Test of the Running of $\alpha_s$ in $\tau$ Decays

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Abstract

The decay rate of the $\tau$ lepton into hadrons of invariant mass smaller than $\sqrt{s_0} > \Lambda_{\text{QCD}}$ can be calculated in QCD assuming global quark–hadron duality. It is shown that this assumption holds with high accuracy for $s_0 > 0.7$ GeV$^2$. From measurements of the hadronic mass distribution, the running coupling constant $\alpha_s(s_0)$ is extracted in the range $0.7$ GeV$^2 < s_0 < m_{\tau}^2$, where its value changes by a factor 2. At $s_0 = m_{\tau}^2$, the result is $\alpha_s(m_{\tau}^2) = 0.329 \pm 0.030$, corresponding to $\alpha_s(m_Z^2) = 0.119 \pm 0.004$. The running of the coupling constant is in excellent agreement with the QCD prediction based on the three-loop $\beta$-function.
The scale dependence of coupling constants is one of the key features of renormalizable quantum field theories. In QCD, the effective coupling constant $\alpha_s(Q^2)$ is predicted to decrease with the momentum transfer $Q^2$, so that the strong interaction becomes weak at high energies, a property referred to as asymptotic freedom [1]. This prediction has been tested by comparing data obtained from experiments performed at different energy scales [2]: low-energy measurements of $\alpha_s (Q \sim 1.6-10 \text{ GeV})$ come from tests of deep-inelastic sum rules, scaling violations in deep-inelastic scattering, hadronic $\tau$ decays, and $\Upsilon$ spectroscopy and decays; at higher energies ($Q \sim 30-100 \text{ GeV}$), the most reliable determinations come from measurements of the total cross section, jet rates and event shapes in $e^+e^-$, $p\bar{p}$ and $ep$ collisions. The running of $\alpha_s$ can also be studied in single high-energy experiments at $p\bar{p}$ and $ep$ colliders, where a large range of $Q$ values can be probed simultaneously [3].

Here we propose a test of the scale dependence of the strong coupling constant in the low-energy region $0.7 \text{ GeV}^2 < Q^2 < m_\tau^2$, where the value of $\alpha_s$ changes by a factor 2. Our method is based on integrals of the invariant mass distribution in hadronic $\tau$ decays. It provides a unique opportunity to test one of the most important predictions of QCD in a single experiment and at low energies, where the effect of the running of $\alpha_s$ is most pronounced.

We shall consider the $\tau$ decay rate into hadrons of invariant mass squared smaller than $s_0$, normalized to the leptonic decay rate:

$$\frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons}; s_{\text{had}} < s_0)}{\Gamma(\tau \rightarrow \nu_\tau e^+e^-)} = \frac{s_0}{\int_0^{s_0} ds \frac{dR_\tau(s)}{ds}}, \quad (1)$$

where $dR_\tau/ds$ is the inclusive hadronic spectrum, which has been measured by the ALEPH and CLEO Collaborations [4, 5]. The quantity $R_\tau(s_0)$ can be calculated in QCD using the Operator Product Expansion (OPE) [6]–[8], as long as $s_0 \gg \Lambda^2_{\text{QCD}}$. The fact that the perturbative contributions are known to high order in this case, together with the fortunate occurrence that the power corrections turn out to be very small, guarantee a good convergence of the OPE down to rather low values of $s_0$. However, the use of the OPE relies on the assumption of global quark–hadron duality, since decay rates are defined in the physical region, where the applicability of perturbation theory is not justifiable. Perturbation theory predicts the production of quarks and gluons, whereas hadrons are produced in nature. The assumption of global duality is that cross sections and decay rates admit a QCD description after applying a “smearing” over a sufficiently wide energy interval [9], which in the present case is provided by the integration over the range $0 < s < s_0$. A theoretical argument in favour of this hypothesis is the following: the $\tau$ decay rate into hadrons can be written in terms of moments $M_k^{[J]}$ of the absorptive part of current–current correlation functions $D^{[J]}(s)$ of
angular momentum $J$ [10, 11]. The quantity $R_	au(s_0)$ is given by

$$
\frac{1}{3S_{\text{EW}}} R_\tau(s_0) = \frac{2s_0}{m_r^2} M_0^{(1)}(s_0) - 2 \left( \frac{s_0}{m_r^2} \right)^3 M_2^{(1)}(s_0) + \left( \frac{s_0}{m_r^2} \right)^4 M_3^{(1)}(s_0)
+ \frac{2s_0}{m_r^2} M_0^{(0)}(s_0) - 2 \left( \frac{s_0}{m_r^2} \right)^2 M_1^{(0)}(s_0) + \frac{2}{3} \left( \frac{s_0}{m_r^2} \right)^3 M_2^{(0)}(s_0),
$$

where $S_{\text{EW}} \approx 1.0194$ accounts for electroweak radiative corrections [12]. The functions $D^{(J)}(s)$ are analytic in the complex $s$-plane, with discontinuities on the positive real axis. The structure of the singularities (resonance poles and production thresholds) cannot be described in perturbation theory. However, using the analyticity properties of the correlation functions, the moments can be written as contour integrals along a circle of radius $s_0$ in the complex plane:

$$
\mathcal{M}_k^{(J)}(s_0) = \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s} \left[ 1 - \left( \frac{s}{s_0} \right)^{k+1} \right] D^{(J)}(s).
$$

(3)

The assumption of global duality rests on the observation that to perform the contour integrals requires knowledge of the correlation functions for large (complex) momenta only. Moreover, the integrand in (3) vanishes for $s = s_0$, where the contour touches the cut; hence, the main contributions come from regions far away from the singularities. The question of how accurate the duality assumption is and for what values of $s_0$ it applies is, however, a phenomenological one; it cannot be answered yet from theoretical grounds. Below, we shall investigate this question, comparing experimental data with theoretical predictions based on the duality assumption.

Since the only large mass scale in (3) is $s_0$, the OPE provides an expansion of the moments in powers of $1/s_0$:

$$
\mathcal{M}_k^{(J)}(s_0) = \mathcal{M}_k^{(1)}[\alpha_s(s_0)]_{\text{pert}} \delta_{J=1} + \sum_{n=1}^\infty c_n^{(J)}[\alpha_s(s_0)] \frac{\langle O_{2n} \rangle}{s_0^n}.
$$

(4)

The leading term is given by perturbation theory alone. For dimensional reasons, terms suppressed by powers of $1/s_0$ consist of perturbative coefficients $c_n^{(J)}$ multiplying some dimensionful parameters $\langle O_{2n} \rangle$, such as quark masses or vacuum condensates [7]. This is how nonperturbative effects are incorporated in the OPE. There is no leading term for the moments with $J = 0$, which vanish in the chiral limit and are thus proportional to powers of the light quark masses. For the moments with $J = 1$, the perturbative contribution is

$$
\mathcal{M}_k^{(1)}[\alpha_s(s_0)]_{\text{pert}} = 1 + \sum_{n=1}^\infty d_n^{(k)} \left( \frac{\alpha_s(s_0)}{\pi} \right)^n,
$$

(5)

2
where $\alpha_s(s_0)$ is defined in the $\overline{MS}$ renormalization scheme, $d_1^{(k)} = 1$, and the next three expansion coefficients are given by [10, 11]

\begin{align*}
    d_2^{(k)} &= 1.63982 + \frac{9}{4(k + 1)}, \\
    d_3^{(k)} &= -10.2839 + \frac{11.3792}{k + 1} + \frac{81}{8(k + 1)^2}, \\
    d_4^{(k)} &= K_4 - 155.955 - \frac{46.238}{k + 1} + \frac{94.810}{(k + 1)^2} + \frac{68.344}{(k + 1)^3}.
\end{align*}

The coefficient $K_4$ appears in the perturbative expansion of the Adler function and is not known exactly. An estimate using the principle of minimal sensitivity and the effective charge approach [13] gives $K_4 \simeq 27.5$ [14]. We shall use this result in our analysis. The error due to the truncation of the perturbative series in (5) is of the order of the last term included. The importance of higher-order corrections can also be estimated by summing a gauge-invariant subset of terms to all orders in perturbation theory. Such a class of corrections is provided by the so-called renormalon chains [15], which are the terms of order $\beta_0^{n-1}\alpha_s^n$ in a perturbative series, where $\beta_0$ is the first coefficient of the $\beta$-function. Efficient techniques to resum these contributions have been developed recently [16]. The resummation of renormalon chains for the moments has been discussed in Ref. [11]. In our analysis, we shall compare fixed-order perturbation theory with this partial resummation and take the difference as an estimate of the perturbative uncertainty.

The power corrections in the OPE are proportional to the light quark masses or to vacuum condensates, which are nonperturbative parameters of QCD [7]. We quote the power corrections for the sum of the moments contributing to $R_r(s_0)$. The terms relevant to the numerical analysis are

\begin{equation}
\frac{1}{3S_{EW}} R_r(s_0)|_{\text{power}} = -6 |V_{us}|^2 \frac{m_s^2(s_0)}{m_r^2} \left[ 1 + \frac{s_0}{m_r^2} - \left( \frac{s_0}{m_r^2} \right)^2 + \frac{1}{3} \left( \frac{s_0}{m_r^2} \right)^3 \right]
+ \frac{16\pi^2}{m_r^4} \left[ \langle m_u \bar{\psi}_u \psi_u \rangle + |V_{ud}|^2 \langle m_d \bar{\psi}_d \psi_d \rangle + |V_{us}|^2 \langle m_s \bar{\psi}_s \psi_s \rangle \right]
- \frac{512\pi^3}{27} \frac{\rho \alpha_s \langle \bar{\psi} \psi \rangle^2}{m_r^6} + \ldots,
\end{equation}

where $m_s(s_0)$ is the running strange-quark mass renormalized at the scale $s_0$, $\langle m_q \bar{\psi}_q \psi_q \rangle$ are scale-invariant quark condensates, and $\rho \alpha_s \langle \bar{\psi} \psi \rangle^2$ denotes the four-quark condensate (in the vacuum saturation approximation). More detailed expressions, which are used in our analysis, can be found elsewhere [8, 10, 11]. At tree level the powers of $1/s_0$ appearing in the OPE of the moments in (4) conspire with the powers of $s_0/m_r^2$, which multiply the moments in (2), so that the dominant corrections to $R_r(s_0)$ are suppressed by powers of $1/m_r^2$. This is no longer
the case if radiative corrections to the coefficients of the vacuum condensates are taken into account, but the corresponding effects are very small. The largest contribution of this type is provided by the gluon condensate and is of the form $\alpha_s^2(s_0)/(\alpha_s G^2)/(s_0 m_t^2)$. Even for values as low as $s_0 = 1$ GeV$^2$, this contribution is only of order $10^{-3}$. As a consequence, the power corrections to $R_\tau(s_0)$ are very small down to rather low values of $s_0$. Using standard values of the QCD parameters (which we take from Ref. [11]) we find $-(1.4 \pm 0.5)$% for the right-hand side of (7) at $s_0 = m_t^2$, and $-(1.5 \pm 0.5)$% at $s_0 = 1$ GeV$^2$.

To extract the quantity $R_\tau(s_0)$, we use the data for the hadronic mass distribution reported by the ALEPH and CLEO Collaborations [4, 5], which are shown in Fig. 1a. To obtain $dR_\tau/ds$, we have multiplied the normalized distributions by $R_\tau = R_\tau(m_t^2)$, using the relation $R_\tau = 1/B_e - 1.97256$, where $B_e$ is the leptonic branching ratio. Direct measurements give $B_e = (17.80 \pm 0.06)$% [17], whereas using the tau lifetime, $\tau_\tau = (291.3 \pm 1.6)$ fs [18], we obtain $B_e = \tau_\tau/\tau_\tau (m_\tau/m_e)^5 = (17.84 \pm 0.10)$%. Averaging the two results gives $R_\tau = 3.642 \pm 0.010$. Not shown in the figure is the contribution from $\tau \rightarrow h^-\nu_\tau$ with $h^- = \pi^- =$ or $K^-$, which has a branching ratio of $(11.77 \pm 0.14)$% [17]. Integrating these spectra over $s$ and combining the results weighted by their statistical errors, we obtain the distribution $R_\tau(s_0)$ shown in Fig. 1b. Systematic errors have been estimated by taking the difference between the ALEPH and CLEO data. This is justified, since the dominant sources of systematic errors are different in the two analyses. The systematic errors are then added in quadrature with the statistical ones. Since the errors in the extraction of $R_\tau(s_0)$ are strongly correlated, the result is presented as a band. The two curves show theoretical calculations of the distribution $R_\tau(s_0)$ based on the OPE approach outlined above. They differ in the treatment of higher-order perturbative corrections. The solid line is obtained using fixed-order perturbation theory (FOPT) to order $\alpha_s^4$. The dashed line is obtained by adding to this a resummation of renormalon chains of order $\alpha_s^5$ and higher, using the results of Ref. [11]. We shall refer to this scheme as resummed perturbation theory (RPT), keeping in mind that starting from order $\alpha_s^5$ the resummation is approximate. In obtaining the theoretical curves, we have adjusted the value of $\alpha_s(m_t^2)$ so as to fit the data at $s_0 = m_t^2$. The fact that the total hadronic decay rate is known with high precision allows for a precise determination of $\alpha_s$ at this point, which is limited by the theoretical uncertainty [8]. The central values obtained in the two schemes are $\alpha_s(m_t^2) = 0.329$ (FOPT) and $\alpha_s(m_t^2) = 0.309$ (RPT). Their difference provides an estimate of the uncertainty due to unknown higher-order corrections. Alternatively, the perturbative uncertainty can be estimated by omitting the term of order $\alpha_s^4$ in the fixed-order calculation, which increases the value of $\alpha_s(m_t^2)$ by 4%. Varying the values of the quark masses and

1Given that the ALEPH data are preliminary, our estimate of the systematic errors should be taken with caution. However, since inclusive quantities such as $R_\tau(s_0)$ do not probe details of the hadronic mass distribution but only its gross features, systematic errors play a minor role in our analysis.
nonperturbative parameters within conservative limits changes $\alpha_s(m_\tau^2)$ by up to 2%. Taking fixed-order perturbation theory as the nominal scheme, we find

$$
\alpha_s(m_\tau^2) = 0.329 \pm 0.030,
\alpha_s(m_Z^2) = 0.119 \pm 0.004.
$$

To be conservative, the total error on $\alpha_s(m_\tau^2)$ has been obtained by adding linearly the perturbative uncertainty (±0.020), the nonperturbative uncertainty (±0.006), and the experimental uncertainty (±0.004). For the sake of completeness, we have translated our result into a value of the running coupling constant at the mass of the $Z$ boson, which is conventionally used to compare measurements of $\alpha_s$. For comparison with previous analyses [4, 5], we note that had we used the resummation method of Le Diberder and Pich [19] to estimate unknown higher-order corrections in the perturbative series for $R_\tau$, we would have obtained $\alpha_s(m_\tau^2) = 0.352$, less than 1σ higher than the central value quoted above. The difference of this result with fixed-order perturbation theory provides another estimate of the perturbative uncertainty, which is in agreement with the one given above.

To test the assumption of global quark–hadron duality, we now compare the data for the quantity $R_\tau(s_0)$ at values $s_0 < m_\tau^2$ with the theoretical predictions obtained from the OPE. The possibility of performing such a test has been mentioned by Le Diberder and Pich [10]. Once $\alpha_s(m_\tau^2)$ is determined, the value of the running coupling constant $\alpha_s(s_0)$ is obtained from the solution of the renormalization-group equation

$$
\mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2} = -\alpha_s(\mu^2) \beta(\alpha_s(\mu^2)),
\beta(\alpha_s) = \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \beta_2 \left(\frac{\alpha_s}{4\pi}\right)^3 + \ldots,
$$

where $\beta_0 = 9$, $\beta_1 = 64$ and $\beta_2 = 3863/6$ are the first three coefficients of the β-function, evaluated for $n_f = 3$ light quark flavours. (The value of $\beta_2$ is specific to the $\overline{\text{MS}}$ renormalization scheme.) At three-loop order, the solution is

$$
\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 L} \left[ 1 - \frac{\beta_1 \ln L}{\beta_0^2 L} + \frac{\beta_2}{\beta_0^2} \left( \ln^2 L - \ln L - 1 \right) + \beta_2 \beta_0 \right],
$$

where $L = \ln(\mu^2/\Lambda^2)$, and $\Lambda$ is the scale parameter in the $\overline{\text{MS}}$ scheme, which we determine from the value of $\alpha_s$ at $\mu = m_\tau$. Whereas the theoretical uncertainty is a limiting factor in the determination of $\alpha_s(m_\tau^2)$, it has little influence on the $s_0$ dependence of $R_\tau(s_0)$. For the perturbative part of the calculation this is apparent from the good agreement of the two theoretical curves in Fig. 1b, which refer to values of $\alpha_s(m_\tau^2)$ that differ by 9%. Varying the values of the quark masses and nonperturbative parameters has a negligible effect ($\sim 0.5\%$ at
The reason is that the only dependence on $s_0$ in $R_{\tau}(s_0)$ comes from the quark-mass corrections, which are known with higher accuracy than the vacuum condensates. Hence, the $s_0$ dependence of $R_{\tau}(s_0)$ is predicted essentially without any free parameters. A comparison of the data with the theoretical predictions thus provides a direct test of the assumption of global quark-hadron duality. We find excellent agreement over the entire range $0.7 \text{ GeV}^2 < s_0 < m_0^2$, indicating that in $\tau$ decays global duality holds as soon as the integral over the hadronic mass distribution includes the $\rho$ resonance peak. It is remarkable that once $s_0$ exceeds the value of $0.7 \text{ GeV}^2$, the onset of duality happens almost instantaneously. Since the $\rho$ meson is such a prominent resonance and the shape of the spectrum in the $\rho$ region cannot be predicted by perturbative QCD, this is the best possible scenario that could be expected.

Having determined the region in which global duality holds, we shall now rely on the duality hypothesis and turn to the main focus of our study: a test, at low energies, of the QCD prediction (10) for the running of the coupling constant. $\tau$ decays are an ideal place to study this phenomenon, since the value of $\alpha_s(s_0)$ changes by a factor 2 over the region where duality holds. (This is comparable to the change of $\alpha_s$ in the region between 5 and 100 GeV.) From the measurement of the quantity $R_{\tau}(s_0)$ shown in Fig. 1b, we extract $\alpha_s(s_0)$ as a function of $s_0$ by fitting the data to the theoretical prediction based on the expressions given in (2), (5)–(7). The result is presented by the bands in Fig. 2, the width of which reflects the (correlated) experimental errors in this determination as well as the nonperturbative uncertainty, which is small (of order $\delta \alpha_s/\alpha_s = 2\%$) and has been added to the experimental errors in quadrature. The dominant theoretical uncertainty is the perturbative one; it can be estimated from the difference between the two plots, which refer to fixed-order and resummed perturbation theory, respectively. The curves in Fig. 2 show the QCD predictions for the running coupling constant obtained at one- and three-loop order, normalized to the data at $s_0 = m_0^2$. The observed scale dependence of the coupling constant is in good agreement with the QCD prediction (10) obtained at three-loop order, and it is significantly stronger than the one-loop prediction. To quantify this agreement, we extract from the data the $\beta$-function that describes according to (9) the running of $\alpha_s(s_0)$. Defining $x = \alpha_s(s_0)/4\pi$, we have

$$\frac{4\pi}{\alpha_s^2(s_0)} \frac{d\alpha_s(s_0)}{d \ln s_0} = \frac{\beta(x)}{x} = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots.$$  \hspace{1cm} (11)

We approximate the derivative $d\alpha_s/d \ln s_0$ by a ratio of differences, $\Delta \alpha_s/\Delta \ln s_0$, for a selected set of $s_0$ values chosen such that the differences $\Delta \alpha_s$ are large enough to be significant given the errors in the measurement. For $\alpha_s(s_0)$ in (11) we take the central value of each interval. We use the following $s_0$ values: 0.75, 0.95, 1.35, 2.06, and 3.16 GeV$^2$, corresponding to four intervals of increasing width $\Delta \ln s_0$, but constant $\Delta \alpha_s \approx 0.075$. The results are shown in Fig. 3. The circles
are the data points obtained using fixed-order perturbation theory, while the squares are obtained using resummed perturbation theory. As expected, the two methods give very similar results for the running of the coupling constant. Our estimate of the errors includes the theoretical uncertainty, the error due to the choice of finite intervals in $\alpha_s$, and the experimental errors. The experimental errors in the difference $\Delta \alpha_s = \alpha_s(s_1) - \alpha_s(s_2)$ are mainly due to the errors in the spectrum between $s_1$ and $s_2$. They can be estimated from the relation $\delta \Delta \alpha_s \approx (d \alpha_s / d R_{\tau}) \delta f_{s_1}^s ds (d R_{\tau} / ds)$, where $d \alpha_s / d R_{\tau}$ is obtained from the ratio of the widths of the bands in Figs. 1b and 2. The curves in Fig. 3 show the QCD $\beta$-function at one-, two- and three-loop order in perturbation theory. The data provide clear evidence for the running of the coupling constant. More remarkable, however, is that they prefer a running that is stronger than predicted at one-loop order; all points lie above the line corresponding to the one-loop $\beta$-function. Between the three curves, the one that shows the three-loop prediction provides the best description of the data. Performing a fit of the data with the three-loop $\beta$-function, where $\beta_0 = 9$ and $\beta_1 = 64$ are kept fixed but the three-loop coefficient $\beta_2$ is treated as a parameter, we find $\beta_2^{\exp} / \beta_2^{\text{th}} = 1.6 \pm 0.7$ using fixed-order perturbation theory, and $1.8 \pm 0.8$ using resummed perturbation theory. We believe that such an experimental determination of the $\beta$-function can at present be done only in $\tau$ decays. A future high-precision measurement of $R_{e^+e^-}(s)$ in the region below the charmonium resonances would provide an alternative place for such a study. At higher energies, the value of $\alpha_s$ is too small to see a difference between the three curves in Fig. 3. Measurements in the region $Q \sim 100$ GeV, for instance, correspond to values $x \sim 0.01$, for which the curves are very close to one another. A test of the running of $\alpha_s$ beyond the leading order is only feasible at very low energies, for large values of the coupling constant.

In summary, we have presented a method to measure the running coupling constant $\alpha_s(s)$ in the low-energy region $0.7 \text{ GeV}^2 < s < m_\rho^2$ using $\tau$ decay data obtained in a single experiment. It provides a test of one of the key features of QCD in a region where the effect of the running is most pronounced. The theoretical analysis is based on the OPE and the assumption of global quark–hadron duality. We have tested this assumption and find that it holds, with high accuracy, if the $\tau$ decay rate is integrated over an energy interval large enough to include the $\rho$ resonance peak. Our analysis, which has been performed using existing data on the hadronic mass distribution in $\tau$ decays, provides a test of QCD at scales lower than the lowest ones achievable before (measurements of deep-inelastic sum rules at $Q \sim 1.6$ GeV), and with higher precision than all other single measurements of the running to date. In particular, we have extracted for the first time the $\beta$-function from data, and find that it is in good agreement with the three-loop prediction of QCD. Future refinements of our analysis may use higher-statistic data samples and aim at a better estimate of systematic errors.

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References


Figure 1: (a) The hadronic mass distribution $dR_\tau/ds$ in $\tau$ decays, obtained from data reported by the ALEPH and CLEO Collaborations [4, 5]. The ALEPH data are preliminary. CLEO data from electron-tagged and muon-tagged events have been combined. Shown are the statistical errors after correcting for detector effects. (b) The integrated spectrum $R_\tau(s_0)$. The experimental result, including statistical and systematic errors, is presented as a band. The curves show the theoretical predictions obtained using two different perturbative approximations (see text).
Figure 2: Values of $\alpha_s(s_0)$ extracted from the data on $R_\tau(s_0)$ using fixed-order (FOPT) and resummed perturbative theory (RPT). The dashed lines show a fit to the data obtained using the three-loop $\beta$-function. The dash-dotted lines refer to the one-loop $\beta$-function.
Figure 3: Experimental determination of the $\beta$-function. The circles are obtained using fixed-order perturbation theory, the squares refer to resummed perturbation theory. The curves show the QCD $\beta$-function at one-loop (dash-dotted), two-loop (dashed) and three-loop (solid) order.