AD

MPI-PhT/95-85
September 1995

A Decoupling Theorem for the BPHZL-Scheme

MATHIAS ASCHENBRENNER*

*Max-Planck-Institut für Physik
Werner-Heisenberg-Institut
Föhringer Ring 6
D-80805 München, Germany*

509548

Abstract

Conditions are stated, which are sufficient for the heavy-mass-suppression of BPHZL-subtracted Feynman-integrals containing propagators of “heavy fields”. This result generalizes the Decoupling Theorems of Ambjørn, Manoukian and Landsman to cases, where massless fields (e.g. gauge fields) are present.

*e-mail: maa@mppmu.mpg.de

1 Introduction

In this paper, a Decoupling Theorem is proved, which may be used to study the Appelquist–Carazzone–Decoupling [1] in renormalized perturbation theory when massless fields are present.

Using quite different techniques, Ambjørn [2], Manoukian [3] and Landsman [4] could verify Decoupling Theorems, which are valid for Feynman–integrals formulated in momentum–subtraction schemes. Without relying on Feynman–diagrams, Girardello and Zaffaroni [5], Ball and Thorne [6] and Kim [7] recently derived analogous results by exploiting the perturbative solution of the “exact renormalization group”. All these proofs, however, seem to apply only to massive field theories¹.

The methods, to be found in the literature, which deal with decoupling–phenomena in the presence of massless fields turn out to be rather heuristic. In addition, these treatments are either incomplete ([8] consider only convergent Feynman–integrals) or based on considerably strange renormalisation–prescriptions ([9] use a combination of minimal pole subtraction and additional momentum subtraction off the origin), which at least – neglecting problems of consistency or (IR–)convergence – will make the use of e.g. the Action Principle rather tedious.

A rigorous proof of a Decoupling Theorem, which also covers situations where massless fields are present, seemed to be lacking. To fill this gap, Manoukian’s method – a combination of momentum–space techniques with Weinberg’s [10] and Fink’s [11] class of functions $B_n(I)$ – is suitably generalized to the BPHZL–framework [12] [13], a momentum–subtraction scheme, which allows the treatment of massless particles. It is proved, that under rather general conditions, a BPHZL–subtracted Feynman–integral associated to a Feynman–diagram Σ belongs as a function of the “heavy–mass” scaling–parameter η to the class $B_1(\emptyset)$; as a consequence its absolute value may be bounded as

$$\left| \int d^{4m} k R_{\Sigma, \epsilon}(p, k, \eta M, \dots) \right| \leq A_{\Sigma, \epsilon} \eta^\alpha (\ln \eta)^\beta \quad \text{for } 1 < b < \eta.$$

Provided certain conditions are satisfied, the exponents α and β (a non–negative integer) satisfy upper bounds, which depend on specific features of Σ . For example, α is less equal -1 , if Σ contains “heavy” propagators; this implies the “manifest–decoupling–property”: integrals containing propagators of “heavy fields” do not contribute in the “heavy–mass–limit”.

Due to this property, the BPHZL–scheme is very convenient to prove the decoupling of “heavy fields” in a given field–theoretic model. The results, presented here,

¹Only [5] could be a counterexample to this statement: It is not clear to the author, however, whether the removal of the IR–cut–off – the real problem in the presence of massless fields – is considered in this paper.

are general enough to treat BPHZL-subtracted contributions to Green-functions and Green-functions with one possibly non-minimally subtracted insertion:

$$\left\langle T F_{a_1} \dots F_{a_t} \left\{ \frac{1}{[e_{k,2} \Delta_k]_{\delta_k}^{g_k}} \right\} \prod_V [e_V \Delta_V]_4^4 \right\rangle_{(0)} \quad (1)$$

Therefore, using the Action Principle, the Appelquist–Carazzone–Decoupling may be studied in all quantities relevant in the framework of Renormalized Perturbation Theory: Green-functions, partial differential equations, Ward-, Slavnov- and Zimmermann-identities. Only the limit $\epsilon \rightarrow 0$ in case of Minkowski-metric – the “distributional limit” [14] [15] [16] – is not included.

The paper is organized as follows: It starts with a short review of BPHZL-renormalisation (section 2).

After giving some definitions (“heavy-mass-parameters M ”, “heavy lines”, “heavy fields”, “decomposed integrand” $\mathcal{R}J$), conditions concerning the M -dependence of propagators, vertices and formal integrands are stated, which are sufficient for “heavy-mass-suppression” of Feynman-integrals involving “heavy lines” (section 3).

The behaviour of the BPHZL-subtracted integrand J under asymptotic scalings of the “heavy-mass-parameters” and subsets of loop-momenta is considered in section 4; bounds on the corresponding “upper” degrees are formulated (Theorem 3) and proved, using well known BPHZ-techniques [17] [18].

These bounds would already imply the announced Decoupling Theorem, provided the BPHZL-subtracted integrand were a member of a suitable class of functions $B_n(I)$. In the presence of massless fields, this is not possible. However, every BPHZL-subtracted integrand may be associated with a $B_n(I)$ -function, which has “analogous” scaling-properties and – upon integration – majorizes the Feynman-integral under consideration. This is the content of Theorem 4, which is stated and proved in section 5.

Combining the results of the preceding sections, the Decoupling Theorem and several Corollaries are derived in section 6.

The definition and certain properties of “upper” and “lower” degrees and the class of functions $B_n(I)$ [10] [11] [3] are relegated to Appendices.

2 A short review of BPHZL-renormalization

The Feynman-integrals under consideration may be represented in momentum space as:

$$\int d^{4m}k \mathcal{R} I_{\Sigma, \epsilon}(p, k, s, \dots) \quad (2)$$

$\mathcal{R}$ is the BPHZL-realisation of Bogoliubov's R -operator, performing subtractions on the "naive" integrand

$$I_{\Sigma, \epsilon} \equiv f(\Sigma) \prod_{z \in \Sigma} e_z P_z(\{l_z\}, s^{z_i}) \prod_{L \in \Sigma} \langle T F_{a_L} F_{b_L} \rangle_{(0), \epsilon(l_L, \dots)}, \quad (3)$$

which results from eq. (1) as a generalized Wick-contraction of the vertices Δ , with free 2-point-functions $\langle T F_a F_b \rangle_{\dots}$; this contraction may be associated with a certain Feynman-diagram Σ . For most of the considerations, it will be sufficient to treat One-Particle-Irreducible (1 PI) diagrams, denoted Λ ; if necessary, modifications needed for the study of arbitrary diagrams Σ will be indicated. The statistical factor $f(\Sigma)$, which is irrelevant for the purpose of this paper, will be omitted from now on.

The vertices Δ_V and the special insertion Δ_k may be represented as²:

$$\forall z \in \{V, k\} : \Delta_z = \int d^4x (s-1)^{a_z} (\partial_{\nu_j})^{b_{zj}} \prod_{F_l \in \mathcal{F}} \{F_l(x)\}^{c_{zl}} \quad (4)$$

a_z , c_{zl} and the components of the multi-index b_{zj} are non-negative integers. In addition, b_{zj} contains the information, on which fields the derivatives ∂_{ν_j} act; $|b_{zj}|$ denotes the order of the multi-index b_{zj} .

In momentum space, the propagators are assumed to be of the form

$$\langle T F_a F_b \rangle_{(0), \epsilon(l_L, s^L, \dots)} = \frac{Q_{a,b}(l_L, m_r, s^L m_r, \tilde{m}, M, \dots)}{\prod_{l \in \mathcal{L}_{a,b}} \left\{ l_L^2 - \mathcal{M}_L^2 + i\epsilon \left(\vec{l}_L^2 + \mathcal{M}_L^2 \right) \right\}}, \quad (5)$$

with the polynomial $Q_{a,b}$ and the "mass-function" $\mathcal{M}_L^2$, which is a real and in $s^L \in [0, 1]$ non-negative, homogeneous function (degree 2) of the dimensionful parameters. In general, $\mathcal{M}_L^2$ can have an s^L -dependence

$$\begin{aligned} \mathcal{M}_L^2 &= (s^L - 1)^2 \mathcal{M}_{L,I}^2 + (s^L)^2 \mathcal{M}_{L,II}^2 + \mathcal{M}_{L,III}^2 \\ \mathcal{M}_{L,I}^2 + \mathcal{M}_{L,III}^2 &> 0 \quad \mathcal{M}_{L,II}^2 : s^L\text{-independent} \end{aligned} \quad (6)$$

and may in addition contain dimensionless parameters (e.g. the gauge-parameter). l_L denotes the momentum flowing in line L (analogous: l_V). It can be represented as a linear combination of the loop-momenta (basis: k_j^Λ ; $j = 1, \dots, m$) and the external momenta (basis: p_e^Λ ; $e = 1, \dots, n$) of the graph Λ ; the momentum-routing is done in an admissible way [17] [19].

$$l_L \rightarrow l_L^\Lambda = p_L^\Lambda + k_L^\Lambda : \quad k_L^\Lambda = \sum_{j=1}^m \alpha_{L,j} k_j^\Lambda ; \quad p_L^\Lambda = \sum_{e=1}^n \beta_{L,e} p_e^\Lambda \quad (7)$$

The set of l_L^Λ spans the space of inhomogeneous linear forms $\mathcal{L}(\Lambda)$ in the loop-momenta k_j^Λ [20]. Elements of $\mathcal{L}(\Lambda)$ are considered as **linear (in-)dependent**, if

²The the factors $s, 1, (s-1)$ have UV-dimension 1, 0, 1 and IR-dimension 0, 0, 1. Looking at eqs. (11) and (12), it is obvious, that s -factors may be eliminated in favour of factors $(s-1)$ and 1.

this is true for the corresponding homogeneous parts k_L^Λ . Consequently a set of elements of $\mathcal{L}(\Lambda)$ is called a **basis of $\mathcal{L}(\Lambda)$** , if the homogeneous parts constitute a basis of the loop-momenta of the diagram Λ .

It is assumed, that the following criteria (due to Lowenstein [13]) are fulfilled:

- UV-dimensions $d(F_a)$ and IR-dimensions $r(F_a)$ will be chosen, such that
 - for all fields F_a , the constraints

$$0 < r(F_a) ; \quad d(F_a) \leq r(F_a) \quad (8)$$

- and for all combinations (a,b) with nontrivial propagator, the following inequalities³ are valid:

$$\begin{aligned} 0 &\geq 4 - d(F_a) - d(F_b) + d_{a,b} ; & d_{a,b} &\equiv \overline{\deg}_{p,s} \langle T F_a F_b \rangle_{(0),\epsilon} \\ 0 &\leq 4 - r(F_a) - r(F_b) + r_{a,b} ; & r_{a,b} &\equiv \underline{\deg}_{p,s-1} \langle T F_a F_b \rangle_{(0),\epsilon} \end{aligned} \quad (9)$$

- In addition, the minimal UV-dimension d_z and maximal IR-dimension r_z of the vertex Δ_z , defined as

$$\left. \begin{aligned} d_z &= \sum_r a_{zr} + \sum_j |b_{zj}| + \sum_l c_{zl} d(F_l) \\ r_z &= \sum_r a_{zr} + \sum_j |b_{zj}| + \sum_l c_{zl} r(F_l) \end{aligned} \right\} \quad \forall : z \in \{V, k\} \quad (10)$$

are constrained:

$$d_V \leq 4 \quad r_V \geq 4 , \quad (11)$$

for all (UV: 4, IR: 4)-vertices $V \in \lambda$ and

$$\delta_k + 1 \geq \varrho_k \quad r_k \geq \varrho_k \geq 3, \frac{5}{2} \quad d_k \leq \delta_k \quad (12)$$

for the “special” insertion Δ_k . Under certain conditions, it is possible to violate the lower bound $r_k \geq 3$ mentioned by Lowenstein [13]: e.g. a spinorial insertion may have $r_k = 5/2$ [21] [22].

The UV- and IR-subtraction-degrees for a 1 PI-(sub)diagram λ are given by

$$\delta(\lambda) = 4 - \sum_l N_l^\lambda d(F_l) \oplus (\delta_k - 4) ; \quad \varrho(\lambda) = 4 - \sum_l N_l^\lambda r(F_l) \oplus (\varrho_k - 4) \quad (13)$$

N_l^λ denotes the number of external, amputated fields F_l of the diagram λ , and the term $\oplus \dots$ only contributes, if λ contains the insertion $k\Delta_k$.

³For a definition of $\underline{\deg}_{p,s-1}$ and $\overline{\deg}_{p,s}$, see Appendix A.

The action of the operator $\mathcal{R}$ on the formal integrand $I_{\Lambda,\epsilon}$ and the relevant definitions are briefly reviewed; for a more detailed account see the original papers [17], [18] and [13].

A **Λ -forest** W is a possibly empty set of non-overlapping⁴ 1 PI-subdiagrams λ , $\lambda \subseteq \Lambda$. The set of all Λ -forests W will be denoted $\mathcal{F}_W(\Lambda)$.

$I_{\Lambda,\epsilon}(W)$ defines a special parametrisation of the formal integrand $I_{\Lambda,\epsilon}$ “with respect to the Λ -forest W ”: the linear forms l_L and l_{V_i} , the parameters s^L resp. s^{V_i} are expressed as functions in the variables k_j^λ , p_e^λ and s^λ – see eq. (7) – of that minimal element $\lambda \in W \cup \Lambda$, which contains the line L resp. the vertex V_i . Explicitly:

$$I_{\Lambda,\epsilon}(W) = \prod_{\gamma \in W \cup \Lambda} \{I_{\bar{\gamma}(W),\epsilon}\}(k_j^\gamma, p_e^\gamma, s^\gamma, \dots)$$

$\bar{\gamma}(W)$ is obtained from γ by shrinking all maximal subdiagrams γ_t , $\gamma_t \subset \gamma$; $\gamma_t \in W$ to points; those “generalized vertices” in $\bar{\gamma}(W)$ acquire a factor 1. $I_{\bar{\gamma}(W),\epsilon}$ is again of the form shown in eq. (2).

The **substitution-operator** S_γ replaces the variables k_j^λ , p_e^λ and s^λ of a graph λ , $\lambda \subset \gamma$, by functions of the independent variables of the graph γ : k_j^γ , p_e^γ and s^γ .

$$S_\gamma(k_j^\lambda, p_e^\lambda, s^\lambda) = (k_j^\lambda(k_l^\gamma), p_e^\lambda(k_l^\gamma, p_f^\gamma), s^\lambda(s^\gamma))$$

The substitution for p and k is constrained by energy-momentum conservation at the vertices and the requirement, that loop-momenta k^λ only depend on the loop-momenta k^γ of the graph γ .

The **subtraction-operator** τ_γ has the form:

$$\tau_\gamma = t_{p,s}^{\delta(\gamma)} + t_{p,s-1}^{\varrho(\gamma)-1} - t_{p,s-1}^{\varrho(\gamma)-1} t_{p,s}^{\delta(\gamma)} \quad (14)$$

For an integer x and a function $f(y)$ sufficiently regular, $t_y^x f$ represents the Taylor series of f around $y = 0$ up to order x . If x is negative, $t_y^x f$ is set identically zero. “ p ” denotes a basis of the external momenta of the graph γ , “ s ” stands for the variable s^γ . The subtraction-degrees $\delta(\gamma)$ and $\varrho(\gamma)$ are defined in eq. (13).

Finally, the BPHZL-subtracted integrand $R_{\Lambda,\epsilon}$ is given by **Zimmermann’s forest-formula**:

$$R_{\Lambda,\epsilon} = S_\Lambda \sum_{W \in \mathcal{F}_W(\Lambda)} \prod_{\gamma \in W} (-\tau_\gamma S_\gamma) I_{\Lambda,\epsilon}(W)$$

In the product $\prod_{\gamma \in W}$ the following order of factors is understood: if $\lambda \subset \gamma$, the factor $(-\tau_\gamma S_\gamma)$ is left of $(-\tau_\lambda S_\lambda)$; for $\lambda \cap \gamma = \emptyset$ the order is irrelevant.

There are equivalent representations of $R_{\Lambda,\epsilon}$, which are more suitable for the treatment of its large-mass-behaviour. Starting point is the concept of a **complete forest** [17]. Let S be a linear subspace of $\mathcal{L}(\Lambda)$. A Λ -forest C is called **complete with respect to the linear subspace S** , if the following properties are fulfilled (Λ is 1 PI):

⁴Diagrams λ and γ are non-overlapping, if either $\gamma \cap \lambda = \emptyset$, $\gamma \subset \lambda$ or $\lambda \subset \gamma$ is true.

- $\Lambda \in C$
- $\gamma \in C$: For **all** lines $L \in \bar{\gamma}(C)$ it is true
either: $l_L \in S$ (“ $\bar{\gamma}(C) \parallel S$ ”) or: $l_L \notin S$ (“ $\bar{\gamma}(C) \not\parallel S$ ”)

l_L is understood as an element of $\mathcal{L}(\Lambda)$, the space of inhomogeneous linear forms.

The set of all complete Λ -forests C with respect to S is denoted by $\mathcal{C}_S(\Lambda)$.

$R_{\Lambda, \epsilon}$ can now be defined by a recursion, that uses the set $\mathcal{C}_S(\Lambda)$. The index ϵ will be omitted “below” $R_\Lambda(C)$.

$$R_{\Lambda, \epsilon} \equiv \sum_{C \in \mathcal{C}_S(\Lambda)} R_\Lambda(C) \quad (15)$$

$$R_\Lambda(C) \equiv (1 - \tau_\Lambda) Y_\Lambda(C) \quad Y_\Lambda(C) \equiv I_{\bar{\Lambda}(C)} \prod_k S_\lambda f_{\lambda_k} Y_{\lambda_k}(C)$$

The last equation is iteratively used for all (sub-)graphs $\lambda \in C$. k labels all maximal subgraphs λ_k of λ in C . f_{λ_k} replaces the subtraction-operator τ_{λ_k} of the forest-formula:

$$f_{\lambda_k} = \begin{cases} 1 - \tau_{\lambda_k} & \bar{\lambda}(C) \parallel S \text{ and } \bar{\lambda}_k(C) \not\parallel S \\ \tau_{\lambda_k} & \text{otherwise} \end{cases} \quad (16)$$

After performing all substitutions, $R_{\Lambda, \epsilon}$ is a function in the natural variables of Λ .

$$R_{\Lambda, \epsilon}(k^\Lambda, p^\Lambda, s^\Lambda, \dots) = \left(\prod_{z \in \Lambda} e_z \right) \frac{Q(l_t, \epsilon, \dots)}{\prod_t \{l_t^2 - \mathcal{M}_t^2 + i\epsilon(\cdot)\}} \quad (17)$$

Some properties of $R_{\Lambda, \epsilon}$ are summarized in

Theorem 1 ([13] [20]) *The BPHZL-subtracted integrand $R_{\Lambda, \epsilon}$ fulfils the slightly modified requirements of the Power-Counting-Theorem [20]: For every transformation*

$$k^\Lambda \longrightarrow (u_1, \dots, u_m) \text{ or } (v_1, \dots, v_m) \text{ or } (u_1, \dots, u_a, v_{a+1}, \dots, v_m),$$

$$u, v \in \mathcal{L}(\Lambda), \quad 1 < a < m,$$

with nonvanishing determinant, the following inequalities are satisfied for all $s \in [0, 1]$:

$$\begin{aligned} 0 &> \overline{\deg}_{v|u} R_{\Lambda, \epsilon}(p, k(u, v, p), \dots) + 4(m - a) ; & \text{if } (m - a) > 0 \\ 0 &< \underline{\deg}_{u|v} R_{\Lambda, \epsilon}(p, k(u, v, p), \dots) + 4a ; & \text{if } a > 0 \end{aligned} \quad (18)$$

Therefore, the integral

$$\int d^{4m} k R_{\Lambda, \epsilon}(p, k(u, v, p), \dots)$$

converges absolutely for $s \in [0, 1]$. $\epsilon > 0$ and non-exceptional p . For $\epsilon > 0$, the integral is a lorentz-non-covariant complex-valued function in the external momenta p .

Definition 1 Starting from the BPHZL-subtracted integrand $R_{\Lambda,\epsilon}$ in eq. (17), $\mathbf{R}_{\Lambda,E}$ is obtained, if in the denominator scalar products wrt. the Minkowskian metric are replaced with their euclidean counterparts (metric: $-\delta_{a,b}$; $a, b = 0, 1, 2, 3$)

$$l_{t,E}^2 \equiv (l_L^0)^2 - (\vec{l}_L)^2 \longrightarrow -l_{t,E}^2 \equiv -(l_L^0)^2 - (\vec{l}_L)^2,$$

and ϵ is set to zero.

The quantity $\mathbf{R}_{\Lambda}^E$ results, if in $R_{\Lambda,\epsilon}$ the manipulations just described are done in the denominator and in the numerator.

Corollary 2 Results similar to Theorem 1 can be formulated for the quantities $R_{\Lambda,E}$ and R_{Λ}^E .

Proof: Assuming Theorem 1 as an hypothesis, Corollary 2 can be proved using the Power-Counting-Theorem and the following arguments:

Due to the special form of the propagators eq. (5), Minkowskian denominators $N_{t,\epsilon}$ can be bounded by their euclidean counterparts $N_{t,E}$ [15]:

$$\frac{1}{\sqrt{1+\epsilon^2} (l_{L,E}^2 + \mathcal{M}_L^2)} \leq \frac{1}{|l_L^2 - \mathcal{M}_L^2 + i\epsilon(l_L^2 + \mathcal{M}_L^2)|} \leq \left(\frac{1}{\epsilon} + \sqrt{1 + \frac{1}{\epsilon^2}} \right) \frac{1}{(l_{L,E}^2 + \mathcal{M}_L^2)}$$

To treat the numerators, it is sufficient to observe, that the “upper” and “lower” degrees of the Minkowskian and the euclidean version of scalar products are identical and independent of ϵ .

Therefore, the absolute convergence of the $R_{\Lambda,\epsilon}$ -integral for $\epsilon > 0$ implies the absolute convergence of the $R_{\Lambda,E}$ - resp. the R_{Λ}^E -integral and vice versa. ■

3 Assumptions concerning the M -dependence

Aim of this paper is to study the behaviour of BPHZL-subtracted Feynman-integrals, when a set of mass-parameters M_r , $r \in \{1, \dots, R\}$, is scaled: $M_r \rightarrow \eta M_r$ for $1 < \eta < \infty$, especially $\eta \rightarrow \infty$. This set of “**heavy-mass-parameters**” is denoted $\mathbf{M}$.

All remaining dimensional parameters will be contained either in the sets $\mathbf{m}_j$ (mass-parameters associated with the s -factors) or $\widetilde{\mathbf{m}}$ (represents all dimensionful parameters neither connected to s nor to $\mathbf{M}$: e.g. the normalisation-scale $\kappa^2, \dots$).

A propagator depending on “heavy-mass-parameters” M_r (“ M -dependent”) is called a “**heavy line**”. A field F_a is by definition a “**heavy field**”, if all its non-trivial propagators $\langle T F_a F_b \rangle_{\dots}$ are “heavy lines”.

Conditions concerning the M -dependence of propagators, vertices Δ_z and formal integrands are stated, which are sufficient for the “heavy-mass-suppression” of BPHZL-subtracted Feynman-integrals with “heavy lines” in the limit $\eta \rightarrow \infty$.

3.1 Decomposition of the integrand $R_{\Lambda,\epsilon}$ wrt. to M

It is rather convenient to decompose the BPHZL-subtracted integrand⁵ $\mathcal{R} I_{\Lambda,\epsilon}$ into the parts E_1^Λ and $\mathcal{R} J_{\Lambda,\epsilon}$

$$R_{\Lambda,\epsilon} \equiv \mathcal{R} I_{\Lambda,\epsilon} \equiv E_1^\Lambda \mathcal{R} J_{\Lambda,\epsilon} \equiv E_1^\Lambda \bar{R}_{\Lambda,\epsilon} \quad ; \quad I_{\Lambda,\epsilon} \equiv E_1^\Lambda J_{\Lambda,\epsilon} \quad (19)$$

by appropriately splitting the prefactors e_z of the vertices $\triangle_z$; E_1^Λ is defined as the product of the $e_{z,1}$ -factors:

$$e_z \equiv e_{z,1} e_{z,2} \quad E_1^\Lambda \equiv \prod_{z \in \Lambda} e_{z,1} \quad (20)$$

As an intermediate step, the Feynman-integral based on $\mathcal{R} J_{\Lambda,\epsilon}$ will be studied. It is this part, which contains the nontrivial “heavy-mass-behaviour”; due to eqs. (20) and (70), the contributions of the factor E_1^Λ are easily taken into account.

On the other hand, the freedom involved in this decomposition can be used to define a subtracted integrand $\mathcal{R} J_{\Lambda,\epsilon}$ with rather favourable properties. The decomposition will be chosen such that

- the operator $\mathcal{R}$ commutes with the factor E_1^Λ ; all individual factors $e_{V,1}$, $V \in \lambda$, are by definition irrelevant in the process of renormalization.
- if possible⁶, $J_{\Lambda,\epsilon}$ fulfils the properties **A1a** – **A3b**, stated below. In this case,
 - $J_{\Lambda,\epsilon}$ is a rational and homogeneous function of the dimensionful parameters and variables (**A1a**).
 - $J_{\Lambda,\epsilon}$ is “minimally BPHZL-subtracted with respect to its engineering dimension $d_{\odot}(J_{\Lambda,\epsilon})$ ”; see Appendix A.2 (**A1b**).
- $J_{\Lambda,\epsilon}$ is of order $\hbar^0$. All contributions of counterterms (which may also contain non-rational parts; i.e. logarithms) are by definition contained in E_1^Λ .
- the “heavy-mass-bounds” for $R_{\Lambda,\epsilon}$ are as sharp as possible.
The bound derived in this paper is optimized, if a decomposition is chosen, which saturates condition **A3**.

An explicit definition of the split eq. (20), which is in accordance with the properties just described, is straightforward [23].

⁵According to Definition 1, similar decompositions are possible for $R_{\Lambda,E}$ and R_Λ^E ; the treatment of general diagrams Σ is analogous.

⁶Of course, eq. (21) of condition **A1a**, eq. (23) of condition **A1b** and condition **A2** are not influenced by this split.

3.2 The conditions A1 – A5

First, the conditions **A1a** – **A3b** are stated; they prove to be sufficient for the “heavy-mass-suppression” of the Feynman-integral $\int d^{4m}k \mathcal{R} J_{\Lambda, \epsilon}$ with “heavy lines”.

A1a: All propagators $\langle T F_{a_L} F_{b_L} \rangle_{\dots}$, $L \in \lambda$, and all coefficients $e_{z,2}$, $z \in \Lambda$, are homogeneous functions of the dimensionful parameters and variables. Their dependence

$$\langle T F_{a_L} F_{b_L} \rangle_{(0), \epsilon} = \langle T F_{a_L} F_{b_L} \rangle_{(0), \epsilon}(l_L, M, m_j, sm_j \equiv w_j, \tilde{m}) \quad (21)$$

$$e_{z,2} = e_{z,2}(M, m_j, \tilde{m}) \quad (22)$$

is such that $J_{\Lambda, \epsilon}$ is a rational, homogeneous function in the variables and parameters $p, k, sm_j \equiv w_j, m_j, \tilde{m}$ and M .

It is emphasized, that the “heavy mass-parameters” M do not appear in combination with the s -parameters (sM or $(s-1)M$).

A1b: In addition, the engineering dimension of all propagators $\langle T F_{a_L} F_{b_L} \rangle_{\dots}$, $L \in \lambda$, is required to satisfy:

$$d_{\odot}(\langle T F_{a_L} F_{b_L} \rangle_{(0), \epsilon}) \equiv d(F_{a_L}) + d(F_{b_L}) - 4 \quad (23)$$

For the choice of UV-dimensions $d(F_a)$, this can lead to additional constraints besides eqs. (8) and (9).

The engineering dimension $d_{\odot}(e_{z,2})$ is constrained for all $z \in \Lambda$:

$$d_{\odot}(e_{z,2}) \equiv \delta_z - d_z$$

A2: In momentum space, all propagators have a form consistent with eqs. (5), (6) and (21). Here, additional requirements are formulated for M -dependent “mass-functions” $\mathcal{M}_L^2$: For all $s \in [0, 1]$, $\mathcal{M}_L^2$ does not vanish, if $M > 0$, and under scalings $M \rightarrow \eta M$, $\eta \rightarrow \infty$, it behaves like $\varsigma_l \eta^2$, with ς_l positive.

A3a: For all formal integrands $J_{\lambda, \epsilon}$, $\lambda \subseteq \Lambda$ 1 PI and nontrivial, the following statements are true:

1. $J_{\lambda, \epsilon}$ without “heavy lines” has a vanishing “upper M -degree”; it is M -independent.
2. $J_{\lambda, \epsilon}$ with “heavy lines” has a negative “upper M -degree”.

The “upper” M -degree $\overline{\deg}_M$ abbreviates $\overline{\deg}_{M_1, \dots, M_R}$.

A3b: The factor $e_{z,2}$ multiplying the vertex Δ_z has an “upper M -degree”, which is bounded by:

$$0 \leq \overline{deg}_M (e_{z,2}) \leq \begin{cases} \delta_k - d_k ; & z \equiv k \\ 4 - d_V ; & z \equiv V \end{cases}$$

Remark 1 Condition **A1a** and eq. (66) imply, that the BPHZL-subtracted integrand $\mathcal{R} J_{\Lambda,\epsilon}$ is again a homogeneous function in the variables and parameters $p, k, sm_j \equiv w_j, m_j, \tilde{m}$ and M . The “upper” degrees wrt. the variables w and s may be identified ■

If a Feynman-integral $\int d^{4m}k \mathcal{R} J_{\Sigma,\epsilon}$ associated to a One-Particle-Reducible diagram Σ is treated, the conditions **A1a** – **A3b** have to be supplemented:

Each connected component of Σ can be represented as a generalized tree-diagram with “full” propagators (free propagators, possibly with 1 PI 2-point-functions inserted) as “lines” and 1 PI n -point-functions, $n \geq 3$, as “generalized vertices”. This generalized tree will be decomposed in nontrivial 1 PI-parts λ , “trivial trees” τ (trivial 1 PI-parts with free propagators not amputated) and “remaining propagators”.

The following conditions apply to “trivial trees” and “remaining propagators” of Σ :

A4a: All “formal integrands” $J_{\tau,\epsilon}$ associated to “trivial trees” τ of Σ satisfy⁷:

1. $J_{\tau,\epsilon}$ without “heavy lines” has vanishing “upper” M -degree; it is M -independent.
2. $J_{\tau,\epsilon}$ with “heavy lines” has negative “upper” M -degree.

A4b: All M -dependent “remaining propagators” fulfil the inequality:

$$\overline{deg}_M \left(\langle T F_a F_b \rangle_{(0),\epsilon} \right) \leq -1$$

The conditions **A1a** – **A3b** together with condition **A5** are sufficient for “heavy-mass-suppression” of Feynman-integrals with “heavy lines” based on $R_{\Lambda,\epsilon}$. For general diagrams Σ , condition **A4** has to be added.

A5: The factor E_1^Λ , considered as a function of the “heavy-mass scaling-parameter” η , belongs to the class $B_1(\emptyset)$; for a definition and properties see Appendix B.

The corresponding asymptotic and logarithmic-asymptotic coefficients fulfil:

$$\alpha(E_1^\Lambda) \leq 0 ; \quad \beta(E_1^\Lambda) \text{ finite}$$

⁷In [24], this behaviour of “trivial trees” was extensively studied in models with (broken) supersymmetry.

4 (v, M) –scaling properties

In this section, the behaviour of the BPHZL–subtracted integrand⁸ $\bar{R}_{\Lambda, \epsilon}$ under scalings

$$\begin{aligned} M_r &\longrightarrow \eta M_r & \text{and} & & k &\longrightarrow \eta k & 1 < b < \eta \\ \forall : M_r &\in M & & & \forall : k &\in v \end{aligned}$$

is studied. v denotes a possibly empty linear subspace of the loop–momenta k_j^Λ , $j = 1, \dots, m$, of the diagram Λ .

For this purpose, a representation of the integrand $\bar{R}_{\Lambda, \epsilon}$ according to eq. (15), with an appropriately chosen linear subspace S^v , is suitable. Given v , S^v will be constructed as follows: Via an orthogonal transformation, the loop–momenta k_j^Λ ; $j = 1, \dots, m$; are mapped to a new basis $u_1, \dots, u_r, v_{r+1}, \dots, v_{4m}$, with $v_{r+1}, \dots, v_{4m}$ being a basis of the linear subspace v . Then, a basis for S^v is given by $u_1, \dots, u_r$.

The part k_L of the line–momentum l_L – see eq. (7) – is represented wrt. this basis as

$$k_L = U_L(u) + V_L(v)$$

Let γ be an element of a complete Λ –forest C wrt. S^v . Is $\bar{\gamma}(C) \parallel S^v$, then V_L vanishes for all lines $L \in \bar{\gamma}(C)$. Lines L being v –dependent ($V_L \neq 0$) necessarily fulfil: $L \in \bar{\gamma}(C) \not\parallel S^v$.

The scaling–behaviour of the subtracted integrand is characterized by a bound on the “upper (v, M) –degree” due to

Theorem 3 *A BPHZL–subtracted integrand $\bar{R}_{\Lambda, \epsilon}$ as described in section 2 and eq. (19) is considered. In addition, it is assumed, that $\bar{R}_{\Lambda, \epsilon}$ fulfils the conditions **A1** and **A3b**. The “upper” (v, M) –degree of $\bar{R}_{\Lambda, \epsilon}$ is bounded as:*

$$\overline{\deg}_{v, M} \bar{R}_{\Lambda, \epsilon} \stackrel{*}{\leq} - \underline{m}^{S^v}(\Lambda) + \underline{N}^{S^v}(\Lambda) \quad (24)$$

Analogous bounds may be formulated for the subtracted integrands $\bar{R}_\Lambda^E$ and $\bar{R}_{\Lambda, E}$.

Equ. (24) still requires some explanation: Let C be a complete Λ –forest wrt. S^v and $\lambda \in C$. Then, $\underline{m}^{S^v}(\lambda)$ may be identified with the dimension of that subspace of $\mathcal{L}_\Lambda$, spanned by the lines $\not\parallel S^v$:

$$\underline{m}^{S^v}(\lambda) \equiv \sum_{\lambda_i \in C : \bar{\lambda}_i \not\parallel S^v ; \lambda_i \subseteq \lambda} 4 \, m(\bar{\lambda}_i)$$

$m(\bar{\lambda}_i)$ counts the number of independent loops of the graph $\bar{\lambda}_i$. The symbol $\stackrel{*}{\leq}$ stands for “=”, if $\underline{m}^{S^v}(\lambda) = 0$; otherwise it means: “<”.

⁸Due to Definition 1, all results of this section are also true for $\bar{R}_\Lambda^E$ and $\bar{R}_{\Lambda, E}$.

$\underline{N}^{S^v}(\lambda)$ sums the “upper” M -degrees of all diagrams $\bar{\lambda}_i \parallel S^v$, $\lambda_i \subseteq \lambda$.

$$\underline{N}^{S^v}(\lambda) \equiv \sum_{\lambda_i \in C : \bar{\lambda}_i \parallel S^v : \lambda_i \subseteq \lambda} \overline{deg}_M J_{\bar{\lambda}_i(C)} \quad (25)$$

The decomposition of Λ in its (sub-)graphs $\bar{\lambda} \parallel S^v$ resp. $\not\parallel S^v$ only depends on the linear subspace S^v , hence on v , but not on the choice of the Λ -forest C . Therefore, the quantities $\underline{N}^{S^v}(\Lambda)$ and $\underline{m}^{S^v}(\Lambda)$ are defined without ambiguity.

4.1 Properties of the subtraction-operator

Some properties of the subtraction-operator τ , relevant for the proof of Theorem 3, are stated below. These are suitably modified versions of the properties ii), iii) and v) to vii) listed in [18]; the original labeling is kept:

- ii) τ is a linear map, which is independent of the choice of basis for the external momenta p . According to eq. (14), τ is represented as a combination of Taylor-operators.

Let F be an abbreviation for $Y_\lambda(C)$ defined in eq. (15).

- iii) (τF) is a polynomial of degree δ in the variables p, w .

If (τF) is not identically zero, eq. (64) implies:

$$\text{v)} \quad \overline{deg}_{p,v,w,M}(\tau F) \leq \overline{deg}_{p,v,w,M} F$$

$$\text{vi)} \quad \overline{deg}_{v,M}(\tau F) \leq \overline{deg}_{v,M} F$$

Let F represent $Y_\lambda(C)$, $\lambda \subseteq \Lambda$, $\lambda(C) \not\parallel S^v$ with $(1 - \tau)F$ not identical zero:

- vii) The following bound on the “upper (v, M) -degree” of $(1 - \tau)F$ will be needed:

$$\overline{deg}_{v,M} \{(1 - \tau)F\} \leq \overline{deg}_{v,p,w,M} F - \max(0, \delta + 1) \quad (26)$$

Property vii) is proved in Appendix A.4.

4.2 Proof of the Theorem

For the proof of Theorem 3, a recursion analogous to eq. (15) is used: all the quantities appearing in this “new” recursion carry a bar “ $\bar{}$ ”, to indicate, that the factors E_i are separated. The following inequalities are chosen as **induction hypotheses** for $\bar{Y}_\lambda(C)$:

$$\underline{\bar{\lambda}(C) \parallel S^v} : \quad \overline{deg}_{v,M} \bar{Y}_\lambda(C) \stackrel{*}{\leq} -\underline{m}^{S^v}(\lambda) + \underline{N}^{S^v}(\lambda) \quad (27)$$

$$\underline{\bar{\lambda}(C) \not\parallel S^v} : \quad \overline{deg}_{p,w,v,M} \bar{Y}_\lambda(C) \leq \delta(\lambda) - \underline{m}^{S^v}(\lambda) + \underline{N}^{S^v}(\lambda) \quad (28)$$

For each complete Λ -forest C (wrt. S^v), the proof will be done by induction with respect to its elements $\lambda \in C$.

Starting the induction

The induction hypotheses eqs. (27) and (28) are shown to be true for the minimal elements σ in C . $\bar{Y}_\sigma(C)$ is identical to the formal integrand J_σ .

$\sigma \parallel S^v$:

In this case $\underline{m}^{S^v}(\sigma)$ is zero; therefore “=” applies in eq. (27). By direct calculation:

$$\overline{deg}_{v,M} J_\sigma = \overline{deg}_M J_\sigma = \underline{N}^{S^v}(\sigma)$$

The induction hypothesis eq. (27) applies to $\sigma \parallel S^v$!

$\sigma \not\parallel S^v$:

In this case: $\underline{N}^{S^v}(\sigma) = 0$. As equation (62) is proved in Appendix A.2, only the validity of the last inequality in

$$\delta(\sigma) - \underline{m}(\sigma) = \overline{deg}_{p,v,w,m,\tilde{m},M} J_\sigma \geq \overline{deg}_{p,v,w,M} J_\sigma$$

remains to be shown. In J_σ , the denominator and the numerator are analyzed separately. Due to $\sigma \not\parallel S^v$, the degree of the denominator is already saturated by the variables “ v ”; the conditions **A1** and **A3b** guarantee, that the numerator satisfies an analogous inequality.

Therefore, the induction hypothesis eq. (27) is fulfilled in the case $\sigma \not\parallel S^v$.

Step of induction

The induction hypotheses are assumed to be true for all $\lambda \in C$, $\lambda \subseteq \gamma_i$. γ_i denotes all maximal subdiagrams of γ , $\gamma \in C$. It is shown, that the hypotheses of induction apply to γ as well. This step is formally equivalent to [18] (the ϵ -indices will be suppressed; $\bar{\gamma}$ stands for $\bar{\gamma}(C)$, $\bar{Y}_\gamma$ for $\bar{Y}_\gamma(C)$, ...):

$\bar{\gamma} \parallel S^v$:

1. $\overline{deg}_{v,M} J_{\bar{\gamma}} = \overline{deg}_M J_{\bar{\gamma}}$
2. $\bar{\gamma}_i \parallel S^v : \overline{deg}_{v,M} S_\gamma \tau_{\gamma_i} Y_{\gamma_i}^* \leq -\underline{m}^{S^v}(\gamma_i) + \underline{N}^{S^v}(\gamma_i)$
3. $\bar{\gamma}_i \not\parallel S^v : \overline{deg}_{v,M} S_\gamma (1 - \tau_{\gamma_i}) Y_{\gamma_i} \leq \min(\delta(\gamma_i), -1) - \underline{m}^{S^v}(\gamma_i) + \underline{N}^{S^v}(\gamma_i)$

The additivity of $\underline{m}^{S^v}(\gamma_i)$ and $\underline{N}^{S^v}(\gamma_i)$ for disjoint γ_i directly leads to eq. (27) for γ . Obviously “=” applies, if $\underline{m}^{S^v}(\gamma)$ is zero: If there is a diagram $\bar{\gamma}_i \not\parallel S^v$, $\underline{m}^{S^v}(\gamma_i)$ is non-zero and a negative term “ $\min(\delta, -1)$ ” appears at least once (therefore: “ $<$ ”).

$\bar{\gamma} \not\parallel S^v :$

$$1. \overline{deg}_{p,v,w,M} J_{\bar{\gamma}} \leq \delta(\bar{\gamma}) - m(\bar{\gamma})$$

This is analogous to the proof of the induction hypotheses for the minimal elements $\sigma \in C$, $\sigma \not\parallel S^v$.

$$2. \bar{\gamma}_i \parallel S^v : \overline{deg}_{p,v,w,M} S_{\gamma} \tau_{\gamma_i} \dot{Y}_{\gamma_i} \leq \delta(\gamma_i) - \underline{m}^{S^v}(\gamma_i) + \underline{N}^{S^v}(\gamma_i)$$

$$3. \bar{\gamma}_i \not\parallel S^v : \overline{deg}_{p,v,w,M} S_{\gamma} \tau_{\gamma_i} \dot{Y}_{\gamma_i} \leq \delta(\gamma_i) - \underline{m}^{S^v}(\gamma_i) + \underline{N}^{S^v}(\gamma_i)$$

For all $\lambda \subseteq \Lambda$, the bounds eq. (27) and eq. (28) are valid by induction.

Transition from $\bar{Y}_{\Lambda}(C)$ to $\bar{R}_{\Lambda,\epsilon}$

As the set of complete Λ -forests with respect to S^v is finite, it is sufficient to prove the validity of inequality (24) for all $\bar{R}_{\Lambda}(C)$, $C \in \mathcal{C}_S(\Lambda)$, separately.

$$\begin{aligned} \underline{\bar{\Lambda}} \parallel S^v : \quad & \overline{deg}_{v,M} \bar{R}_{\Lambda}(C) = \overline{deg}_{v,M} \{(1 - \tau_{\Lambda}) \bar{Y}_{\Lambda}(\Lambda)\} \leq \\ & \leq \max(\overline{deg}_{v,M} \bar{Y}_{\Lambda}(\Lambda), \overline{deg}_{v,M} \tau_{\Lambda} \bar{Y}_{\Lambda}(\Lambda)) \stackrel{*}{\leq} -\underline{m}^{S^v}(\Lambda) + \underline{N}^{S^v}(\Lambda) \end{aligned} \quad (29)$$

This follows from eq. (27), applied to Λ , $\bar{Y}_{\Lambda}$, $\bar{\Lambda} \parallel S^v$, eq. (60) and property vi).

$$\begin{aligned} \underline{\bar{\Lambda}} \not\parallel S^v : \quad & \overline{deg}_{v,M} \bar{R}_{\Lambda}(C) = \overline{deg}_{v,M} \{(1 - \tau_{\Lambda}) \bar{Y}_{\Lambda}(\Lambda)\} \leq \\ & \leq \delta(\Lambda) - \underline{m}^{S^v}(\Lambda) + \underline{N}^{S^v}(\Lambda) - \max(0, \delta(\Lambda) + 1) \leq \\ & < -\underline{m}^{S^v}(\Lambda) + \underline{N}^{S^v}(\Lambda) \end{aligned} \quad (30)$$

Here, eq. (28) and property vii) are used.

Since $\bar{\Lambda} \not\parallel S^v$ enforces $\underline{m}^{S^v}(\Lambda) \neq 0$, the eqs. (29) and (30) can be unified:

$$\forall v, \emptyset \subseteq v \subseteq \mathbb{R}^{4m} : \quad \overline{deg}_{v,M} \bar{R}_{\Lambda,\epsilon}(C) \stackrel{*}{\leq} -\underline{m}^{S^v}(\Lambda) + \underline{N}^{S^v}(\Lambda)$$

This bound on the “upper” (v, M) -degree of every summand $\bar{R}_{\Lambda}(C)$, eq. (15), can directly be transferred to the subtracted integrand $\bar{R}_{\Lambda,\epsilon}$. ■

With the help of eq. (24), the scaling-behaviour of $\bar{R}_{\Lambda,\epsilon}$ for “lorentz-variant” subspaces v_n is examined: a linear subspace will be called “lorentz-variant”, if it cannot be spanned by a set of lorentz-four-vectors (analogous: “euclidean variant”).

For every v_n , a **minimal lorentz-invariant completion** $\bar{v}_n$ is obtained by adding

suitable components of four-vectors to the basis of v_n . If v_n is “lorentz-variant”, the following inequality applies:

$$\dim(v_n) < \dim(\overline{v_n}) \equiv \underline{m}^{S^{\overline{v_n}}}(\Lambda) \quad (31)$$

Line-momenta l_L eq. (7) can always be written als a linear combination of four-vectors. Therefore, the sets of complete forests wrt. the linear subspaces S^{v_n} and $S^{\overline{v_n}}$ are identical: $\mathcal{C}_{S^{v_n}}(\Lambda) \equiv \mathcal{C}_{S^{\overline{v_n}}}(\Lambda)$. According eq. (24) this implies, that the “upper” (v_n, M) - and $(\overline{v_n}, M)$ -degree of $\bar{R}_{\Lambda, \epsilon}$ are both bounded from above by :

$$\stackrel{*}{\leq} \underline{m}^{S^{\overline{v_n}}}(\Lambda) + \underline{N}^{S^{\overline{v_n}}}(\Lambda) \quad (32)$$

Eqs. (31) and (32) will imply, that the behaviour of the Feynman-integrals under scalings $M \rightarrow \eta M$, $\eta \rightarrow \infty$, is determined by the “lorentz-invariant” subspaces $\overline{v_n}$ only; see eq. (57).

5 Construction of the $B_n(I)$ -bound $r_{\Lambda, E}$

It is the aim of this section to construct an upper bound on the Feynman-integral, which allows to exploit properties of the class of functions $B_n(I)$ [10][11][3]⁹. This will be done by replacing the integrand $\bar{R}_{\Lambda, E}$ ¹⁰ by a suitable $B_n(I)$ -function $r_{\Lambda, E}$.

If $\bar{R}_{\Lambda, E}$ were a function in a suitable class $B_n(I)$, the information encoded in Theorem 3 would already be sufficient to characterize the “heavy-mass behaviour” of the corresponding Feynman-integral. However, integrands $\bar{R}_{\Lambda, E}$ containing massless propagators, cannot be in this class: for special values of the loop-momenta, poles are present, a fact, which is incompatible with the $B_n(I)$ -properties.

Nevertheless, a connection between the integrand $\bar{R}_{\Lambda, E}$ and a suitable $B_n(I)$ -function can be established:

Theorem 4 *To every subtracted integrand $\bar{R}_{\Lambda, E}$, which fulfils condition **A2**, a non-negative function $r_{\Lambda, E}$, member in the class $B_{4m+1}(\mathbb{R}^{4m})$ wrt. the variables¹¹ M (set of “heavy mass-parameters”) and $I \equiv \mathbb{R}^{4m}$ (loop-momenta), can be constructed with the following properties:*

1. The “upper” v - and (v, M) -degrees of $\bar{R}_{\Lambda, E}$ and $r_{\Lambda, E}$ are equal for all linear subspaces v , $\emptyset \subseteq v \subseteq \mathbb{R}^{4m}$.

⁹The definition of this class and a short review of its properties may be found in Appendix B.

¹⁰The derivation of the $B_n(I)$ -bound is most natural in the case of euclidean denominators; therefore, throughout this section, the subtracted integrand $\bar{R}_{\Lambda, E}$ is treated. As the proof of Corollary 2 shows, analogous results can be obtained in the cases $\bar{R}_{\Lambda, \epsilon}$, $\epsilon > 0$, and $\bar{R}_{\Lambda}^E$.

¹¹All other variables and parameters $\bar{R}_{\Lambda, E}$ may depend on are fixed suitably; e.g.: p non-exceptional (euclidean sense), $s \in [0, 1]$, ...

2. The **integral** $|\int d^{4m}k \bar{R}_{\Lambda,E}|$ will be majorized by the corresponding $r_{\Lambda,E}$ -integral.

A similar construction is possible for the subtracted integrands $\bar{R}_{\Lambda,\epsilon}$, $\epsilon > 0$, and $\bar{R}_{\Lambda}^E$.

Remark 2 Condition **A2** is stated only for convenience. It only enters via eq. (48), and is therefore not very essential for the proof ■

Remark 3 The equality of the “upper” v -degrees for $\bar{R}_{\Lambda,E}$ and $r_{\Lambda,E}$ in Point 1 of Theorem 4 guarantees the UV-existence of the $r_{\Lambda,E}$ -integral; the IR-convergence of the integral is obvious, since $r_{\Lambda,E}$ is by definition a $B_{4m+1}(\mathbb{R}^{4m})$ -function ■

Remark 4 The “majorization” of $\bar{R}_{\Lambda,E}$ by $r_{\Lambda,E}$, demanded in point 2, is only formulated for the corresponding integrals. The “heavy-mass-suppression” of the $r_{\Lambda,E}$ -integral will imply the analogous behaviour of the integral $|\int d^{4m}k \bar{R}_{\Lambda,E}|$; see eq. (56) ■

Proof of Theorem 4: $r_{\Lambda,E}$ will be constructed by a suitable elimination of the poles of $\bar{R}_{\Lambda,E}$. The strategy is quite similar to the proof of the IR-Power-Counting Theorem [20] and involves several steps:

5.1 Localization of all potential poles

At first, the potential poles of $\bar{R}_{\Lambda,E}$ are localized, and in each case, a suitable parametrisation of the integrand is defined.

According to its definition, the integrand $\bar{R}_{\Lambda,E}$ has the following form:

$$\bar{R}_{\Lambda,E} = E_2 \frac{Q(l_t, M, \epsilon, \dots)}{\prod_t \{l_{t,E}^2 + \mathcal{M}_t^2\}} \equiv E_2 \frac{Q(l_t, M, \epsilon, \dots)}{\prod_t N_{t,E}(l_t, \dots)} \quad (33)$$

As $Q(\dots)$ is polynomial, $\bar{R}_{\Lambda,E}$ can only have poles where at least one of the euclidean denominators $N_{t,E}$ is zero: for this denominator “ t ”, the vanishing of the line-momentum $l_{t,E}$ and the mass-function $\mathcal{M}_t^2$ is a necessary, but not sufficient condition for the existence of a pole.

To isolate all potential poles, a decomposition of the region of integration $\mathbb{R}^{4m}$ will be done as follows [20]: Let N denote the number of denominators “ t ” in $\bar{R}_{\Lambda,E}$. Then, every denominator can uniquely labeled by an index $j \in \{1, \dots, N\}$. Choose a positive constant r^2 and let ξ be any subset of the set of indices: $\xi \subseteq \{1, \dots, N\}$.

A possibly empty subset $X_\xi \subset \mathbb{R}^{4m}$ of the integration-region may be associated to ξ by the prescription:

$$(k_1^\Lambda, \dots, k_m^\Lambda) \in X_\xi \Leftrightarrow \begin{cases} (l_{j,E})^2 \leq r^2 & j \in \xi \\ (l_{j,E})^2 \geq r^2 & j \notin \xi \end{cases} \quad (34)$$

Defining $\Xi \equiv \{\xi | \xi \subseteq \{1, \dots, N\}\}$, the sets X_ξ , $\xi \in \Xi$, obviously have the properties:

- $X_\xi \cap X_{\xi'} \simeq \emptyset$, if $\xi \neq \xi'$
This relation is true modulo sets of measure zero (“ $\simeq$ ”).
- $\sum_{\xi \in \Xi} X_\xi = \mathbb{R}^{4m}$

Therefore, the X_ξ constitute a disjoint decomposition of the space of loop-momenta modulo sets of measure zero.

As the integral $\int d^{4m}k \bar{R}_{\Lambda,E}$ is absolutely convergent, it is invariant under linear, possibly inhomogeneous transformations (determinant 1) of the variables k^Λ . Therefore, a suitable parametrization will be used to represent the restriction of $\bar{R}_{\Lambda,E}$ to loop-momenta $(k_1^\Lambda, \dots, k_m^\Lambda) \in X_\xi$: Via the line-momenta $l_{j,E}$, $j \in \xi$, every ξ defines a lorentz-invariant subspace $\mathcal{T}_\mathcal{L}(\xi)$ (dimension: $0 \leq 4a \leq 4m$) of the linear forms $\mathcal{L}(\Lambda)$. The line-momenta $l_{j,E}$ may be classified as:

type 1: $j \in \xi$ and by definition $l_{j,E} \in \mathcal{T}_\mathcal{L}(\xi)$.

type 2: $j \notin \xi$, but $l_{j,E} \in \mathcal{T}_\mathcal{L}(\xi)$.

type 3: $j \notin \xi$ and $l_{j,E} \notin \mathcal{T}_\mathcal{L}(\xi)$.

As new variables of integration, line-momenta $l_{t_i,E}$ of type 1, which constitute a basis of the subspace $\mathcal{T}_\mathcal{L}(\xi)$ of linear forms $\mathcal{L}(\Lambda)$, are chosen.

$$l_{t_i,E} = \sum_{j=1}^m \alpha_{t_i,j} k_{j,E}^\Lambda + \sum_{e=1}^n \beta_{t_i,e} p_{e,E}^\Lambda \equiv u_i \quad i \in \{1, \dots, a\}$$

Obviously the u -variables are bounded: $0 \leq u_i^2 \leq r^2$. In addition, the u -variables are subject of further constraints resulting from eq. (34); the momenta p^Λ are fixed:

$$\text{type 1: } l_{j,E}^2(u_i, p_{\dots,E}^\Lambda) \leq r^2 \quad j \in \xi, j \notin \{t_i | i = 1, \dots, a\}$$

$$\text{type 2: } l_{j,E}^2(u_i, p_{\dots,E}^\Lambda) \geq r^2$$

The set of u -variables, which solves the inequalities above, is denoted $\mathcal{U}(\xi)$.

To form a basis of the loop-momenta, the variables u_i will be supplemented by choosing suitable line-momenta $l_{j_k,E}$ “of type 3” as further variables of integration:

$$l_{j_k,E} \equiv v_k \quad k \in \{1, \dots, m-a\}$$

With this choice, the v -variables are unbounded: $v_k^2 \geq r^2$. Line-momenta $l_{j,E}$ “of type 3”, which are not contained in this basis can be represented as linear combinations of the u - and v -variables and the external momenta p^Λ . These line-momenta have to fulfil in addition the inequalities:

$$\forall j \in \{1, \dots, N\} \setminus (\xi \cup \{j_k | k = 1, \dots, m-a\}) : l_{j,E}^2(u, v, p) \geq r^2. \quad (35)$$

In general, the set of v -variables, that solves these inequalities, depends on u ; therefore a sensible definition is:

$$\forall u \in \mathcal{U}(\xi) : l_{j,E}^2(u_i, v_k, p_{\cdot,E}^\Lambda) \geq r^2 \iff v \in \mathcal{V}_u(\xi) \quad (36)$$

Using T_ξ , a linear inhomogeneous transformation with determinant 1 [20], X_ξ may be parametrized in the variables u and v :

$$X_\xi \ni (k_1^\Lambda, \dots, k_m^\Lambda) \xrightarrow{T_\xi} (u_1, \dots, u_a, v_1, \dots, v_{m-a}) \in X_\xi(u, v) \quad (37)$$

Using the sets $\mathcal{U}(\xi)$ and $\mathcal{V}_u(\xi)$, $X_\xi(u, v)$ is expressed as:

$$X_\xi(u, v) = \bigcup_{u \in \mathcal{U}(\xi)} (u, \mathcal{V}_u(\xi)) \quad (38)$$

5.2 Upper bound on the contribution “ ξ ”

A first estimate on the absolute value of the integral is a trivial consequence of its absolute convergence.

$$\left| \int d^{4m}k R_{\Lambda,E} \right| \leq \int d^{4m}k |\bar{R}_{\Lambda,E}| = \sum_{\xi \in \Xi} \int_{X_\xi} d^{4m}k |\bar{R}_{\Lambda,E}| \quad (39)$$

Further examinations will be done for each contribution “ ξ ” separately. The integrands $|\bar{R}_{\Lambda,E}|$, restricted to the regions X_ξ , will be the building-blocks of the $B_{4m+1}(\mathbb{R}^{4m})$ -funktion $r_{\Lambda,E}$ described in Theorem 4:

I: If $|\bar{R}_{\Lambda,E}|$ has **no poles** in the region X_ξ , its restriction to X_ξ may be chosen as a direct contribution to $r_{\Lambda,E}$; see eqs. (50) to (53).

II: If **poles** are present in the region X_ξ , the integrand has to be modified suitably.

In case II, it is useful, to change to the corresponding (u, v) -variables and to decompose the integrand in a v -dependent (“ v -sector”) and a v -independent part (“ u -sector”); in this parametrization poles only appear in the u -sector. Then, the corresponding u - and v -integrals will be factorized, and finally, the u -integrand will be replaced by a suitably chosen “averaged and bounded” form.

(u, v) -Decomposition of the integrand $|\bar{R}_{\Lambda,E}|$

The polynomial Q in the numerator can be represented in the form [20]

$$Q(l_{t_1}, l_{t_2}, M, \dots) = \sum_i (u)^{(i)} q^{(i)}(v, p^\Lambda, M, \dots), \quad (40)$$

with u -independent polynomials $q^{(i)}$. The monomial $u_1^{i_1} \dots u_a^{i_a}$ is abbreviated as $u^{(i)}$. Each denominator $N_{t,E}$ is characterized by the labels $(a_1, \dots, a_4)$, which refer to the following properties: If $N_{t,E}(a_1, a_2, a_3, a_4)$ is

- **v -dependent:** $a_1 = v$; otherwise: $a_1 = 0$.
- **u -dependent:** $a_2 = u$. Only if a u -dependence can definitely be excluded, a_2 is zero. For 1 PI-diagrams, v -independent denominators necessarily depend on u and vice versa: $a_1 = 0$ implies $a_2 = u$, and $a_2 = 0$ leads to $a_1 = v$.
- **M -dependent:** $a_3 = M$; otherwise: $a_3 = 0$.
- **“massive”:** $a_4 = 1$; otherwise: $a_4 = 0$. Due to condition **A2**, $a_3 = M$ implies $a_4 = 1$.

Using eq. (40), the triangle inequality and the fact, that $N_{t,E}(a_1, a_2, a_3, a_4)$ is positive semidefinite, an upper estimate on the contribution “ ξ ” in eq. (39) is:

$$\int_{X_\xi} d^{4m} k |\bar{R}_{\Lambda,E}| \leq |E_2| \sum_i \int_{U(\xi)} d^{4a} u \frac{|(u)^{(i)}|}{\prod N_{t,E}(0, u, a_3, a_4)} \int_{V_u(\xi)} d^{4(m-a)} v \frac{|q^{(i)}(v, p^\Lambda, \dots)|}{\prod N_{t,E}(v, u, a_3, a_4)} \quad (41)$$

The IR- and UV-convergence conditions eq. (18), which are satisfied by $\bar{R}_{\Lambda,E}$, combined with the inequalities

$$\forall i: \underline{\deg}_u Q \leq \underline{\deg}_u (u)^{(i)} \quad ; \quad \overline{\deg}_v Q \geq \overline{\deg}_v q^{(i)}$$

guarantee the absolute convergence of every integral “ ξ, i ” in eq. (41).

Factorization and bound on the v -integral

In general, the v -integral contains two sources of u -dependence: the region of integration $V_u(\xi)$ and the denominators $N_{t,E}(v, u, a_3, a_4)$.

The u -dependence of the region of integration $V_u(\xi)$ is eliminated by integrating over V_ξ instead of $V_u(\xi)$:

$$V_\xi \equiv \bigcup_{u \in U_\xi} V_u(\xi) \supseteq V_{u'}(\xi) \quad \forall u' \in U_\xi \quad (42)$$

V_ξ is by definition u -independent; as the v -integrand is positive semidefinite, the integration over the region V_ξ gives an upper estimate on the v -integral.

Simply neglecting the u -contributions in $N_{t,E}(v, u, a_3, a_4)$ may lead to violations of inequality (36); for massless denominators $N_{t,E}(v, u, 0, 0)$ this manipulation even could generate poles in the v -sector. The following strategy avoids this problem:

1. First, for all massless denominators $N_{t,E}(v, u, 0, 0)$ an auxiliary mass Υ is introduced. Then, inequality (36) leads to the upper estimate [20]:

$$\frac{1}{N_{t,E}(v, u, 0, 0)} \leq \frac{1}{l_{t,E}^2 + \Upsilon^2} \left(1 + \frac{\Upsilon^2}{r^2} \right)$$

2. After this manipulation all denominators $N_{t,E}(v, u, \dots)$ are massiv; for simplicity, the mass-function is again denoted by $\mathcal{M}_t^2$. With the inequality [25]

$$\frac{1}{l_{t,E}^2 + \mathcal{M}_t^2} \leq \frac{1}{V_{t,E}^2 + \mathcal{M}_t^2} \underbrace{\left(1 + \left| \frac{l_{t,E} - V_{t,E}}{\mathcal{M}_t} \right| + \frac{(l_{t,E} - V_{t,E})^2}{\mathcal{M}_t^2} \right)}_{\equiv F_t(l-V, \mathcal{M}_t^2)}, \quad (43)$$

the u -dependence of the denominator is eliminated.

This will lead to the following substitutions in the v -sector:

$$\begin{aligned} \frac{1}{N_{t,E}(v, u, 0, 0)} &\xrightarrow{\leq} \frac{1}{\mathcal{N}_{t,E}(v, 0, 0, 1)} K_t(l - V, \Upsilon^2) \\ \frac{1}{N_{t,E}(v, u, a_3, 1)} &\xrightarrow{\leq} \frac{1}{\mathcal{N}_{t,E}(v, 0, a_3, 1)} F_t(l - V, \mathcal{M}_t^2) \end{aligned} \quad (44)$$

The denominators on the r.h.s. are defined as:

$$\mathcal{N}_{t,E}(v, 0, 0, 1) \equiv V_{t,E}^2 + \Upsilon^2 \quad ; \quad \mathcal{N}_{t,E}(v, 0, a_3, 1) \equiv V_{t,E}^2 + \mathcal{M}_t^2 \quad (45)$$

The correction-terms $F_t(l - V, \mathcal{M}_t^2)$, defined in eq. (43), and

$$K_t(l - V, \Upsilon) \equiv \left(1 + \frac{\Upsilon^2}{r^2} \right) F_t(l - V, \Upsilon^2) \quad (46)$$

are v -independent, but may depend on u . Therefore, they will contribute to the u -integral.

The substitutions eq. (44) in the v -integrand do neither alter its positive semidefiniteness nor its properties with respect to v and (v, M) -scalings; again the v -integral is absolutely convergent.

This leads to the following upper bound on the v -integral:

$$\begin{aligned} &\int_{\mathcal{V}_u(\xi)} d^{4(m-a)}_v \frac{|q^{(i)}(v, p^\Lambda, \dots)|}{\prod N_{t,E}(v, u, 0, 0) \prod N_{t,E}(v, u, a_3, 1)} \leq \\ &\leq \underbrace{\prod K_t(., \Upsilon^2) \prod F_t(., \mathcal{M}_t^2)}_{u\text{-dependent}} \int_{\mathcal{V}_\xi} d^{4(m-a)}_v \frac{|q^{(i)}(v, p^\Lambda, \dots)|}{\prod \mathcal{N}_{t,E}(v, 0, a_3, 1)} \end{aligned} \quad (47)$$

Bound on the u -integral

As the integral over $\mathcal{V}_\xi$ in the second line of eq. (47) is u -independent, the u - and v -integrals are already factorized; therefore, the singular u -sector can be replaced by its average over $\mathcal{U}(\xi)$.

Due to point 1 in Theorem 4, it is necessary to extract the M -factors in the denominators $N_{t,E}(0, u, M, 1)$ before performing the u -average. Exploiting condition **A2** and the fact, that the u -integration region $\mathcal{U}(\xi)$ is bounded, the following substitution will be suitable:

$$\frac{1}{N_{t,E}(0, u, M, 1)} \xrightarrow{\leq} \frac{1}{\mathcal{M}_t^2} \quad (48)$$

This substitution and the K - and F -factors will not alter the IR-behaviour. Therefore, the first two u -integrals in eq. (49) converge absolutely; the last identity defines $\mathbf{c}_\xi^{(i)}$.

$$\int_{\mathcal{U}(\xi)} d^{4a}u \frac{|(u)^{(i)}| \prod K_t(\cdot) \prod F_t(\cdot)}{\prod N_{t,E}(0, u, 0, a_4) \prod N_{t,E}(0, u, M, 1)} \leq \int_{\mathcal{U}(\xi)} d^{4a}u \frac{|(u)^{(i)}| \prod K_t(\cdot) \prod F_t(\cdot)}{\prod N_{t,E}(0, u, 0, a_4) \prod \mathcal{M}_t^2} \equiv \int_{\mathcal{U}(\xi)} d^{4a}u \frac{c_\xi^{(i)}}{\prod \mathcal{M}_t^2} \quad (49)$$

Definition of the upper bound on contribution “ ξ ”

To conclude section 5.2, an upper bound C_ξ on the contribution “ ξ ” is defined:

$$C_\xi \equiv \int_{\bar{X}_\xi} d^{4m}k \bar{Z}_{\Lambda,E}^\xi(k, p, \dots) , \quad (50)$$

with the following definition of the integrand

$$\bar{Z}_{\Lambda,E}^\xi(k, p, \dots) \equiv \begin{cases} |R_{\Lambda,E}(k, p, \dots)| ; & \text{case I} \\ \frac{\sum_i c_\xi^{(i)} |q^{(i)}(v, p^\Lambda, \dots)| |E_2|}{\prod \mathcal{M}_t^2 \prod \mathcal{N}_{t,E}(v, 0, a_3, a_4)} \Big|_{(u,v) \equiv \hat{T}_\xi(k,p)} \Big|_{\bar{X}_\xi} ; & \text{case II} \end{cases} \quad (51)$$

and the region of integration

$$X_\xi \equiv \begin{cases} X_\xi ; & \text{case I} \\ \hat{X}_\xi \equiv (\hat{T}_\xi)^{-1}(U(\xi) \times \mathcal{V}_\xi) ; & \text{case II} \end{cases} \quad (52)$$

The region $(U(\xi) \times \mathcal{V}_\xi)$ is defined in eqs. (38) and (42). Obviously, the linear and invertible parameter-transformation T_ξ , eq. (37), can be extended to

$$\hat{T}_\xi : \hat{X}_\xi \longrightarrow (U(\xi) \times \mathcal{V}_\xi) , \text{ with } \hat{T}_\xi \Big|_{X_\xi} \equiv T_\xi$$

5.3 Definition of $r_{\Lambda,E}$ in the region X_ξ

The function $r_{\Lambda,E}$ is now constructed by suitably patching the contributions ξ' , localized in the region $\bar{X}_{\xi'}$. In the case II, slight complications may arise due to the extensions

eq. (42): in general, overlaps of the regions $\bar{X}_{\xi'}$ and X_{ξ} are possible. Therefore, the following definitions are convenient: For all $\xi \in \Xi$, let

$$\bar{\chi}_{\xi}(\mathbf{k}) \equiv \begin{cases} 1; & k \in \bar{X}_{\xi} \\ 0; & \text{“otherwise”} \end{cases}$$

be the **characteristic function** of the set $\bar{X}_{\xi}$, eq. (52). The functions $\chi_{\xi'|\xi}(\mathbf{k})$, $\xi' \in \Xi$, provide a **decomposition of $\bar{\chi}_{\xi}$** with respect to the sets $X_{\xi'}$, $\xi' \in \Xi$:

$$\bar{\chi}_{\xi}(k) \equiv \sum_{\xi' \in \Xi} \chi_{\xi'|\xi}(k) \quad ; \quad \text{supp } \chi_{\xi'|\xi}(k) \subseteq X_{\xi'}$$

The upper bounds $C_{\xi'}$ can now be represented as:

$$C_{\xi'} \equiv \int_{\bar{X}_{\xi'}} d^{4m}k \, \bar{Z}_{\Lambda,E}^{\xi'}(.) \equiv \int_{\mathbb{R}^{4m}} d^{4m}k \, \sum_{\xi \in \Xi} \left\{ \bar{Z}_{\Lambda,E}^{\xi'}(k, p, \dots) \chi_{\xi|\xi'}(k) \right\}$$

Summing all $C_{\xi'}$, $\xi' \in \Xi$, the restriction of $r_{\Lambda,E}$ to X_{ξ} is defined as:

$$\left. r_{\Lambda,E}(k, p, \dots) \right|_{X_{\xi}} \equiv \sum_{\xi' \in \Xi} \bar{Z}_{\Lambda,E}^{\xi'}(k, p, \dots) \chi_{\xi|\xi'}(k) \quad (53)$$

In this way, $r_{\Lambda,E}$ is defined on every set X_{ξ} as a countable superposition of certain integrands $\bar{Z}_{\Lambda,E}^{\xi'}$.

All contributions “ ξ' to ξ ” in eq. (53) have compatible scaling-properties: $\chi_{\xi|\xi'}(k)$ is non-vanishing only for “suitable” combinations $\xi|\xi'$. Looking back to section 5.1, the extension eq. (42) can only lead to a violation of the inequalities (35):

$$\forall j \in \{1, \dots, N\} \setminus (\xi \cup \{j_k | k = 1, \dots, m-a\}) : l_{j,E}^2(u, v, p) \geq r^2.$$

Using the definition of the sets X_{ξ} , eq. (34), an overlap of the regions $\hat{X}_{\xi'}$ and X_{ξ} , with $\xi' \subset \xi$, cannot be not excluded: in subsets of the region $\hat{X}_{\xi'}$, there may be line-momenta $l_{j,E}$, $j \notin \xi'$, which fulfil $l_{j,E}^2 \leq r^2$. The (u, v) -parametrization described in section 5.1 then leads to the following relation for the corresponding v -sectors:

$$\text{“}\xi' \text{ contributes to } \xi\text{”} \implies \xi' \subset \xi \implies v_{\xi} \subseteq v_{\xi'} \quad (54)$$

5.4 Proof of the $B_{4m+1}(\mathbb{R}^{4m})$ -properties

In view of Theorem 4, a check of the $\tilde{v}$ - and $(\tilde{v}, M)$ -scaling-behaviour of $r_{\Lambda,E}$ and its $B_{4m+1}(\mathbb{R}^{4m})$ -properties wrt. to the variables k^{Λ} and M remains to be done:

By construction, $r_{\Lambda,E}$ has no poles for $M > 0$ (this corresponds to a suitable choice of the region W in Definition 3). In addition, $r_{\Lambda,E}$ is integrable wrt. the linear subspace $I \simeq \mathbb{R}^{4m}$, spanned by the loop-momenta k^Λ .

Depending on the constant r^2 and the values of the fixed variables $k_\perp^\Lambda \perp \tilde{v}$ and p^Λ , the behaviour of $r_{\Lambda,E}$ under $\tilde{v}$ - and $(\tilde{v}, M)$ -scalings is determined by the properties of $r_{\Lambda,E}$ in one of the regions $X_{\xi_1}, \dots, X_{\xi_l}$. All those regions – if represented in the corresponding (u, v) -parametrization – are characterized by v -sectors v_{ξ_j} , which contain $\tilde{v}$; due to eq. (54), all contributions “ ξ ” to the region X_{ξ_j} have even bigger v -sectors $v_{\xi'}$:

$$\tilde{v} \subseteq v_{\xi_j} \subseteq v_{\xi'}$$

As the substitutions eq. (44) leave the $\tilde{v}$ - and $(\tilde{v}, M)$ -scaling-behaviour of all v_{ξ} -sectors unchanged, the corresponding “upper” degrees of $r_{\Lambda,E}$ and $\bar{R}_{\Lambda,E}$ are identical. For the regions X_{ξ_j} , $j \in \{1, \dots, l\}$ and for suitable $1 < b_i < \eta_i$ ($i = 1, 2$), this leads to the following estimates:

$$\begin{aligned} r_{\Lambda,E} \left(\eta_1 \eta_2 \vec{B}_{\tilde{v}} + \eta_2 \vec{L}_M + \vec{C} \right) &\leq A_{\xi_j} \eta_1^{\alpha^{\Lambda,E}(\{\vec{B}_{\tilde{v}}\})} \eta_2^{\alpha^{\Lambda,E}(\{\tilde{v}, \vec{L}_M\})} \\ r_{\Lambda,E} \left(\eta_1 \eta_2 \vec{L}_M + \eta_2 \vec{B}_{\tilde{v}} + \vec{C} \right) &\leq A_{\xi_j} \eta_1^{\alpha^{\Lambda,E}(\{\vec{L}_M\})} \eta_2^{\alpha^{\Lambda,E}(\{\tilde{v}, \vec{L}_M\})} \end{aligned}$$

The vectors $\vec{L}_M$ and $\vec{B}_{\tilde{v}}$ denote an embedding of the heavy-mass-parameters M and a basis of the subset $k_\perp^\Lambda \cong \tilde{v}$ in the $(4m+1)$ -dimensional vector “ $\vec{X}$ ” of Definition 3. $\vec{C}$ contains the components $k_\perp^\Lambda$ and – if necessary – a component in $\vec{L}_M$, such that the variables M are nonvanishing for all admissible η ’s.

The asymptotic exponents $\alpha(\dots)$ can be identified with appropriate “upper” degrees of $\bar{R}_{\Lambda,E}$; the logarithmic-asymptotic coefficients $\beta(\dots)$ vanish¹²:

$$\begin{aligned} \alpha^{\Lambda,E} \left(\left\{ \vec{L}_M \right\} \right) &\equiv \overline{deg}_M \bar{R}_{\Lambda,E} & \alpha^{\Lambda,E} (\{\tilde{v}\}) &\equiv \overline{deg}_{\tilde{v}} \bar{R}_{\Lambda,E} \\ \alpha^{\Lambda,E} \left(\left\{ \tilde{v}, \vec{L}_M \right\} \right) &\equiv \overline{deg}_{\tilde{v},M} \bar{R}_{\Lambda,E} & \beta^{\Lambda,E} (\dots) &\equiv 0 \end{aligned} \quad (55)$$

Therefore, the function $r_{\Lambda,E}$, eq. (53), fulfils all the requirements of Theorem 4. ■

6 The Decoupling Theorem

It is assumed, that $\bar{R}_{\Lambda,E}$ fulfils the conditions **A1** – **A3**. Then, due to **A2**, Theorem 4 may be applied to bound the corresponding integral; see eq. (39):

$$\left| \int d^{4m} k \bar{R}_{\Lambda,E}(\eta M, k, \dots) \right| \leq \int d^{4m} k r_{\Lambda,E}(\eta M, k, \dots) \equiv f(\eta M, \dots) \quad (56)$$

¹²The constants $c_\xi^{(i)}$ are defined via an η -independent integral.

Using Theorem 9, part **B**, and Theorem 4, $f(\eta M, \dots)$ is shown to be an element in the class of functions $B_1(\emptyset)$ wrt. the variable η : by definition, there exist constants $A_{\Lambda, E}$ and $b > 1$, such that an inequality of the form

$$f(\eta M, \dots) \leq A_{\Lambda, E} (\eta)^{\alpha_I^{\Lambda, E}(\vec{L}_M)} \sum_{0 \leq \gamma \leq \beta_I^{\Lambda, E}(\vec{L}_M)} (\ln \eta)^\gamma$$

is valid for $\eta > b$. In the case at hand, equ. (73) may be reformulated as:

$$\alpha_I^{\Lambda, E}(\vec{L}_M) = \max_{\emptyset \subseteq v \subseteq \mathbb{R}^{4m}} \left\{ \alpha^{\Lambda, E}(\vec{L}_M, v) + \dim(v) \right\} \quad (57)$$

Due to Theorem 9, part **C**, eq. (74), the logarithmic asymptotic coefficient $\beta_I^{\Lambda, E}(\vec{L}_M)$ is bounded at least by $4m$. Of course, an M -independent integrand $r_{\Lambda, E}$ implies the vanishing of $\beta_I^{\Lambda, E}(\vec{L}_M)$ ¹³.

Provided the conditions **A1** and **A3b** are valid, the following upper bound on $\alpha_I^{\Lambda, E}(\vec{L}_M)$ can be derived with the help of Theorem 3, eqs. (55) and (57):

$$\alpha_I^{\Lambda, E}(\vec{L}_M) = \max_{\emptyset \subseteq v \subseteq \mathbb{R}^{4m}} \left\{ \leq \underline{N}^{S^v}(\Lambda) - \underline{m}^{S^v}(\Lambda) + \dim(v) \right\}$$

Eqs. (31) and (32) imply, that only “lorentz-invariant” subspaces $v \equiv \bar{v}$ can provide “maximizing contributions” to eq. (57). The braces are bounded as:

$$\left\{ \begin{array}{ll} < \underline{N}^{S^v}(\Lambda) : & \underline{m}^{S^v}(\Lambda) \neq 0 \\ \leq \underline{N}^{S^v}(\Lambda) : & \underline{m}^{S^v}(\Lambda) = 0 \end{array} \right\} (v)$$

If in addition condition **A3a** is fulfilled, $\underline{N}^{S^v}(\Lambda)$, eq. (25), is non-positive; $\alpha_I^{\Lambda, E}(\vec{L}_M)$ vanishes only a diagram Λ without “heavy lines”.

$$\alpha_I^{\Lambda, E}(\vec{L}_M) = \begin{cases} \leq -1 ; & \Lambda \text{ with “heavy lines”} \\ = 0 ; & \Lambda \text{ without “heavy lines”} \end{cases}$$

These results are collected in

Theorem 5 (Decoupling Theorem) *A nontrivial Feynman-integral associated to the BPHZL-subtracted integrand $\bar{R}_{\Lambda, E}$ (Λ is 1 PI) is considered. $\bar{R}_{\Lambda, E}$, defined in section 2 and eq. (19), fulfils the conditions **A1**, **A2** and **A3**.*

¹³This is a direct consequence of [11] and the UV-convergence-condition eq. (72): For M -independent integrands $r_{\Lambda, E}$, the following inequality is obvious:

$$\overline{\deg}_{v, M} r_{\Lambda, E} \equiv \overline{\deg}_v r_{\Lambda, E} = \begin{cases} < -\dim(v) ; & v \not\equiv \emptyset \\ = 0 ; & v \equiv \emptyset \end{cases}$$

Therefore, eq. (57) leads to $\alpha_I^{\Lambda, E}(\vec{L}_M) = 0$, and there is a single “maximizing subspace”: $v \equiv \emptyset$.

Then, the corresponding Feynman-integral is a $B_1(\emptyset)$ -function in the variable η (appearing in ηM), $\eta > b > 1$, provided the other variables are fixed suitably; see footnote 11. The following inequality applies

$$\left| \int d^{4m}k \mathcal{R} J_{\Lambda,E}(\eta M) \right| \leq A_{\Lambda,E} (\eta)^{\alpha_{\Lambda,E}} \sum_{0 \leq \gamma \leq \beta_{\Lambda,E}} (\ln \eta)^\gamma, \quad (58)$$

with (logarithmic-) asymptotic coefficients $\alpha_{\Lambda,E}$ and $\beta_{\Lambda,E}$ constrained by:

$$\begin{aligned} \alpha_{\Lambda,E} &\leq -1 & \beta_{\Lambda,E} &\leq 4m & ; & \Lambda \text{ with "heavy lines"} \\ \alpha_{\Lambda,E} &= 0 & \beta_{\Lambda,E} &= 0 & ; & \Lambda \text{ without "heavy lines"} \end{aligned} \quad (59)$$

In the limit $\eta \rightarrow \infty$, contributions of diagrams Λ with “heavy lines” are suppressed. Analogous results can be proved for the modifications $\bar{R}_\Lambda^E$ and $\bar{R}_{\Lambda,\epsilon}$, $\epsilon > 0$.

Remark 5 The behaviour of the Feynman-integral $\int d^{4m}k \mathcal{R} J_{\Lambda,E}(\eta M)$ in the limit $\eta \rightarrow \infty$ is only determined by the asymptotic M - and (v, M) -scaling-behaviour of the integrand. The presence of massless fields only leads to technical problems, which are circumvented with the help of Theorem 4. As the proof of that Theorem shows, all cases of massive/massless “light fields” lead to the same (logarithmic-) asymptotic coefficients $\alpha_{\Lambda,E}$ and $\beta_{\Lambda,E}$ ■

Remark 6 Due to the inequality $\eta > b > 1$, the “sum γ ” can be bounded as:

$$\sum_{0 \leq \gamma \leq \beta} (\ln \eta)^\gamma \leq C(b) (\ln \eta)^\beta$$

In eq. (58), this sum may be replaced by the highest power of $(\ln \eta)$, provided the constant $A_{\Lambda,E}$ is suitably modified ■

The Decoupling Theorem is formulated for nontrivial 1 PI-diagrams Λ . Obviously, it may be generalized to arbitrary diagrams Σ . The application of Theorem 5 to all nontrivial 1 PI parts, of condition **A4a** to all “trivial trees” and **A4b** to “remaining propagators” leads to

Corollary 6 *The Decoupling Theorem is also valid for general diagrams Σ , provided the conditions **A1**, **A2** and **A3** apply to all 1 PI-subdiagrams of Σ , and **A4** is true for all “trivial trees” and “remaining propagators” of Σ :*

The Feynman-integral associated to $\bar{R}_{\Sigma,E}$ is again a $B_1(\emptyset)$ -function in the variable η and fulfils inequalities analogous to eqs. (58) and (59).

Contributions of diagrams Σ with “heavy lines” are suppressed in the limit $\eta \rightarrow \infty$.

Analogous conclusions are true for $\bar{R}_\Sigma^E$ and $\bar{R}_{\Sigma,\epsilon}$, $\epsilon > 0$.

For a 1 PI-diagram Λ_L , which by definition contributes to a vertex-function

$$\left. < T F_1 \dots F_n > \right|_{1 \text{ PI}}$$

of the “light sector” (there are no “heavy fields” among $F_1, \dots, F_n$), the topological relation eq. (63) implies: There are “heavy lines” in Λ_L if and only if there are vertices Δ_z , $z \in \Lambda_L$, which contain “heavy fields”. Therefore:

Corollary 7 *The hypotheses of Theorem 5 are assumed. If a 1 PI-diagram Λ_L (as described above) contains vertices with “heavy fields”, the Feynman-integrals associated to $\bar{R}_{\Lambda_L}^E$, $\bar{R}_{\Lambda_L, E}$ and $\bar{R}_{\Lambda_L, \epsilon}$, $\epsilon > 0$ will be suppressed in the limit $\eta \rightarrow \infty$. If Corollary 6 applies, analogous conclusions follow for general, nonamputated diagrams Σ .*

As a simple consequence of eq. (70) one gets

Corollary 8 *Feynman-integrals associated to the BPHZL-subtracted integrand $R_{\Lambda, E}$ are considered. $R_{\Lambda, E}$ is assumed to be decomposeable, such that the factors $\bar{R}_{\Lambda, E}$ resp. E_1^Λ fulfil the conditions **A1**, **A2** and **A3** resp. **A5**; see eq. (19). Similar to Theorem 5, the corresponding Feynman-integral is a $B_1(\emptyset)$ -function in the variable η ; inequalities formally equivalent to eqs. (58) and (59) may be obtained. Using condition **A5**, the results of Corollaries 6 and 7 can be generalized to $R_{\Sigma, E}$, $R_{\Lambda_L, E}$ and its modifications, provided the Corollaries apply to the corresponding “ $\bar{R}$ ”-quantities.*

The author benefitted from many discussions with Prof. Sibold. He thanks Prof. Zimmermann for helpful hints and encouragement. Financial support by the Max-Planck-Gesellschaft is gratefully acknowledged.

A “Upper” and “lower” degrees

A.1 Definition and simple properties

Definition 2 *A function f , rational in the variables $(x_1, \dots, x_a, y_1, \dots, y_b) \equiv (x, y)$, has the “upper” and “lower” x -degrees g and h*

$$g \equiv \overline{\deg}_{x|y} f(x, y) \quad \text{resp.} \quad h \equiv \underline{\deg}_{x|y} f(x, y),$$

if the equations

$$\lim_{\lambda \rightarrow \infty} \lambda^{-g} f(\lambda x, y) \neq 0, \infty \quad \text{resp.} \quad \lim_{\lambda \rightarrow 0} \lambda^{-h} f(\lambda x, y) \neq 0, \infty$$

are valid for almost all values (x, y) . Occasionally the subscript $|y$ will be suppressed.

As a simple consequence of this definition, the following rules for the “upper” and “lower” x -degrees of products, quotients and sums of rational functions A , B and f_i , $i \in \mathcal{I}$ ($\mathcal{I}$: a finite index-set), apply:

$$\begin{aligned}\overline{deg}_x(\prod_i f_i) &= \sum_i \{\overline{deg}_x f_i\} & \underline{deg}_x(\prod_i f_i) &= \sum_i \{\underline{deg}_x f_i\} \\ \overline{deg}_x\left(\frac{A}{B}\right) &= \overline{deg}_x A - \overline{deg}_x B & \underline{deg}_x\left(\frac{A}{B}\right) &= \underline{deg}_x A - \underline{deg}_x B \\ \overline{deg}_x(\sum f_i) &\leq \max_i \{\overline{deg}_x f_i\} & \underline{deg}_x(\sum f_i) &\geq \min_i \{\underline{deg}_x f_i\}\end{aligned}\quad (60)$$

In the last line, equality holds, if the summands f_i are linear independent.

If a rational function F is **homogeneous of degree d** with respect to a set of variables $x_1 \dots x_j$, i.e.:

$$F(\lambda x_1, \dots, \lambda x_j) = (\lambda)^d F(x_1, \dots, x_j),$$

there is a relation, connecting suitable “upper” and “lower” degrees: For any decomposition (a, b) of the parameters $x_1 \dots x_j$, the following equality holds:

$$d = \overline{deg}_{a|b} F + \underline{deg}_{b|a} F \quad (61)$$

A.2 Engineering dimension

Condition **A1a** implies, that the formal integrand $J_{\lambda, \epsilon}$ is a homogeneous function of the dimensionful variables and parameters p , k , w_j , m_j , $\tilde{m}$ and M of degree

$$d_{\odot}(J_{\lambda, \epsilon}) \equiv \overline{deg}_{p, k, w_j, m_j, \tilde{m}, M} J_{\lambda, \epsilon}$$

$d_{\odot}$ will be called the **engineering dimension**. Due to condition **A1b**, the engineering dimension $d_{\odot}$ and the UV-subtraction-degree $\delta(\lambda)$, eq. (13), of the 1 PI-diagram λ are related: “ $J_{\lambda, \epsilon}$ is minimally UV-subtracted wrt. its engineering dimension”.

$$d_{\odot}(J_{\lambda, \epsilon}) \equiv \overline{deg}_{p, k, w_j, m_j, \tilde{m}, M} J_{\lambda, \epsilon} \equiv \delta(\lambda) - 4m(\lambda) \quad (62)$$

This holds even in the presence of an oversubtracted insertion $k \Delta_k$.

The proof of eq. (62) is straightforward and uses eqs. (3) and (19), condition **A1b** and the topological relation for the field F_a :

$$\sum_{b \neq a} I_{ab}(\lambda) + 2 I_{aa}(\lambda) = \sum_{z \in \lambda} c_{za} - N_a^\lambda \quad (63)$$

$I_{ab}(\lambda)$ denotes the number of propagators $\langle T F_a F_b \rangle_{\dots}$ in λ . The objects on the r.h.s. are explained in eqs. (4) and (13).

A.3 Effects of Taylor-operators

Let T be either $t_{p,s-1}^{e-1}$, $t_{p,s}^\delta$ or the subtraction-operator τ of eq. (14). As TF and $(1-T)F$ are linear independent, eq. (60) implies:

$$\overline{deg}_x F = \overline{deg}_x \{(1-T+T)F\} \geq \begin{cases} \overline{deg}_x \{TF\} ; & \{TF\} \neq 0 \\ \overline{deg}_x \{(1-T)F\} ; & \{(1-T)F\} \neq 0 \end{cases} \quad (64)$$

Acting with $(1-t_y^\delta)$ on a function $F(x,y)$, which is regular in $y=0$ for almost all x , leads to the following bound:

$$\max(0, \delta+1) \leq \underline{deg}_{y|x} \{(1-t_y^\delta)F(x,y)\} ; \quad \text{if } (1-t_y^\delta)F(x,y) \neq 0 \quad (65)$$

Let F be a homogeneous function in the massive parameters and variables of degree $d_\odot F$; the action of T resp. $(1-T)$ does not change its degree of homogeneity, provided F is not mapped to zero:

$$d_\odot F = \begin{cases} d_\odot \{TF\} ; & \text{if } \{TF\} \neq 0 \\ d_\odot \{(1-T)F\} ; & \text{if } \{(1-T)F\} \neq 0 \end{cases} \quad (66)$$

A.4 Proof of property vii) in section 4.1

According to eqs. (15) and (16), F stands for $Y_\lambda(C)$ with $\lambda \subseteq \Lambda$, $\bar{\lambda}(C) \not\parallel S^v$. Since all maximal subdiagrams $\lambda_i \subset \lambda$ are subtracted with τ , the denominators corresponding to lines $L \in \bar{\lambda}_i$, $\bar{\lambda}_i(C) \parallel S^v$, are independent of p, w and v . A decomposition

$$F = \frac{\sum_j A_j B_j}{DE} \quad (67)$$

results [18], with B_j being a monomial in the variables (p, w, v) ; A_j and D are (p, w, v) -independent, and E is a product of v -dependent factors $\{l_i^2 + \mathcal{M}_i^2 + i\epsilon(\cdot)\}$. Due to

$$1 - \tau = (1 - t_{p,s-1}^{e-1}) (1 - t_{p,s}^\delta)$$

and eq. (64), the inequality

$$\overline{deg}_{v,M} \{(1-\tau)F\} \leq \overline{deg}_{v,M} \{(1-t_{p,w}^\delta)F\}$$

follows, and, using eq. (67) and the linear independence of the terms j , one arrives at

$$\overline{deg}_{v,M} \{(1-t_{p,w}^\delta)F\} = \max_j \left\{ \overline{deg}_{v,M} \frac{A_j}{D} + \overline{deg}_{v,M} (1-t_{p,w}^\delta) \frac{B_j}{E} \right\}$$

With the decomposition $(p, w | m, \tilde{m}, M, u, v) \equiv (p, w | x)$ of the massive parameters and eqs. (61), (65) and (66), the inequality

$$d_\odot \left(\frac{B_j}{E} \right) - \max(0, \delta+1) \geq \overline{deg}_{x|p,w} \left\{ (1-t_{p,w}^\delta) \frac{B_j}{E} \right\}$$

is derived, valid if $(1 - t_{p,s}^\delta) B_j/E \neq 0$ (otherwise there is no contribution j). The properties of B_j/E and A_j/D , specified above, guarantee:

$$\begin{aligned} \overline{\deg}_{x|p,w} \left\{ (1 - t_{p,w}^\delta) \frac{B_j}{E} \right\} &\geq \overline{\deg}_{v,M} \left\{ (1 - t_{p,w}^\delta) \frac{B_j}{E} \right\} \\ d_\odot \left(\frac{B_j}{E} \right) &= \overline{\deg}_{v,p,w,M} \left(\frac{B_j}{E} \right) \quad \overline{\deg}_{v,M} \left(\frac{A_j}{D} \right) = \overline{\deg}_{v,p,w,M} \left(\frac{A_j}{D} \right) \end{aligned}$$

Property vii) results for suitable F . ■

B The class of functions $B_n(I)$

Throughout sections 5 and 6, the definition and properties of the class of functions $B_n(I)$ are used. This class was essentially developed by Weinberg [10] and Fink [11]; most of the material presented here is taken from Manoukian's book [3].

Definition 3 ([3]) *A function $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$, belongs to a class $B_n(I)$ of functions, provided the following properties are true: For every linear subspace $T \subseteq \mathbb{R}^n$, every choice of linear independent vectors $\vec{L}_1, \dots, \vec{L}_p \in T$, $p \leq n$, and a bounded region $W \subset \mathbb{R}^n$ (f is finite in W) there exist coefficients $\alpha(T) \in \mathbb{R}$, $\beta(T) \in \mathbb{N}_0$ and a positive constant A , such that the inequality*

$$\begin{aligned} &\left| f \left(\vec{L}_1 \eta_1 \dots \eta_t + \vec{L}_2 \eta_2 \dots \eta_t + \vec{L}_t \eta_t + \vec{C} \right) \right| \leq \\ &\leq A \eta_1^{\alpha(\{\vec{L}_1\})} \dots \eta_t^{\alpha(\{\vec{L}_1, \dots, \vec{L}_t\})} \sum_{a_1, \dots, a_t} (\ln \eta_{\pi_1})^{a_1} \dots (\ln \eta_{\pi_t})^{a_t} \end{aligned}$$

is satisfied for any $\eta_i > b_i > 1$, $i \in \{1, \dots, t\}$, and any $\vec{C} \in W$. The sums are over all non-negative integers $a_1, \dots, a_t$, which fulfil:

$$\forall 1 \leq k \leq t : \sum_{i=1}^k a_i \leq \beta \left(\left\{ \vec{L}_{\pi_1}, \dots, \vec{L}_{\pi_k} \right\} \right) \quad (68)$$

The β 's are arranged in increasing order; $(\pi_1, \dots, \pi_t)$ denotes the corresponding permutation of $(1, \dots, t)$:

$$\beta \left(\left\{ \vec{L}_{\pi_1}, \dots, \vec{L}_{\pi_l} \right\} \right) \leq \beta \left(\left\{ \vec{L}_{\pi_1}, \dots, \vec{L}_{\pi_k} \right\} \right) \Leftrightarrow l \leq k$$

“ I ” in $B_n(I)$ refers to a linear subspace of $\mathbb{R}^n$. If $|f(\vec{x})|$ is finite, this implies, that $|f(\vec{x} + \vec{y})|$ is finite for all finite $\vec{y} \in I$.

The constants A , b_i are η -independent, but may depend on the choice of $\vec{L}_i$ and W .

B.1 Products and sums of $B_n(I)$ -functions

It is shown, that the class of functions $B_n(I)$ is “closed” under multiplication and summation. Upper estimates on the resulting asymptotic and logarithmic-asymptotic coefficients for the product/sum are indicated. As an example, the class of functions $B_1(\emptyset)$ is treated; a discussion of the general case is quite analogous.

Let $\mathcal{I}$ be a finite index-set, and f_i , $i \in \mathcal{I}$, be functions in the class $B_1(\emptyset)$. Then, for W , a bounded interval in $\mathbb{R}^1$ and $L \in \mathbb{R}$, chosen according to Definition 3, there exist constants A_i , α^i and β^i such that

$$\forall i \in \mathcal{I}, C \in W, 1 < b_i < \eta : |f_i(\eta L + C)| \leq A_i (\eta)^{\alpha^i} \sum_{0 \leq \gamma \leq \beta^i} (\ln \eta)^\gamma$$

is valid. The **product** may be bounded as ($\eta > \max_i b_i$):

$$\left| \prod_{i \in \mathcal{I}} f_i(\eta L + C) \right| \leq A' (\eta)^\alpha \sum_{0 \leq \gamma \leq \beta} (\ln \eta)^\gamma ; \quad \alpha \equiv \sum_i \alpha^i ; \quad \beta \equiv \sum_i \beta^i \quad (69)$$

An upper estimate on the **sum** of $B_1(\emptyset)$ -functions is given by ($\eta > \max_i b_i$):

$$\left| \sum_{i \in \mathcal{I}} f_i(\eta L + C) \right| \leq A' (\eta)^\alpha \sum_{0 \leq \gamma \leq \beta} (\ln \eta)^\gamma ; \quad \alpha \equiv \max_{i \in \mathcal{I}} \alpha^i ; \quad \beta \equiv \max_{j; \alpha^j = \alpha} \beta^j \quad (70)$$

Linearly dependent summands f_i actually may lead to an asymptotic coefficient α' less than α and a logarithmic-asymptotic coefficient β' greater than β . However, any $B_1(\emptyset)$ -function with (logarithmic-)asymptotic coefficients α' and β' can be majorized by a corresponding function with coefficients α and β .

B.2 Integrals of $B_n(I)$ -functions

The integral of a $B_n(I)$ -function f over a k -dimensional linear subspace $I \subseteq \mathbb{R}^n$, spanned by an orthonormal basis $\vec{L}_1, \dots, \vec{L}_k$, is considered ($\vec{X} \in I_\perp$; $\mathbb{R}^n = I \oplus I_\perp$):

$$\int_{-\infty}^{\infty} dy_1 \dots \int_{-\infty}^{\infty} dy_k f(\vec{X} + y_1 \vec{L}_1 + \dots + y_k \vec{L}_k) \equiv f_{\vec{L}_1, \dots, \vec{L}_k}(\vec{X}) \quad (71)$$

The relevant properties are described in

Theorem 9 ([3]) A: *The integral eq. (71) is absolutely convergent, if the following inequality is satisfied:*

$$D_I = \max_{\emptyset \neq S' \subseteq I} \{ \alpha(S') + \dim(S') \} < 0 \quad (72)$$

$\alpha(S')$ is the asymptotic coefficient of the integrand f wrt. the subspace S' .
If this inequality is valid, the integral depends only on the subspace I , but not on the choice of vectors $\vec{L}$.

$$f_{\vec{L}_1, \dots, \vec{L}_k}(\vec{X}) \equiv f_I(\vec{X})$$

B: If **A** applies, then $f_I(\vec{X})$ is an element of the class of functions $B_{n-k}(\emptyset)$ with asymptotic coefficients ($S \subseteq I_\perp$):

$$\alpha_I(S) = \max_{S'=S \oplus I_1; I_1 \subset I} \{\alpha(S') + \dim(S') - \dim(S)\} \quad (73)$$

C: If the logarithmic-asymptotic coefficients β of the integrand vanishes, the following upper bound for the logarithmic-asymptotic coefficient $\beta_I(S)$ of the integral follows:

$$S \subseteq I_\perp \quad \beta_I(S) \leq \dim(I) = k \quad (74)$$

Remark 7 Condition (72) is the original formulation of Weinberg’s Power-Counting-Theorem [10] for euclidean metric and massive fields ■

Remark 8 For the purpose of this paper, it is sufficient to know, that $\beta_I(S)$ is finite for finite-dimensional integrals. Therefore, the “exact” form of the logarithmic-asymptotic coefficient $\beta_I(S)$, determined by a rather lengthy recursive procedure [11] [3], will not be needed.

It is conjectured, that the bound (74) may be improved in the case of Feynman-integrals: As already indicated, only “lorentz-invariant” integrations can saturate eq. (73). Most probably, this implies, that already the number of independent four-dimensional integrations (m) will lead to an upper bound on $\beta_I^\Lambda(\vec{L}_M)$ in eq. (59), case: “with heavy lines” ■

References

- [1] T. Appelquist and J. Carazzone. Infrared singularities and massive fields. *Phys. Rev.*, D 11:2856, 1975.
- [2] J. Ambjørn. On the decoupling of massive particles in Quantum Field Theory. *Comm. Math. Phys.*, 67:109, 1979.
- [3] E. B. Manoukian. *Renormalization*. Academic Press, New York, 1983.
- [4] N. P. Landsman. Large-mass and high-temperature behaviour in perturbative Quantum Field Theory. *Comm. Math. Phys.*, 125:643, 1989.
- [5] L. Girardello and A. Zaffaroni. Exact Renormalization Group Equation and decoupling in Quantum Field Theory. *Nucl. Phys.*, B 424:219, 1994.

- [6] R. D. Ball and R. S. Thorne. The Decoupling Theorem in Effective Scalar Field Theory. Preprint hep-th/9404156, 1994.
- [7] C. Kim. A renormalization group flow approach to decoupling and irrelevant operators. Preprint hep-th/9402117, 1994.
- [8] W. E. Caswell and A. D. Kennedy. Asymptotic behavior of Feynman integrals: Convergent integrals. *Phys. Rev.*, D 28:3073, 1983.
- [9] T. Shimert and C. C. Chiang. Heavy-mass power-counting theorem and a proof of decoupling in momentum-subtraction scheme. *Phys. Rev.*, D 29:241, 1984.
- [10] S. Weinberg. High-energy behavior in Quantum Field Theory. *Phys. Rev.*, 118:838, 1960.
- [11] J. P. Fink. Asymptotic estimates of Feynman integrals. *Journ. of Math. Phys.*, 9:1389, 1968.
- [12] J. H. Lowenstein and W. Zimmermann. On the formulation of theories with zero-mass propagators. *Nucl. Phys.*, B 86:77, 1975.
- [13] J. H. Lowenstein. Convergence theorems for renormalized Feynman integrals with zero-mass propagators. *Comm. Math. Phys.*, 47:53, 1976.
- [14] K. Hepp. Proof of the Bogoliubov-Parsiuk theorem on Renormalization. *Comm. Math. Phys.*, 2:301, 1966.
- [15] W. Zimmermann. The power counting theorem for Minkowski metric. *Comm. Math. Phys.*, 11:1, 1968.
- [16] J. H. Lowenstein and E. Speer. Distributional limits of renormalized Feynman integrals with zero-mass propagators. *Comm. Math. Phys.*, 47:47, 1976.
- [17] W. Zimmermann. Convergence of Bogoliubov's method of renormalization in momentum space. *Comm. Math. Phys.*, 15:208, 1969.
- [18] M. Gomes, J.H. Lowenstein, and W. Zimmermann. Generalization of the momentum-space subtraction procedure for renormalized perturbation theory. *Comm. Math. Phys.*, 39:81, 1974.
- [19] J. H. Lowenstein. BPHZ Renormalization. In *Renormalization Theory*, Dordrecht-Holland, 1976. Reidel Publishing Company.
- [20] J. H. Lowenstein and W. Zimmermann. The power counting theorem for Feynman integrals with massless propagators. *Comm. Math. Phys.*, 44:73, 1975.
- [21] T.E. Clark, O. Piguet, and K. Sibold. Renormalization theory in superspace. *Ann. of Phys.*, 109:418, 1977.

- [22] T.E. Clark, O. Piguet, and K. Sibold. Infrared anomaly induced by spontaneous breakdown of supersymmetry. *Nucl. Phys.*, B 119:292, 1977.
- [23] M. Aschenbrenner. *Entkopplung schwerer Freiheitsgrade in einem $N=1$ SYM-Modell*. PhD thesis, Technische Universität München, 1994.
- [24] E. K. Katehou and G. G. Ross. Decoupling in supersymmetric theories. *Nucl. Phys.*, B 299:484, 1988.
- [25] Y. Hahn and W. Zimmermann. An elementary proof of Dyson's power counting theorem. *Comm. Math. Phys.*, 10:330, 1968.