x, 1) \langle \phi(1) \rangle^0 C_{\phi \bar{\psi}}^0(1, 2, 2) G^0(2, y) + \dots] + \dots \right] \quad (35)
\end{aligned}$$

In second order we do not write down all terms because of the great number of diagrams but only some well-known and some new and interesting terms. In Fig. 2 we represent this perturbation series in graphical manner using the following representation: a solid line for the fermion Green function, a dashed line for the boson Green function and a dot for a vertex. The correlation functions are represented by hatched bubbles with a number of points corresponding to the number of correlated fermion (dots) or boson (crosses) field operators. The arrows in the interior of the bubble pointing inward or outward, indicate $\bar{\psi}$ respectively ψ field operators in the correlation function. Arguments coinciding in space-time are indicated by a circle around these points. A single cross represents the correlation C_ϕ^0 .

One recognizes several diagrams in which more than one correlation appears. A good example is the one in second order which contains twice the correlation C_ϕ^0 . Since C_ϕ^0 evolves according to the free field equation, one can interpret C_ϕ^0 as an external field interacting now arbitrarily many times with the propagating fermion. Thus the interpretation problem for the correlations functions is overcome.

6 Summations and linked-cluster theorem

We now take up the second problem mentioned at the end of section 2 and discuss the summation of the perturbation series. A glance at Fig. 2 shows that we have to distinguish whether a cumulant is in the interior of a diagram or connected to an outer point. We distinguish five different irreducible self-

energies depending on how the cumulants in a diagram are connected to the external space-time points of the exact Green function. The first one, Σ_G , contains no cumulants, but only Green functions. The other four contain at least one cumulant. The first of these, Σ_κ^n is not connected to any of the external space-time points. The others are connected either to the first space-time point (Σ_κ^f), to the second one (Σ_κ^s), or to both (Σ_κ^b). The last three self-energies can be reformulated as "internal" correlations using the following trick. Take the differential operators of the free field equation $D_x = i\gamma_\mu \partial_x^\mu - m$ and $\bar{D}_y = -i\gamma_\mu \bar{\partial}_y^\mu - m$ which obey

$$D_x G^0(x, y) = \delta^4(x - y), \quad G^0(x, y) \bar{D}_y = \delta^4(x - y). \quad (36)$$

Then, e.g. for the self-energy with the cumulant connected to the second argument y , we write the identity

$$\Sigma_\kappa^s(x, y) = \int d^4 1 \Sigma_\kappa^s(x, 1) D_1 G^0(1, y), \quad (37)$$

$$= \int d^4 1 \left(\Sigma_\kappa^s(x, 1) \bar{D}_1 \right) G^0(1, y), \quad (38)$$

and define a modified self-energy

$$\tilde{\Sigma}_\kappa^s(x, 1) = \Sigma_\kappa^s(x, 1) \bar{D}_1, \quad (39)$$

which is now not connected to y . In going from Eq. (37) to Eq. (38) a finite system was assumed and such that the surface integral vanishes. In the same way we define

$$\tilde{\Sigma}_\kappa^f(x, y) := D_x \Sigma_\kappa^f(x, y), \quad (40)$$

$$\tilde{\Sigma}_\kappa^b(x, y) := D_x \Sigma_\kappa^b(x, y) \bar{D}_y. \quad (41)$$

All of these modified self-energies are connected to other insertions or to an external point by a Green function like the self-energies Σ and Σ_κ^n . Thus all self-energies can be treated in the same manner and can be collected into a single self-energy Σ

$$\Sigma = \Sigma_G + \Sigma_\kappa^n + \tilde{\Sigma}_\kappa^f + \tilde{\Sigma}_\kappa^s + \tilde{\Sigma}_\kappa^b. \quad (42)$$

We then get back the usual Dyson equation for the Green function with the only difference that the self-energies insertions now contain correlations

$$G(x, y) = G^0(x, y) + \int d^4 1 d^4 2 G^0(x, 1) \Sigma(1, 2) G(2, y). \quad (43)$$

At the end of this section we show that a linked-cluster-theorem is valid for this perturbation expansion of the fermion Green function in which initial correlations in terms of cumulants are included. From diagram technics we know that the disconnected diagrams (*disc*) factorize into a factor of a connected diagram (*conn*) and a factor consisting of closed diagrams (*cl*) which are not connected to any other diagram

$$\langle O_n \rangle_{disc} = \langle O_l \rangle_{cl} \cdot \langle O_m \rangle_{conn}, \quad n = l + m. \quad (44)$$

With $l = 0$ and $\langle O_0 \rangle_{cl} = 1$ we include the connected diagrams in the disconnected diagrams. It is immediately clear that in the connected as well as in the closed diagrams cumulants can appear. If we sum over all disconnected diagrams the above factorization remains valid as in standard quantum field theory since cumulants can appear arbitrarily many times. This is another important simplification in the use of cumulants, because this statement would not have been true if we had used initial correlations in terms of correlations functions defined in Ref. [6]. Thus we write

$$\sum_{a.p.} \sum_{n=2}^{\infty} \langle O_n \rangle_{disc} = \sum_{a.p.} \sum_{l=0}^{\infty} \langle O_l \rangle_{cl} \cdot \sum_{a.p.} \sum_{m=2}^{\infty} \langle O_m \rangle_{conn}. \quad (45)$$

From the Schwinger-Keldysh formalism we can write

$$1 = \langle U_I(-\infty, \infty) U_I(\infty, -\infty) \rangle = \sum_{a.p.} \sum_{l=0}^{\infty} \langle O_l \rangle_{cl}. \quad (46)$$

Thus the factor consisting of the sum over all the closed diagrams is equal to one. We then obtain a linked-cluster-theorem for the perturbation series with cumulants for the 1-particle Green function which states that we have only to sum up connected diagrams.

7 Generating functionals including correlations

In this last section we want to formulate a generating functional from which we can derive the perturbation series with cumulants. We start with a generating functional for the free n -particle Green function. Instead of the generating functions for the moments or cumulants (Eq. (21)) we use functional integrals on the Schwinger-Keldysh contour

$$Z_0[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_C d^4 1 \dots \int_C d^4 n J(1) \dots J(n) \langle O_C(\chi_I(1) \dots \chi_I(n)) \rangle. \quad (47)$$

As before χ denotes all kinds of field operators ϕ , ψ and $\bar{\psi}$ in the interaction picture, so that the functions J represent three different source functions J_ϕ , J_ψ and $J_{\bar{\psi}}$. The contour ordered free Green function is obtained by functional differentiation with respect to the source functions J

$$\langle O_C(\chi_I(1) \dots \chi_I(n)) \rangle = (-i)^n \frac{\delta^n Z_0[J]}{\delta J(1) \dots \delta J(n)} \Big|_{J=0}. \quad (48)$$

In the next step we replace in Eq. (47) the expectation value of the contour ordered product of field operators by their cumulant expansion now in form of a functional integral (compare Eq. (33))

$$Z_0[J] = \exp \left[\sum_{n=1}^{\infty} \frac{i^n}{n!} \int_C d^4 1 \dots \int_C d^4 n J(1) \dots J(n) \kappa_C^0(1, \dots, n) \right]. \quad (49)$$

As stated already at the end of section 4, the Wick theorem with cumulants restricts the general cumulant expansion of Eq. (33) into that of Eq. (31). For the generating functional in the present formalism we therefore obtain

$$\begin{aligned} Z_0[J] = & \exp \left[\int_C d^4 1 J(1) \kappa^0(1) + \frac{1}{2} \int_C d^4 1 d^4 2 J(1) J(2) \overline{\langle O_2 \rangle}(1, 2) \right. \\ & \left. + \sum_{n=3}^{\infty} \frac{1}{n!} \int_C d^4 1 \dots d^4 n J(1) \dots J(n) \kappa^0(1, \dots, n) \right]. \quad (50) \end{aligned}$$

We now turn to the analogous generating functional for the perturbation series of the exact n -point Green function

$$Z[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{C_n} d^4 1 \dots d^4 n J(1) \dots J(n) \langle O_C(\chi_H(1) \dots \chi_H(n)) \rangle, \quad (51)$$

and insert the perturbation expansion for the n -point Green function in analogy to Eq. (8)

$$\begin{aligned} Z[J] = & \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{C_n} d^4 1 \dots d^4 n J(1) \dots J(n) \cdot \\ & \cdot \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{-i}{\hbar} \right)^m \int_{C_n} d^4 1' \dots d^4 m' \langle O_C(H'_I[\chi_I(1')] \dots \\ & \dots H'_I[\chi_I(m')] \chi_I(1) \dots \chi_I(n)) \rangle \quad (52) \\ = & \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{C_n} d^4 1 \dots d^4 n J(1) \dots J(n) \end{aligned}$$

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{-i}{\hbar} \right) \int_C d^4 1' \dots d^4 m H'_I \left[\frac{\delta}{\delta J(1')} \right] \cdot \dots \\ & \dots \cdot H'_I \left[\frac{\delta}{\delta J(m')} \right] \frac{\delta}{\delta J(1)} \dots \frac{\delta}{\delta J(n)} Z_0[J] \Big|_{J=0}. \end{aligned} \quad (53)$$

Using

$$\frac{\delta J(y)}{\delta J(x)} = \delta(x-y) \implies \int d^4 x J(x) \frac{\delta J(y)}{\delta J(x)} = J(y), \quad (54)$$

Eq. (53) can be reformulated as

$$\begin{aligned} Z[J] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar} \right)^n \int_C d^4 1 \dots d^4 n H'_I \left[\frac{\delta}{\delta J(1)} \right] \cdot \dots \\ & \dots \cdot H'_I \left[\frac{\delta}{\delta J(n)} \right] Z_0[J] \end{aligned} \quad (55)$$

$$= \exp \left[-\frac{i}{\hbar} \int_C d^4 x H'_I \left[\frac{\delta}{\delta J(x)} \right] \right] Z_0[J]. \quad (56)$$

Thus a compact expression for the generating functional of the exact Green function can be obtained.

In the last part of this section we want to discuss the relation between connected diagrams for n -point functions and the exact irreducible n -point correlations. From functional approaches in quantum field theory we know that the generating functional for connected diagrams W is equal to the logarithm of the generating functional of non-connected diagrams i.e. the above generating functional $Z[J]$ [13]

$$W[J] = \ln Z[J]. \quad (57)$$

In analogy to Eq. (33), the expansion of the exact Green function into contour ordered exact cumulants κ_C can be written

$$\sum_{n=1}^{\infty} \int_C d^4 1 \dots d^4 n J(1) \dots J(n) \langle O_C(\chi_H(1) \dots \chi_H(n)) \rangle = \quad (58)$$

$$= \exp \left[\sum_{n=1}^{\infty} \int_C d^4 1 \dots d^4 n J(1) \dots J(n) \kappa_C(1, \dots, n) \right]. \quad (59)$$

With this cumulant expansion we find the generating functional for connected diagrams which at the same time is the generating functional of the

exact contour ordered n -point cumulant

$$W[J] = \sum_{n=1}^{\infty} \int_C d^4 1 \dots d^4 n J(1) \dots J(n) \kappa_C(1, \dots, n) \quad (60)$$

$$\kappa_C(1, \dots, n) = \left. \frac{\delta^n W[J]}{\delta J(1) \dots \delta J(n)} \right|_{J=0} = \left. \frac{\delta^n \ln Z[J]}{\delta J(1) \dots \delta J(n)} \right|_{J=0} \quad (61)$$

Therefore, in the diagrammatic representation of the perturbation series, the n -point cumulants are given by a sum only over connected diagrams. This result was already noticed in Ref. [12]. The above result generalizes this result by including initial correlations in form of cumulants into the perturbation expansion, which are present for non-equilibrium conditions of the system.

8 Summary and outlook

We first summarize the results of this paper. It is to be seen as a step into the development of field theoretical methods for strongly interacting system in non-equilibrium situations. An example are heavy ion collisions at relativistic energies which are described in terms of a meson-nucleon effective field theory such as Quantum Hadrodynamics, which we also take as an example here. But applications in other fields as plasma physics, solid state physics, etc. are possible. Generally the appropriate methods for a microscopic treatment of such systems are perturbation expansions and partial resummations. In this paper we start from the results of Refs. [6] and [7] that perturbation expansions of non-equilibrium systems necessarily contain initial correlations which cannot be factorized into single particle Green functions. This is expressed in the fact, discussed in detail in Ref. [7], that a Wick theorem for such systems contains correlation functions which originate from the non-zero expectation values of normal-ordered products.

There are still different ways to express these correlations. It is reasonable to define a correlation function in such a way that correlations of lower order are subtracted out in some way. In Refs. [6,7] the Whiteman functions were completely expressed in terms of such correlations. It was then shown in Ref. [7] that these correlation functions appear at most once in a given diagram of the perturbation expansion.

In this paper we argue that this approach leads to certain difficulties in the interpretation and the summation of the perturbation series. The

first is due to the possibility to interpret the lowest order correlation in the boson field $C_\phi = \langle \phi \rangle$ as an interaction with an external field, which in this way is incorporated in a natural way into the field theory. However, in the formalism of Ref. [7] this interaction with the external field appears only once, which is not what one would expect. A second - and more serious difficulty - occurs in the summation of diagrams. Since correlation terms appear at most once they have to be treated explicitly in summations. This makes cumbersome the formulation of self-energies and a Dyson equation.

Here we propose that these difficulties are circumvented when cumulants are used to express the initial correlations. Cumulants are a well known concept in statistical physics. They are irreducible correlations in the sense that all cumulants of lower order are subtracted out. Thus they are the pure n -point correlations and are, e.g., used in cluster decompositions, in interferometric problems of higher order [9,11]. We show that cumulants can appear arbitrarily many times in the diagrams of the expansion. This solves the problem of interpretation of the interaction with an external field and also allows to define an effective self-energy which can be inserted into a Dyson equation. We also show how to derive in a simple way a linked-cluster theorem for the perturbation series with cumulants which would have been much more involved with the original correlation functions [14]. We thus find a perturbation series with all the customary features and we assert that cumulants are the appropriate tool for the treatment of non-equilibrium systems.

We also show some general results about cumulant expansions. The Wick theorem with cumulants is shown to be a specific cumulant expansion of contour-ordered products with a particularly simple causal structure since only the two-point cumulants, i.e. the free Green functions, have to be contour-ordered. In connection with Ref. [12] we state that the perturbation series of the exact n -point cumulant, which is the irreducibly-correlated part of the exact n -point Green function, consists only of connected diagrams. We also formulate generating functions for cumulants.

With these developments we have made available tools for investigations of non-equilibrium systems including correlations, e.g., in kinetic equations. In particular, it seems to be of interest to develop a transport theory of equal-time quantities because of the easier way of formulating initial conditions. In such equations the above formalism allows to take into account contributions of initial state correlations on the evolution of the 1-particle Green function consistently. In addition one would have to consider the evolution of the correlations functions themselves, thus obtaining a unified

treatment of initial and dynamical correlations and of the interrelations between these and the self-energy terms. By partially summing the perturbation series with cumulants it is possible to define self-consistent integral equations which are well adapted to describe strongly interacting systems like nuclear matter. For such investigations we refer to future papers.

9 Appendix

In this appendix we execute the necessary steps to reformulate Eq. (12) into the Wick theorem, Eq. (31). The procedure is somewhat similar as in Ref. [7]. We first rewrite the factors $\langle O_2 \rangle - W_2$ as

$$\langle O_2 \rangle - W_2 = \langle O_2 \rangle - f - W_2 + f =: \overline{\langle O_2 \rangle} - \overline{W_2}. \quad (62)$$

In the case of fermion field operators, we choose f equal to zero. $\overline{\langle O_2 \rangle}$ then is the free Green function and $\overline{W_2}$ the cumulant κ_2 since amplitudes of single field operators do not appear. In the case of boson field operators f is taken as $\langle \phi \rangle^0 \langle \phi \rangle^0$ i.e. as the product of the expectation values of the boson field operator. $\overline{\langle O_2 \rangle}$ then is the free Green function minus the product of the corresponding field amplitudes and $\overline{W_2}$ is the cumulant κ_2 . Inserting Eq. (30) into Eq. (12) and expanding the multinomial $(\langle O_2 \rangle - W_2)^k$, using Eq. (62), one obtains

$$\begin{aligned} \langle O_n \rangle &= \sum_{a.p.} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - k} \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 2k - 2i} (-1)^P \kappa \{n - 2k - 2i\}'_r \kappa_2^i \\ &\quad \times \sum_{l=0}^k (-1)^l \overline{\langle O_2 \rangle}^{k-l} \kappa_2^l \end{aligned} \quad (63)$$

$$\begin{aligned} &= \sum_{a.p.} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - k} \sum_{l=0}^k \sum_{r=1}^{n-2k-2i} (-1)^P (-1)^l \binom{l+i}{l} \\ &\quad \times \kappa \{n - 2l - 2i\}'_r \overline{\langle O_2 \rangle}^{k-l} \kappa_2^{l+i} \end{aligned} \quad (64)$$

$$\begin{aligned} &= \sum_{a.p.} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{t=0}^s \sum_{i=0}^{s-t} \sum_{r=1}^{n-2s} (-1)^P (-1)^l \binom{l+i}{l} \\ &\quad \times \kappa \{n - 2s\}'_r \overline{\langle O_2 \rangle}^{s-l-i} \kappa_2^{l+i} \end{aligned} \quad (65)$$

$$= \sum_{a.p.} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{t=0}^s \sum_r^{n-2s} (-1)^P \kappa \{n - 2s\}'_r \overline{\langle O_2 \rangle}^{s-t} \kappa_2^t \sum_{l=0}^t (-1)^l \binom{t}{l} \quad (66)$$

$$= \sum_{a.p.} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{n-2s} (-1)^P \kappa\{n-2s\}'_r \overline{\langle O_2 \rangle}^s. \quad (67)$$

In Eq. (63) we used no combinatorial factor since the functions in $(\overline{\langle O_2 \rangle} - W_2)^k$ can be distinguished in Eq. (12) because of their different arguments. However, from Eq. (63) to Eq. (64) we implemented a combinatorial factor to count the number of possibilities whether κ_2 stems from κ_2^i or $\kappa\{n-2k\}_r$ in Eq. (63). From Eq. (64) to Eq. (65) we substituted (k, i) by $(s, i) = (i+k, i)$ with $s = 0 \dots \lfloor \frac{n}{2} \rfloor, i = 0 \dots s$. From Eq. (65) to Eq. (66) we substituted (i, l) by $(t, l) = (i+l, l)$ with $t = 0 \dots s, l = 0 \dots t$. From Eq. (66) to Eq. (67) we used that the sum over l is equal to $\delta_{t,0}$ [7]. In Eq. (67) we obtain the desired expression for the Wick theorem for a non-equilibrium system in which initial corelations appear as cumulants.

10 Literature

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Figure captions

Fig. 1: Perturbation series of the fermion Green function with correlation functions formulated in terms of irreducible self-energies with (Σ_C) and without correlation functions (Σ) and its possible summation.

Fig. 2: Perturbation series of the fermion Green function formulated with cumulants according to Eq. (35) in a graphical representation. The cumulant C_ϕ^0 is represented by a cross, the n -point cumulants by hatched bubbles with dots for correlated fermion and crosses for correlated meson field operators.

Fig. 1

Fig. 2