Ac

STATE RESEARCH CENTER OF RUSSIA
INSTITUTE FOR HIGH ENERGY PHYSICS

IHEP 95-74

R.N.Rogalyov

GEOMETRICAL ASPECTS
OF THE HELICITY FORMALISM

509550

Submitted to *Intern. J. Mod. Phys. A*

Protvino 1995

Abstract

Rogalyov R.N. Geometrical Aspects of the Helicity Formalism: IHEP Preprint 95-74. – Protvino, 1995. – p. 11, refs.: 9.

A bilinear combination of Dirac spinors is expressed via Lorentz vectors. The result is represented in a convenient form, which can give rise to using it for analytical calculations of multiparticle amplitudes. A geometrical interpretation of the introduced vectors is given.

Аннотация

Рогалев Р.Н. Геометрические аспекты спирального формализма: Препринт ИФВЭ 95-74. – Протвино, 1995. – 11 с., библиогр.: 9.

Билинейная комбинация дираковских спиноров выражена через лоренцевские векторы. Результат представлен в форме, удобной для использования при аналитических вычислениях многочастичных амплитуд. Дана геометрическая интерпретация введенных векторов.

1. Introduction

There are many problems in high energy physics where the helicity formalism is considered to be worth. The first one is how to calculate multiparticle helicity amplitudes without squaring. Secondly, this formalism can also be useful in calculating one-loop helicity amplitudes. Then, in considerations of polarized fermions it is helpful to know manifest expressions for transition currents between states with different momenta and spins. The expressions for these currents can easily be derived from a formula of the type

$$w(p_1, n_1)\bar{w}(p_2, n_2) = S + V_\mu\gamma^\mu + T_{\mu\nu}\sigma^{\mu\nu} + A_\mu\gamma^\mu\gamma^5 + P\gamma^5, \quad (1)$$

where symbol $w(p, n)$ stands for the wave function of the fermion with momentum p and spin n , and $S, V_\mu, T_{\mu\nu}, A_\mu, P$ are expressed through the vectors of momentum and spin.

The formulae of this type are known in literature e.g. [4], [5]. However, the expressions for $V_\mu, T_{\mu\nu}, A_\mu$ in these papers contain the denominator

$$\begin{aligned} & \sqrt{(p_1 \cdot p_2 + m_1 m_2)(1 - n_1 \cdot n_2) + (p_1 \cdot n_2)(p_2 \cdot n_1)} \\ & = \sqrt{-m_1 m_2 \delta_{v_1}^{v_1+v_2} \frac{n_1+n_2}{n_1}}, \end{aligned} \quad (2)$$

here $v_i = p_i/m_i$ and $\delta_{pq}^{rs} = p \cdot r \ q \cdot s - p \cdot s \ q \cdot r$. It equals zero when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} = \lambda(n_1 + n_2), \quad (3)$$

which occurs in many practically important cases (in the case of spin-flip amplitudes, say, in the reaction $b \rightarrow s\gamma$). So, it is interesting to have a formula of type (1) which would be manifestly covariant and free of singularities.

The detailed consideration of degenerate case (3) [8] has cleared up the important feature of formula (1): when all the four vectors p_1, p_2, n_1 and n_2 lie in a single 2-plane, the vectors V_μ and A_μ may be orthogonal to this plane. Hence they cannot be expressed through p_i and n_i as linear combinations or with the help of ε -symbols. However, these

vectors and tensor $T_{\mu\nu}$ are determined by p_i and n_i up to a phase factor. All that requires to recollect the geometry of Minkowski space and make a definition of Dirac spinors more precise. That is done in the following section. In the third section twistor- like interpretation of Dirac spinors is given. The last section is devoted to a derivation of formula (1).

2. A General Discussion

2.1. Definitions and notations

Let us consider the well known spinors $u(p, n)$ and $v(p, n)$, which describe solutions of Dirac equation with positive and negative energy. They are eigenvectors of operators $\hat{p}$ and $\hat{n}\gamma^5$ (which can be considered as a full set of commutative variables):

$$\hat{p}u(p, n) = mu(p, n), \quad \hat{p}v(p, n) = -mv(p, n); \quad (4)$$

$$\hat{n}\gamma^5 u(p, n) = u(p, n), \quad \hat{n}\gamma^5 v(p, n) = v(p, n). \quad (5)$$

Axial vector n here is introduced with the help of the formula

$$\bar{u}(p, n)\gamma^\mu\gamma^5 u(p, n) = mn^\mu. \quad (6)$$

The relation

$$u(p, n)\bar{u}(p, n) = \frac{(\hat{p} + m)(1 + \gamma^5\hat{n})}{2} \quad (7)$$

is also valid.

Let ξ be some unit ($\xi^2 = -1$) vector which is orthogonal to p and n simultaneously

$$\xi \cdot p = \xi \cdot n = 0.$$

Then

$$\hat{p}(\hat{\xi}u(p, n)) = -m(\hat{\xi}u(p, n)), \quad \hat{n}\gamma^5(\hat{\xi}u(p, n)) = \hat{\xi}u(p, n),$$

hence $\hat{\xi}u(p, n)$ coincides with $v(p, n)$ up to a factor

$$\hat{\xi}u(p, n) = \lambda v(p, n).$$

It is easy to show that $\bar{\lambda}\lambda = 1$, i.e. $\lambda = e^{i\phi}$. Here and below a symbol $\hat{\zeta}$ denotes the representation of vector ζ in the Clifford algebra: $\zeta = (\zeta^0, \zeta^1, \zeta^2, \zeta^3)$, $\hat{\zeta} = \zeta^0\gamma^0 - \zeta^1\gamma^1 - \zeta^2\gamma^2 - \zeta^3\gamma^3$.

Let's us now introduce vector η with the help of the formula

$$\hat{\eta} = -\frac{i}{m}\hat{p}\hat{n}\gamma^5\hat{\xi},$$

so that $\frac{p}{m}$, n , ξ and η are orthonormal basis with the same orientation as γ^0 , γ^1 , γ^2 , and γ^3 . Then one has

$$\hat{\eta}u(p, n) = ie^{i\phi}v(p, n).$$

Hence the following equations are valid:

$$\hat{\omega}_+ u(p, n) = 0, \quad \hat{\omega}_- v(p, -n) = 0, \quad (8)$$

$$\hat{\omega}_- u(p, -n) = e^{i\phi} v(p, -n), \quad \hat{\omega}_+ v(p, n) = e^{i\phi} u(p, n). \quad (9)$$

Here

$$\omega_{\pm} = \frac{\xi \pm i\eta}{\sqrt{2}}.$$

2.2. Isotropic vectors $\omega_{\pm}$ and their properties

Let us now also introduce them in the other way. Any ordered couple of light-like vectors k_1 and k_2 : $k_1 \cdot k_2 = \frac{1}{2}$ determines another couple of (complex) light-like vectors ω_+ and ω_- , which are orthogonal to both k_1 and k_2

$$\omega_+ \cdot \omega_+ = 0, \quad \omega_- \cdot \omega_- = 0, \quad \omega_+ \cdot \omega_- = -1, \quad (10)$$

$$\omega_+ \cdot k_1 = 0, \quad \omega_+ \cdot k_2 = 0, \quad \omega_- \cdot k_1 = 0, \quad \omega_- \cdot k_2 = 0. \quad (11)$$

It is sufficient to require that the antisymmetric tensors

$$k_1^\mu \omega_+^\nu - \omega_+^\mu k_1^\nu \quad \text{and} \quad k_2^\mu \omega_-^\nu - \omega_-^\mu k_2^\nu$$

be self-dual

$$\hat{k}_1 \hat{\omega}_+ (1 - \gamma^5) = \hat{k}_2 \hat{\omega}_- (1 - \gamma^5) = 0, \quad (12)$$

while forms

$$k_2^\mu \omega_+^\nu - \omega_+^\mu k_2^\nu \quad \text{and} \quad k_1^\mu \omega_-^\nu - \omega_-^\mu k_1^\nu$$

are anti-self-dual

$$\hat{k}_2 \hat{\omega}_+ (1 + \gamma^5) = \hat{k}_1 \hat{\omega}_- (1 + \gamma^5) = 0. \quad (13)$$

Vectors ω_+ and ω_- are determined by k_1 , k_2 and requirements (10) - (13) up to a factor.

- Vector ω_- can be interpreted as the polarization vector of a photon with momentum k_1 and *positive* helicity (in the gauge $k_2 \cdot A = 0$).
- Vector ω_+ can be interpreted as the polarization vector of a photon with momentum k_1 and *negative* helicity (in the gauge $k_2 \cdot A = 0$).

The manifest expression for $\omega_{\pm}$ is

$$\omega_{\pm}^\mu = \frac{1}{\sqrt{2\Delta_3}} (k_1 \cdot k_2 q^\mu - q \cdot k_2 k_1^\mu - q \cdot k_1 k_2^\mu \pm i\epsilon^{\mu q k_1 k_2}). \quad (14)$$

Here

$$\Delta_3 = \delta_{k_1 k_2 q}^{k_1 k_2 q} = k_1 \cdot k_2 (q \cdot q \, k_1 \cdot k_2 - 2 \, k_1 \cdot q \, k_2 \cdot q)$$

and q is an arbitrary vector: $\Delta_3 \neq 0$. In spite of the presence of an arbitrary vector q in (14) both ω_+ and ω_- are almost independent of q : if q varies they acquire only a phase factor. Formula (14) can be rewritten in the equivalent form

$$\hat{\omega}_{\mp} = -\frac{1}{\sqrt{2\Delta_3}} \left(\hat{k}_1 \hat{q} \hat{k}_2 \frac{1 \pm \gamma^5}{2} + \hat{k}_2 \hat{q} \hat{k}_1 \frac{1 \mp \gamma^5}{2} \right). \quad (15)$$

Since ω_+ and ω_- are determined by an ordered pair of light-like vectors, one should write $\omega_+(k_1, k_2)$ and $\omega_-(k_1, k_2)$, notice that

$$\omega_+(k_1, k_2) \sim \omega_-(k_2, k_1).$$

However, if some fixed pair of light-like vectors is mentioned, the dependence of $\omega_{\pm}$ on it can be omitted. If $\mathbf{o}$ and ι are two arbitrary 2-component spinors which satisfy the relation

$$\mathbf{o}_A \iota_B - \iota_A \mathbf{o}_B = \epsilon_{AB}$$

then the vectors

$$k_1 = \mathbf{o}_A \mathbf{o}_A^\dagger, \quad k_2 = \iota_A \iota_A^\dagger, \quad \omega_+ = \sqrt{2} \mathbf{o}_A \iota_A^\dagger, \quad \omega_- = \sqrt{2} \iota_A \mathbf{o}_A^\dagger$$

satisfy all the above-mentioned properties.

2.3. Geometrical interpretation of spinors and the vectors $\omega_{\pm}$

In order to give a geometrical interpretation of these vectors let us remember the Cartan definition of a spinor in n -dimensional complex Euclidean space E_n [2]. The vectors of this space can be represented by $2^{\lfloor \frac{n}{2} \rfloor} \times 2^{\lfloor \frac{n}{2} \rfloor}$ matrixes, symbol $\hat{x}$ denotes the representation of vector x . The basic property of this representation is

$$\hat{x} \hat{y} + \hat{y} \hat{x} = 2 x \cdot y. \quad (16)$$

One should consider the cases of odd and even n separately.

Theorem 1. *If $x \in E_{2\nu+1}$ then the equation*

$$\hat{x} \psi = 0 \quad (17)$$

determines ν -dimensional isotropic plane for each 2^ν component column vector (i. e. spinor) ψ .

Proof. Indeed, if $\hat{x} \psi = 0$ and $\hat{y} \psi = 0$ then, due to (16), $x \cdot y = 0$. Hence this hyperplane is isotropic. The fact that it has exactly ν dimensions can be derived from the manifest construction of the representation (16), which is given in [2], as well as the proof of the theorem.

Theorem 2. *The components of ψ can be interpreted as elements of the Grassman algebra on this ν -plane.*

One can also consider isotropic ν -forms instead of isotropic ν -planes, that is equivalent. The ν -form corresponding to some isotropic ν -plane is the external product of the ν linearly independent vectors from this ν -plane.

Now it is appropriate to recollect the formulae that define $O(2\nu + 1)$ transformations on spinors. Let us start with the observation that

$$q'_\mu = q_\mu + 2q \cdot l l_\mu, \quad (18)$$

where q' is obtained from q as a result of a reflection with respect to 2ν -plane which is orthogonal to a unit spacelike vector l : $l^2 = -1$. The representation of this formula in the Clifford algebra looks as follows:

$$\hat{q}' \equiv S_l(q) = \hat{l}\hat{q}\hat{l}. \quad (19)$$

Let $R_{a_1 a_2} \equiv \rho^2$, where ρ is the boost (or rotation) in (a_1, a_2) plane ($a_1^2 = a_2^2 = -1$), which transforms a_1 into a_2 . Since any rotation is a composition of an even number of reflections, one can easily derive the relations:

$$\begin{aligned} R_{a_1 a_2} \equiv S_{a_2}(S_{a_1}(q)) &= \hat{a}_2 \hat{a}_1 \hat{q} \hat{a}_1 \hat{a}_2 = A \hat{q} A^{-1}, \\ A &= \hat{a}_2 \hat{a}_1. \end{aligned} \quad (20)$$

Hence the action of reflections and rotations on spinors looks as follows:

$$S_l \psi = \pm \hat{l} \psi, \quad R_{l_2 l_1} \psi = \pm \hat{l}_2 \hat{l}_1 \psi. \quad (21)$$

2ν-dimensional case

The space $E_{2\nu}$ can be considered as 2ν -dimensional hyperplane in $E_{2\nu+1}$, which is orthogonal to some vector a . Each spinor $\psi \in E_{2\nu+1}$ can be represented as the sum $\psi = \varphi + \chi$, where $\hat{a}\varphi = \varphi$, $\hat{a}\chi = -\chi$. Only the ν -planes corresponding to eigenspinors of $\hat{a}$ lie in $E_{2\nu}$; an arbitrary isotropic ν -plane intersects $E_{2\nu}$ over some $(\nu - 1)$ -plane.

In the language of external forms this idea can be formulated as follows. An arbitrary ν -form T on $E_{2\nu+1}$ can be uniquely decomposed into two components

$$T = R + a \wedge S,$$

where R is ν -form, S is $(\nu - 1)$ -form, both are defined on $E_{2\nu}$. The two forms R and S give the geometrical interpretation of a spinor in $E_{2\nu}$.

In our case $\nu = 2$, $\hat{a} = \gamma^5$. So, one can consider the spinor $u(p, n)$ as a spinor in 5-dimensional space, however it will not be done in detail here. Concerning the vector ω_+ one can say that it determines the line which is the intersection of the isotropic 2-plane corresponding to a spinor $u(p, n)$ with the (complexified) Minkowski space.

In 4-dimensional space there appears the two-fold ambiguity in the operator of reflection [2]: $S_l = \pm \hat{l}$ or $S_l = \pm \hat{l}\gamma^5$. To agree the denotations of books [1] and [2] with each other, one should choose

$$S_l = \hat{l}\gamma^5. \quad (22)$$

2.4. Agreements and definitions

We have considered the action of the Lorentz group on spinors in detail in order to make a definition of spinors more precise. Let us introduce the basis spinors describing a fermion at rest with a vector of spin directed along z -axes (γ -matrices are chosen in a chiral representation):

$$u(\mathbf{p}, \mathbf{n}) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad u(\mathbf{p}, -\mathbf{n}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

$$v(\mathbf{p}, \mathbf{n}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v(\mathbf{p}, -\mathbf{n}) = \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Spinors $u(\mathbf{p}, \mathbf{n})$ and $v(\mathbf{p}, \mathbf{n})$ correspond to a fermion and antifermion with 4-momentum $\mathbf{p} = (m, 0, 0, 0)$ and the vector of spin $\mathbf{n} = (0, 0, 0, 1)$, spinors $u(\mathbf{p}, -\mathbf{n})$ and $v(\mathbf{p}, -\mathbf{n})$ describe a fermion and antifermion with the same momentum and oppositely directed spin.

Now make a definition of the spinor $u(p, n)$ more precise. This spinor should describe a fermion with arbitrary momentum p and vector of spin n . We set that $u(p, n)$ is the result of the Lorentz transformation Λ applied to $u(\mathbf{p}, \mathbf{n})$

$$u(p, n) = L(\Lambda) u(\mathbf{p}, \mathbf{n}), \quad (23)$$

where Λ satisfies the following requirements:

$$p = \Lambda \mathbf{p}, \quad n = \Lambda \mathbf{n}. \quad (24)$$

A concrete form and arbitrariness in choosing Λ will be discussed in subsequent sections. Formulae (21) and (22) will help us to write an expression for $L(\Lambda)$.

Sometimes one has to deal with a couple of spinors of the type $u(p, n)$ and $u(p, -n)$. In the given paper we set that both $u(p, -n)$ and $u(p, n)$ are obtained with **the same Lorentz transformation** Λ from $u(\mathbf{p}, -\mathbf{n})$ and $u(\mathbf{p}, \mathbf{n})$ correspondingly

$$u(p, n) = L(\Lambda) u(\mathbf{p}, \mathbf{n}), \quad u(p, -n) = L(\Lambda) u(\mathbf{p}, -\mathbf{n}). \quad (25)$$

(An alternative way of thinking: both $u(p, n)$ and $u(p, -n)$ are obtained with *various* transformations from **the same** spinor $u(\mathbf{p}, \mathbf{n})$)

$$u(p, n) = L(\Lambda) u(\mathbf{p}, \mathbf{n}), \quad u(p, -n) = L(\Lambda') u(\mathbf{p}, \mathbf{n}). \quad (26)$$

If one neglects a sign of a spinor, both definitions are equivalent. However for some technical purposes it may be useful to fix a sign of a spinor, in spite of this sign having no physical meaning. In such cases definition (25) will be accepted.

3. Twistor-like Interpretation of Dirac Spinors

Having accepted the agreement about a sign of a spinor one can fix signs in formulae (9). All the similar formulae can also be obtained in the following way. It is easy to prove them in the rest frame of a fermion and then use their manifest covariance. So we get

$$\hat{\omega}_+ u(p, -n) = -\sqrt{2} v(p, -n), \quad (27)$$

$$\hat{\omega}_+ v(p, n) = -\sqrt{2} u(p, n), \quad (28)$$

$$\hat{\omega}_- u(p, n) = \sqrt{2} v(p, n), \quad (29)$$

$$\hat{\omega}_- v(p, -n) = \sqrt{2} u(p, -n), \quad (30)$$

$$\hat{\omega}_+ u(p, n) = \hat{\omega}_+ v(p, -n) = \hat{\omega}_- u(p, -n) = \hat{\omega}_- v(p, n) = 0. \quad (31)$$

Provided with the accepted agreement the following formulae are also valid:

$$\gamma^5 u(p, \pm n) = \mp v(p, \mp n), \quad (32)$$

$$\gamma^5 v(p, \pm n) = \pm u(p, \mp n), \quad (33)$$

$$\hat{n} u(p, \pm n) = -v(p, \mp n), \quad (34)$$

$$\hat{n} v(p, \pm n) = u(p, \mp n). \quad (35)$$

It is helpful to introduce the spinors

$$\varphi_{\pm}(p, n) = u(p, n) \pm v(p, -n), \quad (36)$$

$$\psi_{\pm}(p, n) = u(p, -n) \pm v(p, n). \quad (37)$$

They satisfy the following equations:

$$\left(\frac{\hat{p}}{m} + \hat{n} \right) \varphi_+ = 2\varphi_-, \quad (38)$$

$$\left(\frac{\hat{p}}{m} - \hat{n} \right) \varphi_- = 2\varphi_+, \quad (39)$$

$$\left(\frac{\hat{p}}{m} + \hat{n} \right) \varphi_- = \left(\frac{\hat{p}}{m} - \hat{n} \right) \varphi_+ = 0. \quad (40)$$

$$\left(\frac{\hat{p}}{m} + \hat{n} \right) \psi_+ = 2\psi_-, \quad (41)$$

$$\left(\frac{\hat{p}}{m} - \hat{n} \right) \psi_- = 2\psi_+, \quad (42)$$

$$\left(\frac{\hat{p}}{m} + \hat{n} \right) \psi_- = \left(\frac{\hat{p}}{m} - \hat{n} \right) \psi_+ = 0. \quad (43)$$

$$\hat{\omega}_+ \varphi_+ = \hat{\omega}_+ \varphi_- = 0, \quad (44)$$

$$\hat{\omega}_- \varphi_+ = \sqrt{2} \psi_+, \quad (45)$$

$$\hat{\omega}_- \varphi_- = -\sqrt{2} \psi_-. \quad (46)$$

$$\hat{\omega}_- \psi_+ = \hat{\omega}_- \psi_- = 0, \quad (47)$$

$$\hat{\omega}_+ \psi_+ = -\sqrt{2} \varphi_+, \quad (48)$$

$$\hat{\omega}_+ \psi_- = \sqrt{2} \varphi_-. \quad (49)$$

$$(1 + \gamma^5) \varphi_+ = (1 - \gamma^5) \varphi_- = 0, \quad (50)$$

$$(1 + \gamma^5) \psi_- = (1 - \gamma^5) \psi_+ = 0. \quad (51)$$

Since $(\frac{p}{m} + n)$ and $(\frac{p}{m} - n)$ are light-like vectors, spinors $\varphi_\pm$ and $\psi_\pm$ satisfy the massless Dirac equation. φ_+ and ψ_- describe its right-handed, φ_- and ψ_+ - left-handed solutions. In chiral representation they look as follows:

$$\varphi_+ = \begin{pmatrix} 0 \\ \varphi_+^{\dot{B}} \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \psi_-^{\dot{B}} \end{pmatrix}, \quad \varphi_- = \begin{pmatrix} \varphi_-^A \\ 0 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} \psi_+^A \\ 0 \end{pmatrix},$$

where $\varphi_+^{\dot{B}}$, $\psi_-^{\dot{B}}$, φ_-^A and ψ_+^A are two-component spinors. Let now consider isotropic 2-plane Π :

$$\Pi = \{k | k = \frac{1}{2} \left(\frac{p}{m} - n \right) + \alpha \left(\frac{p}{m} + n \right) + \beta \omega_+; \alpha, \beta \in C\}. \quad (52)$$

The Clifford representation for each element $k \in \Pi$ looks as follows:

$$\hat{k} = \begin{pmatrix} 0 & k_{A\dot{B}} \\ k^{\dot{B}A} & 0 \end{pmatrix},$$

where $k_{A\dot{B}}$ and $k^{\dot{B}A}$ are well known 2×2 matrixes. For each $k \in \Pi$ the following relation is valid:

$$k^{\dot{B}A} \varphi_-^A = \varphi_+^{\dot{B}}. \quad (53)$$

If 2-plane Π belonged to the complexified Minkowski space, one could say that a couple of spinors $Z = (\varphi_+^{\dot{B}}, \varphi_-^A)$ is a twistor which corresponds to the plane Π [3]. The spinor

$$u(p, n) = \begin{pmatrix} \varphi_-^A \\ \varphi_+^{\dot{B}} \end{pmatrix}$$

looks like Z . Its p -dependence is similar to x -dependence of a twistor, so in momentum space one has the twistor-like interpretation of Dirac spinors.

4. Applications and Conclusions

Before searching for $V_\mu, T_{\mu\nu}, A_\mu$ and P , let us consider the peculiar case, when the polarization vectors lie in the 2 - plane of the vectors p_1 and p_2 . Let us denote them N_1 and N_2 . The conditions

$$N_1 \cdot p_1 = N_2 \cdot p_2 = 0, \quad N_1^2 = N_2^2 = -1 \quad (54)$$

determine them up to a sign, for definiteness set

$$N_1 = \frac{1}{\Delta} \left(\frac{p_1 \cdot p_2}{m_1} p_1 - m_1 p_2 \right), \quad N_2 = \frac{1}{\Delta} \left(-m_2 p_1 + \frac{p_1 \cdot p_2}{m_2} p_2 \right), \quad (55)$$

where

$$\Delta = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}. \quad (56)$$

For definiteness, only the case $w = u$, $\bar{w} = \bar{u}$ will be considered here in detail. A state of fermion with the momentum p_i and an arbitrary polarization vector n , in general, can be represented as a superposition

$$u(p_i, n) = \alpha u(p_i, N_i) + \beta u(p_i, -N_i), \quad (57)$$

where the coefficients α and β depend on the polarization n . So, to describe the full set of helicity amplitudes one can restrict the consideration to the following four cases:

$$n_1 = \pm N_1, \quad n_2 = \pm N_2, \quad (58)$$

where N_1 and N_2 are given in formula (55). Let us note that in the case

$$n_1 = N_1, \quad n_2 = N_2, \quad (59)$$

or in the case

$$n_1 = -N_1, \quad n_2 = -N_2 \quad (60)$$

there appears the degenerate case $S = 0$. Further we will see that the things become the most clear just in this degenerate case.

Now the formula for $u(p_1, n_1)\bar{u}(p_2, n_2)$ can be derived in the following way:

$$u(p_1, n_1)\bar{u}(p_2, n_2) = \frac{(\hat{p}_1 + m_1)(1 + \gamma^5 \hat{n}_1)}{2} L(\Lambda)^{-1}, \quad (61)$$

where Λ is the Lorentz transformation which transforms p_1, n_1 into p_2, n_2 . For the cases $n_1 = N_1, n_2 = -N_2$ and $n_1 = -N_1, n_2 = N_2$ it looks as follows:

$$L(\Lambda) = B = \frac{m_2 m_1 + \hat{p}_2 \hat{p}_1}{\sqrt{2(m_2 m_1 + p_2 \cdot p_1)}}. \quad (62)$$

For the cases $n_1 = N_1, n_2 = -N_2$ and $n_1 = -N_1, n_2 = N_2$ one has

$$L(\Lambda) = BR, \quad (63)$$

where

$$R = \frac{e^{i\delta} \hat{\omega}_+ + e^{-i\delta} \hat{\omega}_-}{\sqrt{2}} \hat{N}_1.$$

$e^{i\delta}$ here is an arbitrary phase factor, it appears due to the ambiguity of the required Lorentz transformation and results in the phase ambiguity of the considered spinors. Further we set $e^{i\delta} = 1$.

In order to present the final result in a convenient form the following variables are introduced:

$$\begin{aligned} j_1 &= \frac{\sqrt{p_1 \cdot p_2 + m_1 m_2} + \sqrt{p_1 \cdot p_2 - m_1 m_2}}{\sqrt{2}}, \\ j_2 &= \frac{\sqrt{p_1 \cdot p_2 + m_1 m_2} - \sqrt{p_1 \cdot p_2 - m_1 m_2}}{\sqrt{2}}. \end{aligned} \quad (64)$$

It is useful to introduce a couple of light-like vectors too

$$k_1 = \frac{1}{2\Delta} \left(j_1 p_1 - \frac{m_1}{m_2} j_2 p_2 \right), \quad k_2 = \frac{1}{2\Delta} \left(-\frac{m_2}{m_1} j_2 p_1 + j_1 p_2 \right), \quad k_1 \cdot k_2 = \frac{1}{2}. \quad (65)$$

After some simplifications the formulae one seeks for look as follows:

$$u\bar{u}(\pm, \pm) = \left(j_1 \frac{1 \pm \gamma^5}{2} - j_2 \frac{1 \mp \gamma^5}{2} + m_1 \hat{k}_2 - m_2 \hat{k}_1 \right) \frac{\hat{\omega}^\pm}{\sqrt{2}}, \quad (66)$$

$$u\bar{u}(\pm, \mp) = (j_1 \hat{k}_1 + m_1) \hat{k}_2 \frac{1 \pm \gamma^5}{2} + (j_2 \hat{k}_2 + m_2) \hat{k}_1 \frac{1 \mp \gamma^5}{2}. \quad (67)$$

Left hand sides of these equations are given in a brief form: for instance, the symbol $u\bar{u}(+, -)$ denotes $u(p_1, n_1 = +N_1)\bar{u}(p_2, n_2 = -N_2)$.

Formulae (66) and (67) give the solution of the problem under consideration in the cases $w = u$, $\bar{w} = \bar{u}$; $n_1 = N_1$, $n_2 = -N_2$ and $w = u$, $\bar{w} = \bar{u}$; $n_1 = N_1$, $n_2 = N_2$, respectively. Equations (27) - (33) enable one to derive formulae of type (1) in all other cases.

In conclusion it is important to underline that formulae (66) - (67) which are the main result of the given paper, are quite sufficient for the calculation of the full set of helicity amplitudes. Any Feynman diagram describing multiparticle process has an even number of fermion ends. They should be divided in pairs, then the formulae obtained can be applied to each pair. However, the description of concrete calculations with the help of the mentioned algorithm may be the subject of another paper.

After the work was completed, I have known that formulae similar to (66) - (67) were derived in ref. [9]. I thank A.L. Bondarev for pointing out at this reference.

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Received May, 25, 1995

Р.Н.Рогалев.

Геометрические аспекты спирального формализма.

Оригинал-макет подготовлен с помощью системы $\text{\LaTeX}$.

Редактор Е.Н.Горина.

Технический редактор Н.В.Орлова.

Подписано к печати 25.05.1995. Формат 60 x 84/8. Офсетная печать.
Печ.л. 1,75. Уч.-изд.л. 1,34. Тираж 250. Заказ 389. Индекс 3649.
ЛР №020498 06.04.92.

ГНЦ РФ Институт физики высоких энергий
142284, Протвино Московской обл.

Индекс 3649

ПРЕПРИНТ 95-74, ИФВЭ, 1995