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Casimir energy for a three-piece relativistic string

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ABSTRACT

The Casimir energy for the transverse oscillations of a piecewise uniform closed string is calculated. The string consists of three pieces I, II, III of equal length, endowed with different tensions and mass densities, but adjusted in such a way that the velocity of sound always equals the velocity of light. In this sense the string forms a relativistic mechanical system. In the present paper the string is subjected to the following analysis: the dispersion function is derived and the zero-point energy is regularized using (i) a contour integration technique, being most convenient for the generalization of the theory to the case of finite temperatures, and (ii) the Hurwitz ζ -function technique, being usually the most compact method when the purpose is to calculate the Casimir energy numerically at $T = 0$. The energy, being always non-positive, is shown graphically in some cases as functions of the tension ratios I/II and II/III. The generalization to finite temperature theory is also given.

I. INTRODUCTION

Consider the piecewise uniform string shown in Fig. 1: it consists of three pieces I, II, II, each of equal length $L/3$, L being the total length of the string. We shall consider the Casimir energy of the transverse oscillations of the system, assuming that the material in the string is everywhere relativistic. This means that the velocity of sound v_s is everywhere equal to the velocity of light:

$$v_s = (T_I/\rho_I)^{1/2} = (T_{II}/\rho_{II})^{1/2} = (T_{III}/\rho_{III})^{1/2} = c \quad . \quad (1)$$

Here the T 's are the tensions and the ρ 's are the mass densities for the respective three pieces.

An attractive physical feature of the composite string model is that it can help us to understand the issue of the energy of the vacuum state in two-dimensional quantum field theories in general. There has been some activity on the study of Casimir energy in connection with tachyon problems for strings; see Nesterenko [1], and D'Hoker, Sikivie, and Kanev [2]. The model considered in the present paper was introduced by Brevik and Nielsen some years ago [3]. There, the string was assumed to consist only of two pieces, of lengths L_I and L_{II} . The zero-point energy was regularized by means of an exponential cutoff. Later

on, Li, Shi, and Zhang [4] showed how this system can be regularized in an elegant way by means of the Hurwitz ζ function. Brevik and Elizalde [5] thereafter applied the contour integration method of van Kampen, Nijboer, and Schram [6] (originally introduced in the context of two-plate geometry) in order to regularize the energy. A striking advantage of the contour integration method in comparison with the cutoff method is that there exists a one-to-one correspondence between the degeneracy of eigenfrequencies and the multiplicity of zeros of the dispersion function. The degeneracies are thus built in automatically, in the integral expression for the energy (cf. Eq. (26) below).

References [3-5] were all concerned with the two-piece string. Recently, Brevik and Nielsen [7] considered the analogous problem for a string divided into four equal pieces, of alternating type I and type II material. The zero-point energy was regularized both by use of the contour integration technique, and by the Hurwitz ζ -function technique, and was presented graphically as a function of the tension ratio $x = T_I/T_{II}$. Again, the problem was found to be analytically quite tractable; there was no sign of the troublesome divergences that so often occur in the analogous Casimir calculations for dielectric media having curved boundaries.

The simplicity and analytic tractability of the composite string model make it desirable to investigate the model for various physical configurations. The new string property introduced in the present paper is the appearance of the three pieces I, II, III, i.e., an odd number of pieces in contrast to the even number of pieces considered in the earlier treatments on the composite string. We calculate the dispersion function and eigenvalue spectrum, find the Casimir energy at temperature $T = 0$ using as regularization procedure both the contour integration method and the ζ function method, and show the result graphically for various input values of the tension ratios $x = T_I/T_{II}$ and $y = T_{II}/T_{III}$. Some attention is given also to the generalization of the theory to the case of finite T .

Similarly as in previous works [3-5, 7], it is found that the negativity of the Casimir energy is a characteristic property of the calculated expressions. Thus physically it becomes possible for an initially uniform string to lower its zero-point energy by transforming itself into a nonuniform state. It is not unconceivable that "phase transitions" of this sort played a physical role in the early Universe.

As a general remark, it would be interesting to formulate the theory of our model in a Lagrangian language. That would give us the possibility to study the influence from string thickness, or eventually from external fields, on the Casimir energy. Perhaps it would then

be possible to make the energy positive under certain circumstances.

II. GENERAL FORMALISM

A. Dispersion equation

With reference to Fig. 1, we let σ denote the spatial coordinate along the string. The three junctions are lying at $\sigma = (0, L/3, 2L/3)$. The tension ratios at the junctions are

$$x = \frac{T_I}{T_{II}} , \quad y = \frac{T_{II}}{T_{III}} , \quad \frac{1}{xy} = \frac{T_{III}}{T_I} ; \quad (2)$$

these are shown on the figure. Let $\psi(\sigma, \tau)$ be the transverse displacement of a point σ from its equilibrium position at time τ . The field ψ can be regarded as a scalar, one-dimensional, field. Taking into account the right- and left-moving waves in each of the three regions, we can write

$$\begin{aligned} \psi_I &= \xi_I e^{i\omega(\sigma-\tau)} + \eta_I e^{-i\omega(\sigma+\tau)} , \\ \psi_{II} &= \xi_{II} e^{i\omega(\sigma-\tau)} + \eta_{II} e^{-i\omega(\sigma+\tau)} , \\ \psi_{III} &= \xi_{III} e^{i\omega(\sigma-\tau)} + \eta_{III} e^{-i\omega(\sigma+\tau)} . \end{aligned} \quad (3)$$

There occur here six constants, $\xi_{I,II,III}$ and $\eta_{I,II,III}$, which can be determined by the 3×2 boundary conditions at the three junctions. These conditions are that the transverse displacement ψ , as well as the transverse elastic force $T\partial\psi/\partial\sigma$, must be continuous at each junction. It is convenient to introduce the symbol p , defined as

$$p = \omega L/3 . \quad (4)$$

We obtain the six equations

$$\begin{aligned}
\xi_I e^{ip} + \eta_I e^{-ip} &= \xi_{II} e^{ip} + \eta_{II} e^{-ip} , \\
\xi_I e^{ip} - \eta_I e^{-ip} &= x^{-1} (\xi_{II} e^{ip} - \eta_{II} e^{-ip}) , \\
\xi_{II} e^{2ip} + \eta_{II} e^{-2ip} &= \xi_{III} e^{2ip} + \eta_{III} e^{-2ip} , \\
\xi_{II} e^{2ip} - \eta_{II} e^{-2ip} &= y^{-1} (\xi_{III} e^{2ip} - \eta_{III} e^{-2ip}) , \\
\xi_{III} e^{3ip} + \eta_{III} e^{-3ip} &= \xi_I + \eta_I , \\
\xi_{III} e^{3ip} - \eta_{III} e^{-3ip} &= xy (\xi_I - \eta_I) .
\end{aligned} \tag{5}$$

As in Ref. [7], it is here convenient to introduce 2×2 transfer matrices. First, let us introduce the three column matrices $\underline{\xi}_I$, $\underline{\xi}_{II}$, and $\underline{\xi}_{III}$. Their elements are the respective values of ξ and η ; thus, for example,

$$\underline{\xi}_I = \begin{pmatrix} \xi_I \\ \eta_I \end{pmatrix} . \tag{6}$$

We see that it is possible to write Eqs. (5) in the compact form

$$\underline{\xi}_I = \underline{M}^{(1)} \underline{\xi}_{II} , \quad \underline{\xi}_{II} = \underline{M}^{(2)} \underline{\xi}_{III} , \quad \underline{\xi}_{III} = \underline{M}^{(3)} \underline{\xi}_I , \tag{7}$$

where the three transfer matrices are given by

$$\begin{aligned}
\underline{M}^{(1)} &= \begin{pmatrix} \frac{1+x^{-1}}{2} & , & \frac{1-x^{-1}}{2} e^{-2ip} \\ \frac{1-x^{-1}}{2} e^{2ip} & , & \frac{1+x^{-1}}{2} \end{pmatrix}, \\
\underline{M}^{(2)} &= \begin{pmatrix} \frac{1+y^{-1}}{2} & , & \frac{1-y^{-1}}{2} e^{-4ip} \\ \frac{1-y^{-1}}{2} e^{4ip} & , & \frac{1+y^{-1}}{2} \end{pmatrix}, \\
\underline{M}^{(3)} &= \begin{pmatrix} \frac{1+xy}{2} e^{-3ip} & , & \frac{1-xy}{2} e^{-3ip} \\ \frac{1-xy}{2} e^{3ip} & , & \frac{1+xy}{2} e^{3ip} \end{pmatrix}.
\end{aligned} \tag{8}$$

Under stationary conditions the eigenfrequencies ω are all real; they are determined from the equation

$$\text{Det } (\underline{M} - 1) = 0, \tag{9}$$

with $\underline{M} = \underline{M}^{(1)} \underline{M}^{(2)} \underline{M}^{(3)}$. Some calculation leads to the following expressions for the components of $\underline{M}$:

$$\begin{aligned}
M_{11} &= -\frac{1}{8xy} \left[(1-x)(1+y)(1-xy)e^{ip} + 2x(1-y)^2 e^{-ip} - (1+x)(1+y)(1+xy)e^{3ip} \right], \\
M_{12} &= -\frac{1+y}{8xy} \left[(1-x)(1+xy) e^{ip} + 2x(1-y) e^{-ip} - (1+x)(1-xy) e^{-3ip} \right], \\
M_{22} &= M_{11}^* \quad , \quad M_{21} = M_{12}^* .
\end{aligned} \tag{10}$$

In view of the last line of Eqs. (10), we can write the condition (9) as

$$|M_{11}|^2 - 2\operatorname{Re} M_{11} + 1 - |M_{12}|^2 = 0 . \quad (11)$$

The expression on the left of this equation is essentially the dispersion function, which we shall call $g(\omega)$. For reasons to become clear from the discussion in the next section, we shall define $g(\omega)$ as the expression to the left in (11) multiplied by the factor $F(x,y)/[4(F(x,y) + 1)]$, $F(x,y)$ being the function

$$F(x,y) = \frac{8xy}{(1+x)(1+y)(1+xy) - 8xy} . \quad (12)$$

Inserting the expressions for M_{11} and M_{12} from (10), we get

$$\begin{aligned} g(\omega) &= \frac{F(x,y)}{4[F(x,y)+1]} [|M_{11}|^2 - 2\operatorname{Re} M_{11} + 1 - |M_{12}|^2] = \\ &= \frac{F(x,y) \sin^2 (3p/2) + \sin p \sin 2p}{F(x,y) + 1} . \end{aligned} \quad (13)$$

The dispersion equation $g(\omega) = 0$ determines the eigenvalue spectrum, the tension ratios x and y here serving as input parameters.

We shall give a complete discussion of the eigenvalue spectrum in Sect. III.C, in connection with the ζ -function regularization method. At present, we shall merely analyse a few special cases. This is of interest in connection with the contour integration method to be discussed in Sect. III.B.

B. Uniform string

This case corresponds to $x = y = 1$. According to Eq. (12), $F(1,1) = \infty$ and so the dispersion equation reduces, in view of (11) or (13), to $\sin(3p/2) = \sin(\omega L/2) = 0$. The eigenvalue spectrum thus becomes

$$\omega = \frac{2\pi}{L}n, \quad n = 1, 2, 3, \dots \quad (14)$$

C. The case $x \rightarrow 0$, y arbitrary

Equation (12) yields $F(0, y) = 0$, and so we obtain from (11) or (13) the equation $\sin p \sin 2p = 0$. There are thus two eigenvalue sequences:

$$\omega = \frac{3\pi}{2L} \times \begin{cases} 2n, \\ n, \end{cases} \quad (15)$$

with $n = 1, 2, 3, \dots$, as before.

D. The case $y = 1$, x arbitrary

This case corresponds to a two-piece string; we can simply substitute $L_{\text{II}} + L_{\text{III}} \rightarrow L_{\text{II}} = 2L/3$. From (12) we see that

$$F(x, 1) = \frac{4x}{(1-x)^2} \equiv F(x), \quad (16)$$

where $F(x)$ is the function first introduced in Eq. (26) of [3]. The dispersion function (13) can be expressed as

$$g(\omega) = \frac{F(x)\sin^2(\omega L/2) + \sin(\omega L/3)\sin(2\omega L/3)}{F(x) + 1}, \quad (17)$$

which is in agreement with Eq. (9) of [5] for the case $s \equiv L_{\text{II}}/L_{\text{I}} = 2$.

III. CASIMIR ENERGY

A. Direct calculation in special cases

We define the Casimir energy $E(x,y)$ for given values of x and y as the zero-point energy $E_{I+II+III}$ for the composite string minus the zero-point energy for the uniform string:

$$E(x,y) = E_{I+II+III} - E_{\text{uniform}} . \quad (18)$$

In order to be able to check the general expressions for the energy calculated later on against known cases it is convenient first to reconsider the situations discussed in Sects. II.B and C. The calculation of the Casimir energy for a uniform string is straightforward, and is most easily found by means of the Riemann zeta function:

$$E_{\text{uniform}} = 2 \times \frac{1}{2} \sum_n \omega_n = 2 \frac{\pi}{L} \zeta(-1) = -\frac{\pi}{6L} . \quad (19)$$

Here the prefactor 2 takes into account the degeneracy. In the case of $x \rightarrow 0$, we obtain from (15)

$$E_{I+II+III} = 3 \frac{\pi}{2L} \sum_n n + 3 \frac{\pi}{4L} \sum_n n = 9 \frac{\pi}{4L} \zeta(-1) = -3 \frac{\pi}{16L} , \quad (20)$$

so that in view of (18) and (19)

$$E(0,y) = -\frac{\pi}{48L} , \quad (21)$$

valid for all values of y .

B. Contour integration method

The starting point is the same as in Ref. [5], viz. the so-called argument principle

$$\frac{1}{2\pi i} \oint \omega \frac{d}{d\omega} \ln g(\omega) = \sum \omega_o - \sum \omega_\infty , \quad (22)$$

holding for any meromorphic function $g(\omega)$ whose zeros are ω_o and whose poles are ω_∞ inside the integration contour. We let this contour be a semicircle of large radius R in the right half of the complex ω plane, closed by a straight line from $\omega = iR$ to $\omega = -iR$. We substitute for $g(\omega)$ the dispersion function (13). There are no poles in this case; the last term in (22) is absent. Putting $x = y = 1$, we first have for the uniform string

$$E_{\text{uniform}} = \frac{1}{4\pi i} \oint \omega \frac{d}{d\omega} \ln \sin^2 \left(\frac{3p}{2} \right) d\omega . \quad (23)$$

From (18) we then obtain in general

$$E(x,y) = \frac{1}{4\pi i} \oint \omega \frac{d}{d\omega} \ln \left[\frac{F(x,y) + \frac{\sin p \sin 2p}{\sin^2(3p/2)}}{F(x,y) + 1} \right] d\omega . \quad (24)$$

The contribution from the semicircle vanishes when $R \rightarrow \infty$. We introduce the frequency $\xi = -i\omega$ along the imaginary axis, integrate by parts, and take into account the symmetry about the origin to obtain

$$E(x,y) = \frac{1}{2\pi} \int_0^\infty \ln \left[\frac{F(x,y) + \frac{\sinh(\xi L/3) \sinh(2\xi L/3)}{\sinh^2(\xi L/2)}}{F(x,y) + 1} \right] d\xi . \quad (25)$$

This can be rewritten as

$$E(x,y) = \frac{3}{2\pi L} \int_0^\infty \ln \left[\frac{F(x,y) + \frac{\sinh t \sinh 2t}{\sinh^2(3t/2)}}{F(x,y) + 1} \right] dt . \quad (26)$$

There are three corollaries of interest here. First, for a uniform string we have $E(1,1) = 0$, as it should be according to the construction (18) of the Casimir energy. Secondly, if $x \rightarrow 0$ we get

$$E(0,y) = \frac{3}{2\pi L} \int_0^\infty \ln \frac{\sinh t \sinh 2t}{\sinh^2(3t/2)} dt . \quad (27)$$

Numerical evaluation of this expression turns out to be in agreement with Eq. (21). This corollary was actually the reason behind our particular choice (13) for the dispersion function $g(\omega)$. Thirdly, if $y = 1$ (i.e., a two-piece string, $F(x,1) = F(x)$), we check that Eq. (25) is in agreement with Eq. (14) in [5].

The general expression (26) requires numerical evaluation. Because of the symmetry property

$$E(x,y) = E(y,x) \quad (28)$$

we need only consider the region $x \leq y$ in the xy plane. Note that, for given values of x and y , the product $E(x,y)L$ is a pure number, being the same for all values of L .

Figure 2 illustrates how $E(x,y)L$ varies with x in the interval $x \leq y$ for three different input values of y : $y = (1/2, 1, 2)$. It is seen that the energy always takes its minimum value in the case when $x \rightarrow 0$; then $E(0,y)L = -\pi/48$, in accordance with (21). The cases $y = 1/2$ and $y = 1$ imply a monotonic increase of the energy for increasing values of x , in contradistinction to the case $y = 2$ which shows an energy maximum at $x = \sqrt{1/2}$. The reason for this behaviour is that the function $F(x,2)$ has a maximum located at this value of x .

C. ζ -function method

This regularization method is elegant and usually the simplest method to apply (for a recent general treatise on this kind of regularization, see Ref. [11]). First, let us analyse the eigenvalue spectrum: the expression (13) for $g(\omega)$ can be rewritten as

$$g(\omega) = -2(\cos p - 1) \left[\cos^2 p + \cos p + \frac{F(x,y)/4}{F(x,y) + 1} \right], \quad (29)$$

showing that the spectrum consists of three branches. There is one degenerate branch given by $\cos p = 1$, which means

$$\omega = \frac{6\pi}{L} n, \quad (30)$$

$n = 1, 2, \dots$ Next, there are two nondegenerate simple branches, obtained by solving the quadratic equation for $\cos p$ in (29):

$$\cos p = -\frac{1}{2} \pm \frac{1}{\sqrt{F(x,y)+1}}. \quad (31)$$

The frequencies can be expressed as

$$\omega = \frac{3\pi}{L} \times \begin{cases} \beta_1 + 2n, \\ 2 - \beta_1 + 2n, \end{cases} \quad (32)$$

with $i = 1, 2$; $i = 1$ corresponding to the upper sign in (31) and $i = 2$ corresponding to the lower. Here, β_1 is a number required to lie between $1/2$ (when $F = 0$) and $2/3$ (when $F = \infty$), whereas β_2 lies between $2/3$ ($F = \infty$) and 1 ($F = 0$). The zero-point energy of the composite string becomes

$$\begin{aligned}
E_{I+II+III} = & 2 \times 3 \frac{\pi}{L} \sum_{n=1}^{\infty} n + 3 \frac{\pi}{2L} \sum_{i=1}^2 \sum_{n=0}^{\infty} (2n + \beta_i) + \\
& + 3 \frac{\pi}{2L} \sum_{i=1}^2 \sum_{n=0}^{\infty} (2n + 2 - \beta_i) .
\end{aligned} \tag{33}$$

Here the first term is regularized by means of the Riemann zeta function, in the same way as in subsection A. The two next terms are regularized by means of the Hurwitz ζ function, originally defined as

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s} \quad (0 < a \leq 1, \text{ Re } s > 1) . \tag{34}$$

For practical purposes we need only the following property of the analytically continued function:

$$\zeta(-1, a) = -\frac{1}{2}(a^2 - a + \frac{1}{6}) . \tag{35}$$

Application of (35) in (33), and subtraction of $E_{\text{uniform}} = -\pi/(6L)$, finally lead to the following expression for the Casimir energy:

$$E(x, y) = \frac{\pi}{6L} \left[1 - \frac{9}{2} \sum_{i=1}^2 (1 - \beta_i)^2 \right] . \tag{36}$$

If $x = y = 1$, implying $F = \infty$, we have $\beta_1 = \beta_2 = 2/3$, and so (36) yields $E(1, 1) = 0$, as it should. And if $x \rightarrow 0$, $F = 0$ and $\beta_1 = 1/2$, $\beta_2 = 1$, leading to $E(0, y) = -\pi/(48L)$, in agreement with (21). For general values of x and y , it can be verified numerically that (36) is in agreement with the integral (26). It thus turns out, similarly as in the case of a four-piece

string [7], that the formula obtained from the ζ -function method is the most compact way of expressing the result.

D. Finite temperatures

Let us sketch how the theory of the composite string looks when generalized to finite temperatures T . In this context the integral expression (25) for the Casimir energy becomes most useful. We can obtain the finite temperature energy, called $E(T)$, simply by replacing the integral over imaginary frequencies by a sum:

$$\int_0^\infty d\xi \rightarrow 2\pi k_B T \sum_{n=0}^\infty ' \quad (37)$$

the prime meaning that the $n = 0$ contribution is taken with half weight. We get

$$E(T) = k_B T \sum_{n=0}^\infty ' \ln \left[\frac{F(x,y) + \frac{\sinh(\xi_n L/3) \sinh(2\xi_n L/3)}{\sinh^2(\xi_n L/2)}}{F(x,y) + 1} \right], \quad (38)$$

where $\xi_n = 2\pi n k_B T$ are the Matsubara frequencies.

The expression (38) requires in general numerical evaluation. At low temperatures a large number of Matsubara frequencies is involved, and we do not consider that case further here. The opposite case of high temperatures is however easily tractable. We first note that there are two characteristic frequencies in the system, viz. a thermal frequency $\omega_T = k_B T$, and a geometric frequency which may be defined as $\omega_{\text{geom}} = 2\pi/L$. The limit of "high" temperatures correspond to $\omega_T/\omega_{\text{geom}} \gg 1$, which implies $\xi_n L \gg 4\pi^2 n$. The contribution to (38) from integers $n \geq 1$ is thus very small, and we remain with the $n = 0$ term:

$$E(T) = \frac{k_B T}{2} \ln \frac{F(x,y) + \frac{8}{9}}{F(x,y) + 1} . \quad (39)$$

This is a classical (nonquantal) result. We see that $E(T) < 0$ in the limit of high temperatures. From (38) we check that the Casimir energy is negative at all temperatures.

E. Point mass on a string

By means of a little change in the basic assumptions we can use the above theory to analyse the Casimir energy for a point mass sitting on a uniform string. This is of physical interest since problems of this kind have often been discussed in QCD.

Assume first that $y = 1$, meaning that we return to the case of a two-piece string. Then let us remove the assumption made hitherto of three equal string pieces, and assume instead that $L_I \rightarrow 0$. It corresponds to $s \equiv L_{II}/L_I \rightarrow L/L_I \rightarrow \infty$. Physically this means that the piece I degenerates into a point. (It should be borne in mind, however, that this "point" is of a very special kind, since the velocity of sound in its interior is still required to be equal to c .)

Some information about this case can be obtained by mere inspection, without going into calculations. We must expect that the most important (imaginary) frequencies ξ are of the order of $1/L_I$. This is quite analogous to the well known dominance of frequencies of order $1/a$ in the usual case of two-plate geometry, where a denotes the distance between the plates. Moreover, for dimensional reasons the Casimir energy of a little piece of string embedded in a finite string of different tension has to be inversely proportional to the length L_I of the little string (cf. also the discussion in [3]). Thus $E \propto 1/L_I$, or

$$EL \propto s . \quad (40)$$

Consider now the formalism: we start from Eq. (14) in [5]:

$$E = \frac{1}{2\pi} \int_0^\infty \ln \left| \frac{F(x) + \frac{\sinh(\xi L_I) \sinh(s\xi L_I)}{\sinh^2[(s+1)\xi L_I/2]}}{F(x) + 1} \right| d\xi . \quad (41)$$

As we have just seen, $\xi L_I \sim 1$ for the most dominant frequencies, so that $s\xi L_I$ becomes a large quantity. Two of the hyperbolic functions in (41) can thus be replaced by simple exponential forms, and so we can write the result as

$$\begin{aligned} EL &= \frac{s}{4\pi} \int_0^\infty \ln [1 - q(x)e^{-t}] dt \\ &= \frac{s}{4\pi} \int_{1-q(x)}^1 \frac{\ln u}{1-u} du , \end{aligned} \quad (42)$$

where for brevity $q(x) = [F(x) + 1]^{-1}$. We confirm that (42) is in agreement with the qualitative expression (40).

The integral in (42) is easily calculated numerically, and we show some values of EL/s in Table I, where $F(x)$ serves as input parameter. If in particular $F(x) \rightarrow 0$, i.e., $x \rightarrow 0$, we get analytically from (42)

$$LE = -\frac{\pi}{24} s , \quad s \rightarrow \infty , \quad (43)$$

which is in agreement with the equation [3]

$$LE = -\frac{\pi}{24} \left(s + \frac{1}{s} - 2 \right) , \quad \text{general } s, \quad (44)$$

holding for $x \rightarrow 0$ in the limit $s \rightarrow \infty$.

At finite temperatures, as a check we may calculate

$$E(T) = k_B T \sum_{n=0}^{\infty} \ln [1 - q(x) \exp(-2\xi_n L_I)] \quad , \quad (45)$$

showing that the Casimir energy is still negative.

III. CONCLUSIONS

For given values of the tension ratios $x = T_I/T_{II}$ and $y = T_{II}/T_{III}$, the Casimir energy $E(x,y)$ at $T = 0$ is most conveniently calculated by first evaluating $F(x,y)$ from (12) and thereafter determining the numbers β_i from (31) and (32). Then $E(x,y)$ follows from (36). A noteworthy physical property is that the Casimir energy of the composite string is always negative: the zero-point energy of a string being initially uniform becomes diminished if it transforms itself into a nonuniform state.

Whereas the ζ -function regularization method is usually most convenient for the calculation of $E(x,y)$ at $T = 0$, the integral expression (26), following from the contour integration method, becomes most convenient when generalizing the theory to the case of finite T . Again, the Casimir energy turns out to be negative. If maybe possible that inclusion of some external fields like electromagnetic ones can make this energy to be positive.

It ought to be stressed that the relativistic property of the present string model, as expressed in (1), is essential here. How to regularize the Casimir energy in the case of an ordinary string material, corresponding to a velocity of sound v_s being different from c , is to our knowledge not known.

Finally, we would like to note that the development of a Lagrangian formulation of such a theory may help us to address many related problems, like interaction with external gravitational or electromagnetic fields.

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TABLE I Values of EL/s for single point mass on the string. $F(x) = 4x/(1-x)^2$.

$F(x)$						
	0	0.01	0.1	1	100	∞
EL/s	-0.1309	-0.1264	-0.1053	-0.0463	-0.00079	0

Fig. 1

String of length L divided into three pieces, all of the same length $L/3$.
Tension ratios at the junctions I/II and II/III are x and y .

Fig. 2 Nondimensional Casimir energy versus $x = T_I/T_{II}$ for three different parameter values of $y = T_{II}/T_{III}$. Only intervals $x \leq y$ are shown.