Gravitating Chern-Simons vortices

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Abstract

The construction of self-dual vortex solutions to the Chern-Simons-Higgs model (with a suitable eighth-order potential) coupled to Einstein gravity in (2 + 1) dimensions is reconsidered. We show that the self-duality condition may be derived from the sole assumption $g_{00} = 1$. Next, we derive a family of exact, doubly self-dual vortex solutions, which interpolate between the symmetrical and asymmetrical vacua. The corresponding spacetimes have two regions at spatial infinity. The eighth-order Higgs potential is positive definite, and closed timelike curves are absent, if the gravitational constant is chosen to be negative.

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The well-known Abelian Higgs model [1] can be generalized, when the number of dimensions of Minkowski spacetime is restricted to three, by the addition of a Chern-Simons term, leading to a model which has been shown to admit charged vortex solutions [2]. While these are very complicated, it was shown [3] that if the Maxwell term is omitted from the action\(^1\), and a specific sixth-order potential is chosen, the theory admits self-dual vortex solutions.

The Maxwell-Chern-Simons-Higgs model may again be generalized by coupling it to Einstein gravity. One expects [4] that this gravitating model still admits vortex solutions with a spacetime metric asymptotic to that generated, in (2 + 1)-dimensional general relativity, by a massive spinning particle [5]. It was shown in [6] that the Einstein-Chern-Simons-Higgs (ECSH) model (without the Maxwell term) admits self-dual stationary solutions corresponding to systems of non-interacting vortices, provided a suitable eighth-order potential is chosen. In [7] an Einstein-Maxwell-Chern-Simons-Higgs model with an additional real scalar field was studied, and self-dual vortex solutions were similarly obtained for a suitably chosen potential. When a certain limit in the space of model parameters was taken, this model reduced to the ECSH model, and the multi-vortex solutions of [6] were recovered, with some generalization.

In a recent paper [8], the ECSH model was reinvestigated, and it was shown that, under the same assumptions as in [6], complemented by the assumption of rotational symmetry, the full set of field equations (including the Einstein equations) can be reduced to a set of four first-order differential equations. This does not, however, represent a significant advance over the results of [6] [7], where the solution of the multi-vortex problem was implicitly reduced to that of two (second-order) non-linear Schrödinger equations (see below).

The purpose of this Letter is two-fold. First, the self-dual vortex solutions of [6-8] were obtained under several assumptions, including the condition \(g_{00} = 1\), and the self-duality condition, eq. (11) below. The fact that these \(a \text{ priori} \) unrelated assumptions turn out to be consistent leads us to surmise that one can be deduced from the other. Indeed, we shall show in the following that the sole ansatz of a stationary metric with \(g_{00} = 1\) is enough

\(^1\)Because the Chern-Simons term dominates the Maxwell term at long distances, this may be viewed as an asymptotic approximation.
to lead to the self-dual solutions of [6] [7], which we shall derive in a more transparent, deductive fashion.

Our second motivation is the search for exact vortex solutions. As we have mentioned, the field equations of the ECSH model may be, under the assumption $g_{00} = 1$, partially integrated to a fourth-order differential system, which at first sight can only be solved numerically. However we shall show that a certain “double self-duality” ansatz yields exact “extreme” vortex solutions. A peculiarity of these solutions is their non-Euclidean spatial topology — spatial sections have two disconnected regions at infinity.

The abelian ECSH model is defined by the action

$$I = \int d^3 x \left\{ -\frac{1}{16\pi G} \sqrt{|g|} \left( R - \frac{\mu}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho \right) + \sqrt{|g|} \left( g^{\mu\nu} D_\mu \phi^* D_\nu \phi - V(\phi, \phi^*) \right) \right\},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu \phi = (\partial_\mu + ie A_\mu) \phi$, and $\epsilon^{\mu\nu\rho}$ is the antisymmetric symbol; the Higgs potential $V(\phi, \phi^*)$ shall be specified later on. The resulting Euler-Lagrange equations are

$$R_{\mu\nu} = 8\pi G [D_\mu \phi^* D_\nu \phi + D_\nu \phi^* D_\mu \phi - 2V g_{\mu\nu}],$$

$$\frac{\mu}{2\sqrt{|g|}} \epsilon^{\mu\lambda\nu} F_{\mu\lambda} = ie g^{\nu\rho} (\phi^* D_\rho \phi - D_\rho \phi^* \phi),$$

$$\frac{1}{\sqrt{|g|}} D_\mu (\sqrt{|g|} g^{\mu\nu} D_\nu \phi) = -\frac{\partial V}{\partial \phi^*}.$$

We make the ansatz for stationary solutions

$$ds^2 \equiv N^2 \left( dt + \omega_i dx^i \right)^2 + \gamma_{ij} dx^i dx^j,$$

$$A_\mu dx^\mu \equiv A_0 \left( dt + \omega_i dx^i \right) + \overline{A}_i dx^i,$$

where the fields $N$, $A_0$, $\omega_i$, $\overline{A}_i$ and $\gamma_{ij}$ depend only on the spatial variables $x^i$ ($i = 1, 2$), and the spatial metric $\gamma_{ij}$ will be used to move indices. We shall use in the following the decomposition of the Ricci tensor into 2-scalar, 2-vector and 2-tensor components,

$$R_{00} \equiv N^2 \left[ -N^{-1} N^{jk} + \frac{1}{2} b^2 \right],$$

$$2$$
\[ R^i_0 \equiv \frac{1}{2\sqrt{|\gamma|}} N^{-1} \varepsilon^{ij}(N^2 b)_{ij}, \]  

\[ R^{ij} \equiv -N^{-1} N^{ij} + \frac{1}{2} [R \gamma - b^2] \gamma^{ij}, \]  

where

\[ b \equiv \frac{N}{\sqrt{|\gamma|}} \varepsilon^{ij} \omega_{ji}, \]

is the “twist” field associated with the metric (3), and \( ; j \) is the spatial covariant derivative, as well as the reduced magnetic field

\[ \overline{B} \equiv \frac{N}{\sqrt{|\gamma|}} \varepsilon^{ij} \overline{A}_{ji}. \]

In order for static multi-vortex configurations to extremize the action (1), a balance must be achieved between the various forces acting on the system. In the case \( G = 0 \) — vortices in Minkowski spacetime — the conditions for such a balance are the Bogomol’nyi [9] self-duality conditions. We do not expect this static equilibrium to be altered (at least in lowest order) when the gravitational coupling is switched on, provided the static gravitational force acting on the system vanishes, \( g_{00}(x^i) = \text{constant} \). The inference is that this last condition, which may be written (up to a rescaling of time)

\[ N = 1, \]

would appear to be necessary for (static) multi-vortex solutions to exist. We now show that the condition (7) is also sufficient, and implies the self-duality conditions, if the potential has the appropriate eighth-order form.

The ansatz (7) implies, from the last equation (4) and the Einstein field equations (2), the proportionality relation

\[ D^i \phi^* D^j \phi + D^j \phi^* D^i \phi \propto \gamma^{ij}, \]

where the contravariant components \( D^i \phi \) are related to the reduced covariant derivatives of the Higgs field by \( D^i \phi = \gamma^{ij} \overline{D}_j \phi \). Choosing for convenience conformal spatial coordinates such that

\[ \gamma_{ij} = -e^{2u} \delta_{ij}, \]
we can write the condition (8) as

\[ \mathcal{D}_z \phi \mathcal{D}_z \phi^* = 0 \]  

(10)

(with \( z \equiv x + iy \)), which is solved either by \( \mathcal{D}_z \phi = 0 \) or by \( \mathcal{D}_z \phi^* = 0 \). Changing back to an arbitrary spatial coordinate system, we thus find that the solution of equation (8) is given by the covariant self-duality condition [7]

\[ \mathcal{D}_i \phi = \mp \frac{i}{\sqrt{|\gamma|}} \gamma_{ijk} \mathcal{D}_k \phi. \]  

(11)

Using once this condition, we then find that the current on the right-hand side of the second equation (2) (the field-current relation) for \( \nu = i \) is a gradient, so that this equation can be integrated to

\[ A_0 = \pm \frac{e}{\mu} (\phi^* \phi - \eta^2), \]  

(12)

where \( \eta^2 \) is an arbitrary real constant. The Einstein equation (2) for the mixed component \( R_{0i} \) similarly integrates to

\[ b = -8\pi G \mu (A_0^2 - a^2), \]  

(13)

where \( a^2 \) is a new real constant\(^2\). The second equation (2) for \( \nu = 0 \) then gives the reduced magnetic field in terms of \( A_0 \) and \( b \):

\[ \mathcal{B} = -A_0 \left[ \frac{2e^2}{\mu} \phi^* \phi + b \right]. \]  

(14)

We are now in a position to exhibit the potential for which our ansatz (7) is indeed a solution of the field equations. The consistency condition for this ansatz is a constraint for the component \( R_{00} \) of the Ricci tensor which, on account of equations (12) and (13), yields the potential:

\[ V = \frac{e^4}{\mu^2} \phi^* \phi (\phi^* \phi - \eta^2)^2 - 2\pi G \mu^2 \left[ \frac{e^2}{\mu^2} (\phi^* \phi - \eta^2)^2 - a^2 \right]^2. \]  

(15)

Only two of the field equations (2) remain. The Higgs equation is redundant, by virtue of the Bianchi identities, while the contracted Einstein equation for \( R^{ij} \) yields the spatial two-curvature

\[ R_{ij} = b^2 + 16\pi G \left[ \gamma^{ij} \mathcal{D}_i \phi^* \mathcal{D}_j \phi - 2V \right]. \]  

(16)

\(^2\)This constant, \( -C \) in [7], is absent in [6].
Finally, we briefly summarize the integration of these equations along the lines of [7]. The self-duality conditions (11) can be solved for the $A_i$, leading to the expression of the reduced magnetic field, written using the conformal metric (9),

$$B = \mp \frac{1}{e} e^{-2u} \nabla^2 \ln |\phi|.$$  \hspace{1cm} (17)

We may also choose a time gauge in which $\partial_i \omega_i = 0$ in conformal coordinates, so that $\omega_i$ is the curl of a real potential $\omega$, leading to the twist

$$b = -e^{-2u} \nabla^2 \omega.$$  \hspace{1cm} (18)

Then, the Higgs equation (2) may be used to express the right-hand side of (16) as a linear combination of $\Box (\phi^* \phi)$, $B$ and $b$. These last two fields being covariant laplacians, as well as $R_\gamma = 2e^{-2u} \nabla^2 u$ in conformal coordinates, equation (16) may be integrated to give the metric function

$$\sqrt{|\gamma|} = e^{2u} = |h(z)|^{-2} |\phi|^{16\pi G \eta^2} e^{-8\pi G (|\phi|^2 + 2\mu a^2 \omega)}.$$  \hspace{1cm} (19)

(the arbitrariness of the analytical function $h(z)$ reflects the invariance of the parametrization (9) under conformal transformations). The problem is then reduced to the solution of two coupled non-linear Schrödinger equations for the real functions $|\phi|$ and $\omega$, obtained by identifying (17) to (14) and (18) to (13), respectively.

For solutions to these equations to qualify as vortex solutions, they should obey appropriate boundary conditions. The conditions chosen in [7]

$$b(\infty) = 0, \quad B(\infty) = 0,$$  \hspace{1cm} (20)

ensure the convergence of the integrals giving the net spin and magnetic flux in an asymptotically flat space (and asymptotic flatness itself if $\Box (\phi^* \phi)$ goes to zero at infinity). These two conditions are solved either by

$$|\phi|(\infty) = \eta, \quad A_0(\infty) = 0 \quad \text{with} \quad a = 0,$$  \hspace{1cm} (21)

or by

$$|\phi|(\infty) = 0, \quad A_0(\infty) = \pm a = \mp \frac{e}{\mu} \eta^2.$$  \hspace{1cm} (22)

If the spatial topology is $R^2$, the first set of boundary conditions leads to topological solitons, while the second set leads to non-topological solitons. In
the following, we shall generalize these conditions to allow for the possibility of non-asymptotically flat spatial sections.

To find exact vortex solutions, we assume that the gauge field is also self-dual in the sense of electric-magnetic duality [10], \( E = \varepsilon B \) (with \( \varepsilon^2 = 1 \)). This may be enforced covariantly by assuming that the scalar field \( F_{\mu\nu}F^{\mu\nu} \) vanishes [11].

Taking into account the expression (14) for the reduced magnetic field, this assumption reads, in conformal coordinates,

\[
\frac{1}{2} F_{\mu\nu}F^{\mu\nu} = A_0^2 \left[ \frac{4e^4}{\mu^2} (\phi^* \phi)^2 - e^{-2u} \left( \frac{\nabla A_0}{A_0} \right)^2 \right] = 0. \tag{23}
\]

In the one-vortex case, we chose polar coordinates \( \tau, \theta \) such that

\[
ds^2 = (d\tau + Y(\tau)\,d\theta)^2 - \sigma^2(\tau)\,d\theta^2 - d\tau^2, \quad \phi = R(\tau)e^{i\theta} \tag{24}
\]

(the proper radial distance \( \tau \) is related to the conformal radial distance \( r \) defined from equation (9) by \( d\tau = e^\mu dr \)). The square root of the ansatz (23) then leads to the differential equation

\[
A_0^{-1} \frac{dA_0}{d\tau} = \varepsilon \frac{2e^2}{\mu} \phi^* \phi, \tag{25}
\]

which (taking into account equation (12)) is solved by

\[
R^2 = \frac{\eta^2}{1 + e^{-2e^2\eta^2/|\mu|} (\tau - \tau_0)}, \tag{26}
\]

where \( \tau_0 \) is an integration constant, and we have chosen without loss of generality \( \varepsilon = -\text{sign}(\mu) \). Clearly, the variable \( \tau \) is unbounded; as \( \tau \) varies over the real axis, the Higgs field interpolates monotonously between the symmetric vacuum \( R = 0 \) and the asymmetrical vacuum \( R = \eta \), and may be chosen as new radial coordinate. We then find that, for any value of the parameter \( a \), equation (19), in which we choose \( h(z) = \text{const.} \, z^{1+k} \) (\( k \) constant), is consistent with the two coupled non-linear Schrödinger equations for \( R \) and \( \omega \) mentioned above, which may be solved to yield the explicit metric

\[
ds^2 = (dt + \varepsilon(\sigma - c)\,d\theta)^2 - \sigma^2\,d\theta^2 - \frac{\mu^2}{e^4} \frac{dR^2}{R^2(\eta^2 - R^2)^2}, \tag{27}
\]
(p positive constant), and \(c = -\varepsilon k/8\pi G\mu a^2\). The associated electric potential \(A_0\) is given by equation (12), while the magnetic potential \(A_\theta = \overline{A}_\theta + YA_0\) is determined from equation (11) to be

\[
A_\theta = -\varepsilon cA_0 - \frac{n}{e}.
\]

The metric (27) and the gauge potentials (29) are regular over the range of variation of the Higgs field \(R\).

For a metric of the form (27), \(b = \varepsilon d \ln \sigma / d\tau\), leading to the spatial curvature \(R_\gamma = 2(b^2 + \varepsilon db / d\tau)\). At either endpoint \(R = \eta (\tau = +\infty)\) or \(R = 0 (\tau = -\infty)\), the twist \(b\) goes to a constant, while its first derivative vanishes from equation (25). We then find from equation (4) the asymptotic behaviours of the Ricci tensor

\[
R_{\mu\nu} \simeq \frac{1}{2} b^2(\pm\infty) g_{\mu\nu} \quad (\tau \to \pm\infty),
\]

so that the regular spacetime (27) is, for any value of \(a\), asymptotic to two spacetimes of constant curvature corresponding to negative cosmological constants \(\Lambda_\pm = -l_{\pm}^{-2}\), with \(l_{\pm}^{-1} = b(\pm\infty)/2\) (comparing (30) with the first equation (2), we check that this effective cosmological constant is due to the potential energy \(V(\pm\infty) = -b^2(\pm\infty)/32\pi G\). More precisely, the metric (27) goes over to

\[
ds^2 \simeq (dt + \varepsilon(\sigma - c) d\theta)^2 - \sigma^2 d\theta^2 - \frac{l_\pm^2}{4} \frac{d\sigma^2}{\sigma^2} \quad (\tau \to \pm\infty).
\]

The asymptotic metric (31) is the extreme BTZ solution [12] with negative mass \(M = \varepsilon l_{\pm}^{-1}J = -2c^2/l_{\pm}^2\), viewed in an uniformly rotating frame with angular velocity \(\Omega = \varepsilon l_{\pm}^{-1}\) and with time rescaled by \(t \to (l_{\pm}/2c) t\); the apparent singularity at \(\sigma = 0\) is actually at infinite geodesic distance [13].

The asymptotic behaviours of \(\sigma\) are, for \(G > 0\), \(\sigma(+\infty) = +\infty\) for \(a^2 < 0\), \(\sigma(+\infty) = 0\) for \(a^2 > 0\), and \(\sigma(-\infty) = 0\) for \(a^2 < (e^2/\mu^2)\eta^4\), \(\sigma(-\infty) = +\infty\) for \(a^2 > (e^2/\mu^2)\eta^4\). For \(G < 0\) (the sign of the gravitational constant is not fixed \(a\ priori\) in three dimensions [5]), these asymptotic behaviours are inversed.
with $\sigma(\pm \infty)$ replaced by $1/\sigma(\pm \infty)$. For the particular values $a^2 = 0$, or $a^2 = (e^2/\mu^2)\eta^4$, the metric (27) is asymptotic for $\tau \to +\infty$ (resp. $-\infty$), to the flat metric

$$ds^2_0 = [dt + \varepsilon(\sigma_\infty - c) d\theta]^2 - \sigma_\infty^2 d\theta^2 - d\tau^2$$

(\sigma_\infty constant), which is simply the cylindrical Minkowski metric $ds^2 = dt^2 - \sigma_\infty^2 d\theta^2 - d\tau^2$, viewed in an uniformly rotating frame and time rescaled; the doubly self-dual solution (26)-(29) thus automatically satisfy, for these values of $a^2$, the flat-space boundary conditions (20) at one of its two endpoints. For any value of $a$, our solution interpolates between two inequivalent vacuum configurations, and is thus a topological soliton. However, contrary to the case of vortex solutions with Euclidean spatial topology, the magnetic flux

$$\Phi = A_\theta(+\infty) - A_\theta(-\infty) = \mp e^c |\mu| \eta^2$$

is unquantized for our solution. The reason is that the usual flux quantization follows from the condition that the vector potential is regular at the origin of polar coordinates; this condition does not hold in our cylindrical topology (the apparent flux at $+\infty$, $\Phi_+ \equiv A_\theta(+\infty)$, is from equations (29) and (21) quantized for $a = 0$).

Our solution (26), (12), (29) for the Higgs and gauge fields does not depend on the gravitational constant $G$, and thus survives in the limit $G \to 0$. If we also take in this limit $a^2 \to \infty$ so that $|Ga^2| \equiv 1/4\pi|\mu|l$ stays fixed, we obtain the exact vortex solution (26), (12), (29) sitting on the background constant curvature metric (31), which in the case $a^2$ finite ($l \to \infty$), reduces to the flat cylindrical metric (32). This last solution was previously obtained by Schiff [14] (the squared Higgs field is given in equation (2.20) of [14], and the background metric in equation (2.29)), who pointed out that the Minkowskian flux quantization law is lost. Let us also recall that Comtet and Khare [15] showed that the Maxwell-Chern-Simons-Higgs model with the standard fourth-order potential admits self-dual vortex solutions on a specific background metric with constant curvature, and $g_{00} = 0$ (the constant $k_1$ in equation (11b) of [15] may be taken equal to zero without loss of generality); probably much of the analysis of the present paper could also be carried out starting from the assumption $g_{00} = 0$, instead of (7).

An unpleasant feature of the metric (27) is the occurrence of closed time-like curves (CTCs) for $c \sigma(R) < c^2/2$. All the circles $t = \text{const.}, R = \text{const.}$
are CTCs for $c < 0$, and CLCs (closed timelike lines) for $c = 0$. For $c > 0$, CTCs occur whenever $\sigma(R)$ goes to 0 at one endpoint. So the only CTC-free case is $c > 0$, $G < 0$, $0 \leq a^2 \leq (e^2/\mu^2)\eta^4$; then $\sigma$ goes to $+\infty$ (or a positive constant value) at both endpoints and no CTCs occur provided the constant $p$ in (28) is large enough. Another advantage of the choice $G < 0$ is that the potential (15), which is unbounded from below for $G > 0$ [7], is now positive definite, with the extrema $R = 0$ and $R = \eta$ remaining local minima if $|G|$ is not too large\(^3\).

We have shown that the sole assumption $g_{00} = 1$ leads to self-dual stationary solutions of the Chern-Simons-Higgs model (with a suitable eighth order potential) coupled to Einstein gravity, and derived a set of exact, doubly self-dual solutions. These solutions interpolate between the two vacua (symmetrical and asymmetrical), the corresponding spacetimes being asymptotic to two spacetimes of constant curvature. The next, logical step would be to investigate the existence of self-dual vortex solutions to the Chern-Simons-Higgs model coupled to topologically massive gravity [17], or even to pure Chern-Simons gravity [18].

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\(^3\)Menotti and Seminara have shown [16] that if the weak energy condition is satisfied, there are no CTCs in (2+1) dimensions. The occurrence of CTCs in our solution for $G > 0$ is thus due to the fact that the potential is not positive definite. There is no obvious reason for the existence of CTC-free solutions for $G < 0$, as the weak energy condition for the source $8\pi GT_{\mu\nu}$ is certainly not satisfied in this case ($GT_{00}$ is negative definite).
References