The Vector-Tensor Supermultiplet with Gauged Central Charge

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Abstract

The vector-tensor multiplet is coupled off-shell to an $N = 2$ vector multiplet such that its central charge transformations are realized locally. A gauged central charge is a necessary prerequisite for a coupling to supergravity and the strategy underlying our construction uses the potential for such a coupling as a guiding principle. The results for the action and transformation rules take a nonlinear form and necessarily include a Chern-Simons term. After a duality transformation the action is encoded in a homogeneous holomorphic function consistent with special geometry.

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Off-shell $N = 2$ supermultiplets have at least eight bosonic and eight fermionic components. The smallest of these constitute a variety with two distinct subsets. In the first subset are two multiplets which on shell describe one spin-1, two spin-0, and a doublet of spin-1/2 massless states. These are the vector multiplet [1], which includes a complex scalar, a vector gauge field and a triplet of auxiliary scalars, and the vector-tensor multiplet [2, 3], with a real scalar, a vector gauge field, a tensor gauge field, and a real auxiliary scalar. In the second subset are three multiplets which on shell describe four spin-0 and a doublet of spin-1/2 states. These are the hypermultiplet [4], with four real scalars and four real auxiliary scalars, the tensor multiplet [5], with a triplet of scalars, a tensor gauge field and a complex auxiliary scalar, and the double-tensor multiplet, with two real scalars and two tensor gauge fields. Within a given subset, the alternative field-theoretic formulations are equivalent on-shell in the sense that their linearized field equations lead to the same states. They are, however, inequivalent off-shell. Unlike in $N = 1$ supersymmetry, it is not possible to convert one field representation into another in a way that leaves the full $N = 2$ supersymmetry manifest. This aspect is presumably tied to the presence of an off-shell central charge which is required for all of these multiplets other than the vector and the tensor multiplet. In terms of $N = 1$ supersymmetry, the conversion between different multiplets involves the replacement of a chiral by a tensor (linear) supermultiplet or vice versa.

The off-shell features of these multiplets are crucial for understanding their general couplings. This is directly related to vector and tensor gauge invariances that must be preserved. In the context of local supersymmetry, there is a further restriction since the central charge must also be associated with a local symmetry. These aspects are important when considering string compactifications, where the axion field emerges from a tensor gauge field. In the $N = 2$ effective action this tensor field must be part of an $N = 2$ supermultiplet. For the heterotic string the dilaton-axion complex is contained in a vector-tensor multiplet, which for practical reasons is often converted into a vector multiplet. For type-IIA strings it is contained in a tensor multiplet and for type-IIB in a double-tensor multiplet, either of which can be converted to a hypermultiplet. To understand the systematics of the various couplings of the dilaton-axion complex, it seems advantageous to consider that multiplet which is most closely related to the vertex operators in the underlying string theory in order to fully exploit the restrictions at the level of the effective action, especially when one considers data beyond the spectrum and the Yukawa couplings.

The relevance of these issues is two-fold. Because the dilaton acts as the loop-counting parameter in string perturbation theory, its generic couplings have a direct bearing on the perturbative features of string theory. In this context the $N = 2$ nonrenormalization theorems play an important role. Then there are subtle relations amongst string groundstates, through mirror symmetry and through string-string duality, which are of truly nonperturbative nature. These relations are also described at the level of four-dimensional $N = 2$ supersymmetric effective actions, where the assignment of the dilaton-axion complex to an appropriate supermultiplet forms a crucial ingredient. This is the motivation for the work described in this letter, where we consider the coupling of the vector-tensor multiplet to an $N = 2$ vector multiplet background that is associated with local central charge transformations. Local central charge transformations are necessary in supergravity, as was

\[1\] For a recent discussion of $N = 2$ dilaton assignments, see[6]
first exhibited in [7] for massive hypermultiplets and in [8] for off-shell hypermultiplets in the context of conformal supergravity. We base ourselves on the linearized transformation rules given in [3], from which it is already clear that the gauge field of the central charge transformations cannot be the vector field of the vector-tensor multiplet itself. As a guiding principle for deriving the coupling we require the background fields to couple such that we can consistently assign the multiplet components to a representation of conformal supersymmetry. In this way we hope to incorporate all the crucial features necessary for a coupling to supergravity by means of the superconformal multiplet calculus used in the past.

The vector-tensor multiplet contains a scalar field \( \phi \), a vector gauge field \( V_\mu \), a tensor gauge field \( B_{\mu\nu} \) and a doublet of Majorana spinors \( \lambda_i \). As mentioned above, the multiplet includes a central charge. The vector multiplet background contains a complex scalar \( X \), a spinor doublet \( \Omega_i \), the gauge field \( W_\mu \), now associated with the central charge, and the auxiliary fields \( Y_{ij} \). The supersymmetry transformations, the central charge, and two additional gauge invariances describe an unusual geometry which requires explanation. We begin by discussing the central charge. Infinitesimally, this acts as \( \delta \phi = z \phi \). Successive applications generate a sequence of translations,

\[
\phi \longrightarrow \phi^{(z)} \longrightarrow \phi^{(zz)} \longrightarrow \text{etc,}
\]

and similarly on the remaining fields. As is well known, such a hierarchy arises naturally when starting from a five-dimensional supersymmetric theory with one compactified coordinate, but this interpretation is not essential here. The field \( \phi \) is an independent scalar. In contrast, all other objects in the hierarchy, \( \phi^{(z)}, V^{(z)}_\mu, V^{(z)}_{\mu\nu}, \) etc., are dependent, and are given by particular combinations of the independent fields. This is enforced by a set of constraints, which we exhibit below. The two gauge transformations include a tensor transformation with parameter \( \Lambda_\mu \), under which \( B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu \), and a vector transformation, with parameter \( \theta \), under which \( V_\mu \rightarrow V_\mu + \partial_\mu \theta \). Closure of the algebra requires that \( B_{\mu\nu} \) transform as well under the vector gauge transformation and couple to a Chern-Simons form. The transformation rules are determined by imposing closure of the algebra. Modulo field redefinitions these rules are given by

\[
\begin{align*}
\delta W_\mu &= \partial_\mu z, \\
\delta V_\mu &= \partial_\mu \theta + z V_\mu^{(z)}, \\
\delta B_{\mu\nu} &= \partial_\mu \Lambda_\nu + \theta \partial_\mu V_\nu + z B_{\mu\nu}^{(z)},
\end{align*}
\]

where \( \theta, \Lambda_\mu \) and \( z \) are spacetime-dependent parameters. Note that at the linearized level, the need for the Chern-Simons modification is not apparent since a transformation \( \delta B_{\mu\nu} \propto \partial_\mu V_\nu \) can be regarded as a field-dependent tensor gauge transformation. Imposing closure of the algebra on \( V_\mu \), one readily concludes that \( V_\mu^{(z)} \) is invariant under the \( \theta \) transformation and also that \( B_{\mu\nu}^{(z)} \) takes the following form,

\[
B_{\mu\nu}^{(z)} = \tilde{B}_{\mu\nu}^{(z)} - V_\mu^{(z)} V_\nu^{(z)},
\]

\[\tag{3}\]

\(^2\)We use the chiral notation employed in [8, 9], where, for spinor quantities, upper and lower \( SU(2) \) indices \( i, j, \ldots \) denote chiral components. For the spinors used in this letter, the positive chirality spinors are \( \Omega_i, \lambda_i \) and \( c^i \) and thus satisfy \( \gamma_5 \Omega_i = \Omega_i \), etc. The \( SU(2) \) indices are raised and lowered by complex conjugation. Antisymmetrization is defined with “weight one”, so that, for example, \( \partial_\mu V_\nu = \frac{1}{2} (\partial_\mu V_\nu - \partial_\nu V_\mu) \), and similarly for symmetrization. A dual tensor is defined by \( F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) with \( \varepsilon^{1234} = 1 \), and chiral components are defined by \( F^{\pm}_{\mu\nu} = \frac{1}{2} (F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \).
where $\hat{B}^{(z)}_{\mu\nu}$ is invariant under the $\theta$ transformation. The $z$ and $\theta$ gauge transformations do not commute, but close into a tensor gauge transformation with parameter $\Lambda_\mu \propto z \theta V^{(z)}_{\mu}$. Derivatives covariant with respect to central charge transformations are given by $D_\mu \phi = \partial_\mu \phi - W_\mu \phi^{(z)}$, for instance. The field strengths are

$$
\mathcal{F}_{\mu\nu} = 2\partial_{[\mu} W_{\nu]}
$$

$$
F_{\mu\nu} = 2\partial_{[\mu} V_{\nu]} - 2W_{[\mu} V^{(z)}_{\nu]}
$$

$$
H^\mu = \frac{i}{2} \varepsilon^{\mu\nu\lambda\sigma} \left( \partial_\nu B_{\lambda\sigma} - V_\nu \partial_\lambda V_\sigma - W_\nu \hat{B}^{(z)}_{\lambda\sigma} \right),
$$

(4)

where $\hat{B}^{(z)}_{\mu\nu}$ is defined in (3). These field strengths are invariant under vector and tensor gauge transformations. Under a central charge transformation, $\mathcal{F}_{\mu\nu}$ is invariant and both $F_{\mu\nu}$ and $H^\mu$ are covariant. The field strengths satisfy the following Bianchi identities

$$
D_\mu \tilde{F}^{\mu\nu} = -V^{(z)}_\mu \tilde{F}^{\mu\nu},
$$

$$
D_\mu H^\mu = -\frac{i}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{i}{2} \hat{B}^{(z)}_{\mu\nu} \tilde{F}^{\mu\nu}.
$$

(5)

Notice the appearance of $F_{\mu\nu} \tilde{F}^{\mu\nu}$ in the Bianchi identity for $H^\mu$. This is related to the Chern-Simons form $V_{[\mu} \partial_{\nu} V_{\lambda]}$ appearing in the definition of $H^\mu$ in equation (4), which was also mentioned previously. This coupling is unavoidable if the algebra is to close in the presence of the vector multiplet background, and is responsible for significant nonlinearities in the supersymmetry transformation rules as we will discuss.

The vector-tensor multiplet is an assembly of an $N = 1$ vector multiplet and an $N = 1$ tensor multiplet. Since these multiplets carry different conformal weights, it is impossible to assemble them directly into an $N = 2$ representation of the superconformal algebra. This incompatibility could have been anticipated by observing that kinetic terms for vector and tensor gauge fields are not conformally invariant in the same spacetime dimension. However, the vector multiplet, which provides the gauge field required to gauge the central charge, can simultaneously provide fields to compensate for the difference in conformal weights [8]. Hence this vector multiplet plays a dual role; it provides the gauge field for the central charge and it also enables us to construct transformation rules which are covariant with respect to both scale and chiral transformations. The structure is therefore constrained. In this way we find an extra bonus because the central charge transformations in the algebra become field-dependent so as to become central with respect to the full $N = 2$ superconformal algebra.

If the scalar field of the background vector multiplet is set to a constant and the other components are set to zero, the full supersymmetry is retained, but scale and chiral transformations are broken. In this limit, one obtains a vector-tensor multiplet with modifications which are nonlinear in the vector-tensor fields. One of these modifications is the coupling of the tensor field to the Chern-Simons form. There is a singular limit in which these nonlinear modifications disappear.

The vector multiplet is completely fixed as a superconformal multiplet\(^3\). For instance, $X$ transforms under dilatations with weight $w = 1$ and under chiral $U(1)$ transformations with weight $c = -1$. The vector-tensor multiplet is not so restricted. For instance, by

\(^3\)The Weyl and chiral weights for most multiplets are summarized in the first paper in [9]
multiplying by powers of $|X|$ we can adjust the conformal weight of $\phi$ arbitrarily. We use this freedom to choose that $\phi$ transforms under dilatations with weight $w = 0$ (the chiral weight of $\phi$ must be zero since it is a real field). In a similar manner we adjust the weights of $\lambda_i$ to $w = c = 1/2$. The gauge fields $V_\mu$ and $B_{\mu\nu}$ must have $w = c = 0$; any other assignment would create an obstruction between their corresponding gauge transformations and local scale and chiral transformations. In this letter we omit further details of how we obtained our results. Instead, we simply present the various nonlinear constraints, the supersymmetry transformation rules, and briefly discuss the supersymmetric action and its symmetries. We defer a deeper discussion to a more comprehensive presentation which is forthcoming.

The central charge commutes with all other transformations up to gauge transformations. Therefore by successive application of the central charge one generates an infinite hierarchy of vector-tensor multiplets, as already indicated in (1), whose components have the same weights and have the same transformation rules, modulo gauge transformations. However, the components of these new multiplets are not independent. At the same time one has an infinite set of constraints, required by the closure of the supersymmetry algebra, so that the vector-tensor multiplet has precisely eight bosonic and eight fermionic components. Therefore, with the exception of $\phi^{(z)}$, there are no additional degrees of freedom. The constraints can be concisely summarized by giving the expressions for $V^{(z)}_\mu, B^{(z)}_{\mu\nu}, \lambda^{(z)}_i$ and $\phi^{(zz)}$, from which all other constraints follow by successively applying central charge transformations,

$$V^{(z)}_\mu = \frac{-1}{4\phi|X|^2}H_\mu + \frac{i\phi}{2\partial_\mu \ln \bar{X}} + \frac{i}{2\phi|X|^2} \left\{ (\bar{X}\lambda^i + \phi\dot{\Omega}^i)\gamma_\mu (X\lambda_i + \phi\Omega_i) - \frac{1}{2}\phi^2\dot{\Omega}^i\gamma_\mu \Omega_i \right\},$$

$$B^{(z)}_{\mu\nu} = -V_{[\mu}V^{(z)}_{\nu]} + i\phi\bar{F}_{\mu\nu} - \frac{1}{2}\phi^2F_{\mu\nu} - \varepsilon^{ij}X\lambda_i\sigma_{\mu\nu}\lambda_j - \varepsilon_{ij}\bar{X}\bar{\lambda}^i\sigma_{\mu\nu}\lambda^j,$$

$$\lambda^{(z)}_i = \frac{-1}{4|X|^2} \varepsilon_{ij}\partial_\lambda (2\bar{X}\lambda^j + \phi\dot{\Omega}^j) - \frac{1}{4\phi|X|^2} \varepsilon_{ij}(\partial_\phi)\lambda^j - \frac{i}{8\phi|X|^2} \left\{ \sigma \cdot (F - i\phi\dot{F}) \lambda_i - i\phi\varepsilon_{ik}Y^{kj}\lambda_j \right\}$$

$$-\frac{\phi^{(z)}}{2\phi|X|} (X\lambda_i + 2\phi\Omega_i) - \frac{i}{4|X|^2}\varepsilon_{ij}(\bar{X}\lambda^j + \phi\dot{\Omega}^j),$$

$$\phi^{(zz)} = -\frac{1}{8\phi|X|^2}(D_\mu\phi)^2 - \frac{1}{|X|^2}D^2\phi - \frac{1}{2|X|^2}D_\mu\phi \partial^\mu \ln |X|$$

$$+ \frac{\phi}{8|X|^6} \left\{ (X\partial_\mu\bar{X} - \bar{X}\partial_\mu X)^2 - |X|^2(X\square\bar{X} - \bar{X}\square X) \right\}$$

$$+ \frac{1}{128\phi^3|X|^6} \left( H_\mu + 2i\phi^2|X|^2\partial_\mu \ln \bar{X} \right)^2 + \frac{1}{64|X|^4}\phi^2(F_{\mu\nu} + i\phi\bar{F}_{\mu\nu})^2$$

$$- \frac{(\phi^{(z)})^2}{2\phi} + \frac{\phi}{64|X|^4}Y_{ij}Y^{ij} + \text{fermion terms}. \quad (6)$$

With these constraints the supersymmetry algebra closes upon anticommutation into a spacetime translation, a vector and a tensor gauge transformation, and a central charge transformation. One may notice that the $B^{(z)}_{\mu\nu}$ equation is not invariant under a vector
gauge transformation. This equation can, however, be cast in a $\theta$-invariant form by expressing it in terms of $\hat{B}^{(Z)}_{\mu \nu}$, defined in (3).

The supersymmetry transformation rules for the independent fields, with all of the nonlinear modifications discussed above, are

$$
\delta \phi = \tilde{e}^i \lambda_i + \bar{\epsilon}_i \lambda^i,
\delta V_\mu = i\varepsilon^{ij}\bar{\epsilon} \gamma_\mu \left(2X \lambda_j + \phi \Omega_j\right) - iW_\mu \epsilon^i \lambda_i + h.c.,
\delta B_{\mu \nu} = -\tilde{e}^i \sigma_{\mu \nu} \left(8\phi |X|^2 \lambda_i + 4\phi^2 \bar{X} \Omega_i\right) - i\varepsilon^{ij}\bar{\epsilon} \gamma_\mu V_{[\nu} \left(2X \lambda_j + \phi \Omega_j\right)
+\varepsilon^{ij}\bar{\epsilon} \gamma_\mu \left(4\phi X \lambda_j + \phi^2 \Omega_j\right) - i\epsilon^i \lambda_i W_{\mu \nu} + h.c.,
\delta \lambda_i = \left(\slashed{D}^\phi - i\slashed{V}^{(Z)}\right)\epsilon_i - \frac{i}{2X} \varepsilon_{ij} \sigma \cdot (F - i\phi \mathcal{F}) \epsilon^j + \left(2\varepsilon_{ij} \bar{X} \phi^{(Z)} - \frac{\phi^{(Z)}}{2X} Y_{ij}\right) \epsilon^j
+ \frac{1}{X} \left\{\frac{1}{2\phi} \epsilon^i \left(X \bar{\lambda}_i \lambda_j - \bar{X} \varepsilon_{ik} \bar{\epsilon} \lambda^k \lambda^l\right) - (\bar{\epsilon}^j \Omega_j) \lambda_i - (\tilde{e}^j \lambda_j) \Omega_i\right\}.
$$

(7)

Observe that these transformation rules are not linear in the vector-tensor fields; $\delta B_{\mu \nu}$ is quadratic in these fields and also $\delta \lambda_i$ contains quadratic terms. This nonlinearity is linked to the Chern-Simons modification discussed above. We stress that these nonlinearities are unavoidable if the vector-tensor multiplet is to exist in a superconformal background.

The results presented so far have analogs at all higher levels in the central charge. To elucidate the structure more completely we introduce a notation where, for positive integers $Z$, we have $\mathcal{O}^{(Z)} = (\mathcal{O}^{(1)}, \mathcal{O}^{(2)}, ...,)$, so that

$$
\delta_z \mathcal{O}^{(Z)} = z \mathcal{O}^{(Z+1)}.
$$

(8)

Covariant derivatives are given by $D_\mu \mathcal{O}^{(Z)} = \partial_\mu \mathcal{O}^{(Z)} - W_\mu \mathcal{O}^{(Z+1)}$. The objects $\phi^{(Z)}, V_\mu^{(Z)},$ and $\lambda_i^{(Z)}$ are $\theta$-invariant, but $B_{\mu \nu}^{(Z)}$ are not. The objects

$$
\hat{B}^{(Z)}_{\mu \nu} = B^{(Z)}_{\mu \nu} + \left(V_{[\mu} V_{\nu]}^{(Z)}\right)^{(Z-1)}
$$

(9)

are, however, $\theta$-invariant. Equation (3) is the $Z = 1$ version of this equation. The transformation rules for all $Z \geq 1$ are given by

$$
\delta \phi^{(Z)} = \tilde{e}^i \lambda_i^{(Z)} + \bar{\epsilon}_i \lambda^i,
\delta V_\mu^{(Z)} = i\varepsilon^{ij}\bar{\epsilon} \gamma_\mu \left(2X \lambda_j^{(Z)} + \phi \Omega_j^{(Z)}\right) + i\tilde{e}^i D_\mu \lambda_i^{(Z-1)} + h.c.
\delta \hat{B}^{(Z)}_{\mu \nu} = -\tilde{e}^i \sigma_{\mu \nu} \left(8\phi |X|^2 \lambda_i^{(Z)} + 4\phi^2 \bar{X} \Omega_i^{(Z)}\right) + 2i\varepsilon^{ij}\bar{\epsilon} \gamma_\mu \left(V_{[\nu}^{(Z)} \left(2X \lambda_j^{(Z)} + \phi \Omega_j^{(Z)}\right)\right)^{(Z-1)}
-\varepsilon^{ij}\bar{\epsilon} \gamma_\mu D_{\nu} \left(4\phi X \lambda_j^{(Z)} + \phi^2 \Omega_j^{(Z)}\right)^{(Z-1)} + i\epsilon^i \left(\lambda_i F_{\mu \nu}\right)^{(Z-1)} + h.c.,
\delta \lambda_i^{(Z)} = \left(\slashed{D}^\phi^{(Z)} - i\slashed{V}^{(Z+1)}\right)\epsilon_i - \frac{i}{2X} \varepsilon_{ij} \sigma \cdot (F - i\phi \mathcal{F})^{(Z)} \epsilon^j + \left(2\varepsilon_{ij} \bar{X} \phi^{(Z+1)} - \frac{\phi^{(Z)}}{2X} Y_{ij}\right) \epsilon^j
+ \frac{1}{X} \left\{\frac{1}{2\phi} \epsilon^i \left(X \bar{\lambda}_i \lambda_j^{(Z)} - \bar{X} \varepsilon_{ik} \bar{\epsilon} \lambda^k \lambda^l\right) - (\bar{\epsilon}^j \Omega_j^{(Z)}) \lambda_i - (\tilde{e}^j \lambda_j^{(Z)}) \Omega_i\right\}.
$$

(10)

Aside from completeness, we include these details to indicate a new feature which is present in the general transformation rules. At all levels $Z \geq 1$, a supersymmetry transformation involves objects both at the next higher level, $Z + 1$, and also at the preceding
level, $Z - 1$. This is to be compared with the case of the hypermultiplet, which also involves a central charge hierarchy. In that case supersymmetry transformations link only to objects at higher levels $Z + 1$; they do not involve $Z - 1$. Notice that the transformation rules for $\phi^{(Z)}$ and $\lambda_{i}^{(Z)}$ are simply $z$-transformed versions of the corresponding rules for $\phi$ and $\lambda_{i}$. This reflects the fact that central charge transformations and supersymmetry transformations commute when applied to these objects. In contrast, the transformation rules for $V_{\mu}^{(Z)}$ and $\tilde{B}_{\mu
u}^{(Z)}$ do not share this property with their $Z = 0$ analogs. This reflects the fact that $V_{\mu}$ and $B_{\mu\nu}$ are gauge fields and that central charge transformations and supersymmetry transformations commute into field-dependent vector and tensor gauge transformations when acting on them. One may wonder, since there exists an infinite sequence of multiplets which are joined above and below by supersymmetry, how it can be that a lowest multiplet exists. The answer to this seeming paradox is that, as made clear above, for the lowest lying multiplet, the vector and tensor fields are the gauge fields associated with certain symmetries. The respective gauge transformations then appear in the algebra in place of a linkage to a lower lying multiplet. We stress this point especially; the lowest lying multiplet in the central charge hierarchy is special in this respect.

Using the components of the vector-tensor multiplet $(\phi, V_{\mu}, B_{\mu\nu}, \lambda_{i}, \phi^{(z)})$ and the background vector multiplet $(X, \Omega_{i}, W_{\mu}, Y_{ij})$, we construct a linear multiplet $(L_{ij}, \varphi^{i}, G, E_{\mu})$ by requiring the lowest component $L_{ij}$ to have weights $w = 2$ and $c = 0$ and to transform into a spinor doublet $\varphi_{i}$ according to \[ \delta L_{ij} = \epsilon_{(i} \varphi_{j)} + \epsilon_{ik} \varphi_{jl} \epsilon^{(k} \varphi^{l)}. \] The expression for $L_{ij}$ is an extension of the linearized result presented in [3] and is given by

\[ L_{ij} = \phi \left( X \hat{\lambda}_{i} \lambda_{j} + \hat{X} \varepsilon_{ik} \varepsilon_{jl} \lambda^{k} \lambda^{l} \right) + \frac{1}{3} \phi^{3} Y_{ij}. \] (11)

We should point out that the existence of such a multiplet in the vector multiplet background depends sensitively on the specific nonlinear transformation rules given above. The higher components of the linear multiplet are constructed by successive supersymmetry transformations. We refrain from giving these expressions here, which are complicated but straightforward to compute. From the product of a vector and a linear multiplet one can construct an invariant action as described in [8, 9]. The vector multiplet in this construction must coincide with the vector multiplet that gauges the central charge. In this way we arrive at the Lagrangian

\[
\mathcal{L} = 2|X|^2 \phi (D^\mu \phi)^2 - \frac{2}{3} \phi^3 \left( X \Box X + \hat{X} \Box X \right) \\
- 2|X|^2 \phi (V^{(z)}_{\mu})^2 + \frac{1}{4} \phi \left( F_{\mu\nu} + i \phi \tilde{F}_{\mu\nu} \right)^2 \\
- 8|X|^4 \phi^{(z)} (\phi^{(z)})^2 + \frac{1}{12} \phi^3 Y_{ij} Y^{ij} \\
+ 4|X|^2 \phi \phi^{(z)} W^\mu D_\mu \phi + \frac{1}{3} \phi^3 W^\mu \partial_\nu \mathcal{F}_{\mu\nu} \\
+ \phi W^\mu \left( V^{(z)}_{\nu} (F^{\mu\nu} + i \phi \tilde{F}^{\mu\nu}) - i D_\nu \phi (\tilde{F}^{\mu\nu} + i \phi \mathcal{F}_{\mu\nu}) \right) \\
+ \text{fermion terms}.
\] (12)

Equation (12) describes a supersymmetric action involving the fields of the vector-tensor multiplet which is also invariant under a local central charge transformation. Note that the explicit factors of $W^\mu$ ensure that $\mathcal{L}$ transforms, under the central charge, into a total derivative.
As mentioned above, it is possible to convert this Lagrangian to a (classically equivalent) Lagrangian involving two vector multiplets. Such a duality transformation is performed, in the usual manner, by introducing a Lagrange multiplier field \( a \), which, upon integration, would enforce the Bianchi identity, shown in (5), imposed on the field strength \( H^\mu \),

\[
\mathcal{L}_{\text{multiplier}} = a \left( D_\mu H^\mu + \frac{i}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{i}{2} \tilde{B}^{(z)} \tilde{F}^{\mu\nu} \right).
\]

Including this term, we treat \( H^\mu \) as unconstrained and integrate it out of the action. In equation (12) the \( H^\mu \) dependence is implicit in \( V^\mu(z) \) via the first equation of (6). In terms of \( V^\mu(z) \), the equation of motion for \( H^\mu \) takes on a simple form, given by

\[
V^\mu(z) = \partial_\mu a.
\]

The natural gauge fields in the dual theory are found to be \( W_0^\mu = W_\mu \) and \( W_1^\mu = V_\mu + aW_\mu \), which transform under a combined central charge and gauge transformation as \( \delta W_\mu = \partial_\mu \theta + a \partial_\mu z \) and \( \delta W_1^\mu = \partial_\mu (\theta + az) \). Thus the enigmatic entangling of the gauge and central charge transformations exhibited in (2) satisfyingly disentangles in the dual formulation, leaving us with an abelian gauge structure with field strengths \( \mathcal{F}_{\mu\nu}^I = 2 \partial_\mu W_\nu^I \). The dual action also involves two complex scalars, \( X^0 = X \) and \( X^1 = X(a + i\phi) \). One can verify that \( X^1 \) and \( W_1^\mu \) transform as the scalar and gauge field of a single vector multiplet. This is confirmed by rewriting the bosonic Lagrangian in the dual formulation (integrating out auxiliary fields) in terms of the derivatives of a holomorphic function \( F(X^0, X^1) \),

\[
\mathcal{L} = \frac{i}{2} (\partial_\mu F_I \partial^{\mu} X^I - \partial_\mu X^I \partial^{\mu} F_I) - \frac{i}{8} (F_{IJ} \mathcal{F}_{\mu\nu}^I \mathcal{F}_{\mu\nu}^J - F_{IJ} \mathcal{F}_{\mu\nu}^I \mathcal{F}^{-\mu\nu} \mathcal{F}^{-\mu\nu} J) ,
\]

where a subscript \( I \) denotes differentiation with respect to \( X^I \). This is the generic form for the \( N = 2 \) supersymmetric action of vector multiplets. The function in the case at hand is found to be

\[
F(X^0, X^1) = -\frac{1}{3} \frac{(X^1)^3}{X^0}.
\]

Prior to performing the duality transformation, we could have included \( n - 1 \) additional vector multiplets, labeled by \( I = 2, \ldots, n \). The coupling of these vector multiplets would in principle involve the background vector multiplet as well and would be characterized by another holomorphic function involving \( X^0, X^2, \ldots, X^n \). The dual Lagrangian would be encoded in a function which is the sum of this new function and (16). Hence, in this extended formulation, the appearance of \( X^1 \) would be strongly restricted. In all such cases, the Lagrangian is invariant, up to a total divergence, under

\[
X^1 \to X^1 + bX^0, \quad W_1^\mu \to W_1^\mu + bW_0^\mu ,
\]

where \( b \) is an arbitrary real parameter. This is the generalized Peccei-Quinn symmetry associated with the axion field \( a \).

In order to make contact with \( N = 2 \) heterotic string compactifications, we must couple the above Lagrangian to supergravity. As we have stressed, this coupling should follow straightforwardly within our adopted strategy. Therefore it comes as no surprise that the function (16) is homogeneous of second degree, which is precisely the condition that must be satisfied in order to couple vector multiplets to supergravity. In fact, with the
supergravity couplings included, the theory based on (16) coincides with the dimensional reduction of pure five-dimensional supergravity. Although we find no indication that the coupling of supergravity will lead to surprises, our result is somewhat unexpected from the point of view of string theory. By starting from a vector-tensor multiplet, which is one of the supermultiplets of vertex operators in the compactified heterotic string (cf. the discussion in [3]), one would expect the dilaton field $S = \phi - ia$ to exhibit stringy features. This expectation does not seem fulfilled, however, as the dilaton coupling in our construction is not universal. A vector-tensor multiplet seems unable to couple to vector multiplets other than the one associated with its central charge. Although we can arrange that the dilaton is subject to an $SU(1,1)$ $S$-duality invariance, the special Kähler space does not factorize, so our solution does not meet the conditions of the theorem of [10]. More firm conclusions regarding this issue necessitate further work on the supergravity coupling. The results reported here are a first step in that direction.

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References


