On the Definition of the Partition Function in Quantum Regge Calculus

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Abstract

We argue that the definition of the partition function used recently to demonstrate the failure of Regge calculus is wrong. In fact, in the one-dimensional case, we show that there is a more natural definition, with which one can reproduce the correct results.

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Nowadays, due to the precise measurement of the gauge coupling constants at LEP, it is usual to consider the three interactions which are described quite well by the Standard Model to be unified into a Grand Unified Theory at about the energy scale of $10^{16}$ GeV. On the other hand, it is well known that the gravitational interaction between particles becomes nonnegligible at the Planck scale, which is around $10^{19}$ GeV. These two energy scales are remarkably close to each other considering the ambiguity involved in the derivation of the above values. The most natural interpretation of this fact is that, around such a high energy scale, all the four interactions, including gravity, are unified. There are two possibilities for such unified theories at present. One is string theory and the other is a unification within ordinary field theory including gravity.

The problem we encounter when we try to study quantum gravity within ordinary field theory in four dimensions is that we cannot renormalize it perturbatively. We therefore need to use some kind of nonperturbative approach. In the path integral formalism, the quantization of the geometry is performed by integrating over the metric field. In order to make it well-defined, we need to regularize the theory. There are two types of lattice regularization of quantum gravity. One is dynamical triangulation and the other is Regge calculus. In lattice regularization, the general coordinate invariance is not manifest and whether it is restored in the continuum limit is a crucial problem. Dynamical triangulation is exactly solved in two dimensions [1] and its continuum limit is shown to reproduce Liouville theory [2]. There is also a handwaving argument for the restoration of general coordinate invariance in the continuum limit of dynamical triangulation [3]. From the viewpoint of numerical simulation, however, dynamical triangulation is much harder than ordinary statistical systems since we have to change the lattice structure dynamically, which makes it difficult for us to write a vectorized code. Numerical simulation of four–dimensional dynamical triangulation has been performed now for three years, and although it is promising that a second order phase transition has been found by sweeping the gravitational constant in the Einstein–Hilbert action [4], it is still unclear whether we can take a sensible continuum limit at the critical point.

Obviously we need a larger lattice, and in this respect it is worth while studying the other type of lattice regularization of quantum gravity, namely Regge calculus [5]. In this formalism, the lattice structure is fixed and the fluctuation of geometry is represented with the integration over the link lengths within the constraint of triangle inequalities. The lattice structure can be taken to be a regular lattice, for example. Since this system is nothing but
a kind of ordinary statistical system, it is generally easy to write an efficient code using vectorization. However, the crucial problem of this formalism is that there is no plausible argument for recovering the general coordinate invariance as with dynamical triangulation. Also we have no guiding principle for choosing the measure for the link–length integration. One natural strategy is to study the two–dimensional case and to see if it reproduces the known results of two–dimensional quantum gravity [6, 7, 8, 9].

One of the most fundamental quantities in two–dimensional quantum gravity is the partition function for a fixed total area $A$, which is known to have the following large $A$ behavior.

$$Z(A) \sim A^{\gamma_{str} - 3} e^{\kappa A}$$

(1)

Here, $\kappa$ is subject to renormalization from the cosmological constant in the bare action, whereas the $\gamma_{str}$, which is called string susceptibility, is a universal quantity in the sense that it cannot be changed by changing the bare action. The explicit form for the string susceptibility $\gamma_{str}$ is known as the KPZ formula [10], which has been derived also from Liouville theory [2] in accordance with the known result from matrix models [1]. A fundamental test of Regge calculus is, therefore, to see if we can reproduce the string susceptibility in two–dimensional quantum Regge calculus.

Such an attempt was made by Gross and Hamber [6] a few years ago and by two other groups [7] more recently, from which the conclusion seems to be that Regge calculus fails to reproduce the desired string susceptibility. In Ref. [9], however, we pointed out that the definition of the partition function in Regge calculus is subtle and that the one they used might be wrong. In this paper, we study the one–dimensional case in order to make our arguments concrete. We first show that by using their definition of the partition function the measure is almost uniquely determined by the requirement that it should reproduce the correct continuum result. We then show that, with this special choice of measure, one cannot reproduce the correct Green’s functions. On the other hand, we show that there is a class of measure which reproduces the correct Green’s functions. We also give a natural definition of the partition function with which one can reproduce the correct result for the above class of measure.

Let us first consider one–dimensional quantum gravity in the continuum theory. We consider a line segment parametrized with the parameter $\tau$. Let the ends correspond to $\tau = 0$ and $\tau = 1$ respectively. On the line segment, we put $D$ copies of the matter field
$X^\mu(\tau) \ (\mu = 1, 2, \cdots, D)$ and the metric $g(\tau)$. We consider the action
\begin{equation}
S = \int d\tau \sqrt{g(\tau)} \left( \frac{1}{2} g^{-1}(\tau) \frac{\partial X^\mu(\tau)}{\partial \tau} \frac{\partial X^\mu(\tau)}{\partial \tau} + \frac{1}{2} M^2 X^\mu(\tau) X^\mu(\tau) \right),
\end{equation}
which is invariant under the reparametrization $\tau' = f(\tau)$. In the following we set $M^2 = 0$ for simplicity.

The partition function can be defined as
\begin{equation}
Z(L, X^\mu, Y^\mu) = \int \mathcal{D}g \mathcal{D}X^\mu e^{-S[g, X]} \delta \left( \int_0^1 d\tau \sqrt{g(\tau)} - L \right),
\end{equation}
where the path integral is performed with the constraints $X^\mu(0) = X^\mu$ and $X^\mu(1) = Y^\mu$. The measure for the path integral is defined through the norms
\begin{align}
||\delta g||^2 &= \int d\tau \sqrt{g(\tau)} g^{-2}(\tau)(\delta g(\tau))^2 \\
||\delta X||^2 &= \int d\tau \sqrt{g(\tau)} (\delta X(\tau))^2,
\end{align}
so that it is reparametrization invariant. The partition function can be expressed as
\begin{equation}
Z(L, X^\mu, Y^\mu) = \frac{1}{(\sqrt{2\pi L})^D} \exp \left\{ -\frac{(X^\mu - Y^\mu)^2}{2L} \right\}.
\end{equation}

Let us consider the corresponding quantity in Regge calculus. We divide a line segment into $n$ pieces. The metric degrees of freedom are represented by the link lengths $l_i \ (i = 1, 2, \cdots, n)$ and the matter field $X^\mu_i$ is sited on each node $(i = 0, 1, \cdots, n)$. Let us consider the following quantity.
\begin{align}
Z_{\text{RC}}[n, L, X^\mu, Y^\mu] &= \int \prod_{i=1}^n dl_i \rho_n(l_i) \prod_{i=1}^{n-1} dX^\mu_i \exp \left\{ -\sum_{i=1}^n \frac{(X^\mu_i - X^\mu_{i-1})^2}{2l_i} \right\} \delta(\sum_{i=1}^n l_i - L) \\
&= \prod_{i=1}^n \{dl_i \rho_n(l_i)(2\pi l_i)^{D/2}\} \delta(\sum_{i=1}^n l_i - L) \frac{1}{(\sqrt{2\pi L})^D} \exp \left\{ -\frac{(X^\mu - Y^\mu)^2}{2L} \right\},
\end{align}
where $X^\mu_0 = X^\mu$ and $X^\mu_n = Y^\mu$. One definition of the partition function in terms of the above quantity is
\begin{equation}
Z[L, X^\mu, Y^\mu] = \lim_{n \to \infty} Z_{\text{RC}}[n, L, X^\mu, Y^\mu] \cdot f(n) e^{-g(n)L},
\end{equation}
where $f(n)$ represents the overall renormalization and $g(n)$ represents the renormalization of the cosmological term. This is the definition of the partition function that corresponds to
the one adopted in Refs. [6, 7]. In order that the partition function thus defined may agree with the continuum result (6), the condition

\[
\lim_{n \to \infty} \int \prod_{i=1}^{n} dl_i \varphi_n(l_i) \delta \left( \sum_{i=1}^{n} l_i - L \right) \cdot f(n)^{-1} e^{g(n)L} = 1
\]  

(9)

should be satisfied, where \( \varphi_n(l) = \rho_n(l)(2\pi l)^{D/2} \). Using a Laplace transformation one finds that \( \varphi_n(l) \) should have the following asymptotic behavior for large \( n \).

\[
\varphi_n(l) \sim f(n)^{1/n} \frac{1}{\Gamma \left( \frac{1}{n} \right)} \frac{1}{l^{1-\frac{1}{n}}} e^{-g(n)'},
\]  

(10)

where \( f(n) \) and \( g(n) \) can be taken arbitrarily. We can take, for example,

\[
\varphi_n(l) = \frac{1}{\Gamma \left( \frac{1}{n} \right)} \frac{1}{l^{1-\frac{1}{n}}} e^{-\lambda l}.
\]  

(11)

As is seen here, the large \( n \) limit is, essentially, unnecessary in reproducing the continuum result.

Let us see then if we can reproduce the Green’s function which is defined as

\[
G^{(1)}(L, L', X^\mu, Y^\nu, Z^\nu) = \int \mathcal{D}g \mathcal{D}X^\mu e^{-S[g, X]} \delta \left( \int_0^1 d\tau \sqrt{g(\tau) - L} \right).
\]  

(12)

The path integral is performed with the constraints \( X^\mu(0) = X^\mu \) and \( X^\mu(1) = Y^\mu \) and \( X^\mu(\xi) = Z^\mu \), where \( \int_0^1 \sqrt{g(\tau) d\tau} = L' \). The continuum result is given by

\[
G^{(1)}(L, L', X^\mu, Y^\nu, Z^\nu) = Z(L', X^\mu, Z^\nu) \cdot Z(L - L', Z^\nu, Y^\nu)
\]

\[
= \frac{1}{(\sqrt{2\pi L'})^D} \exp \left\{ -\frac{(X^\mu - Z^\nu)^2}{2L'} \right\} \frac{1}{(\sqrt{2\pi (L - L')})^D} \exp \left\{ -\frac{(Z^\mu - Y^\nu)^2}{2(L - L')} \right\}.
\]  

(13)

One can check explicitly that by integrating over \( Z^\mu \) and \( L' \), one gets \( Z(L, X^\mu, Y^\nu) \).

Let us consider the following quantity in Regge calculus.

\[
G^{(1)}_{\text{RC}}[n, L, L', X^\mu, Y^\nu, Z^\nu]
\]

\[
= \int \prod_{i=1}^{n} dl_i \varrho_n(l_i) \sum_{n' = 1}^{n-1} \int \prod_{i=1}^{n'} dX_i^\mu \prod_{i=n'+1}^{n-1} dX_i^\mu \exp \left\{ -\sum_{i=1}^{n'} \frac{(X_i^\mu - X_{i-1}^\mu)^2}{2l_i} \right\} \cdot \delta \left( \sum_{i=1}^{n'} l_i - L' \right) \delta \left( \sum_{i=n'+1}^{n} l_i - (L - L') \right)
\]

\[
= \int \prod_{i=1}^{n} \left\{ dl_i \varrho_n(l_i)(2\pi l_i)^{D/2} \right\} \sum_{n' = 1}^{n-1} \delta \left( \sum_{i=1}^{n'} l_i - L' \right) \delta \left( \sum_{i=n'+1}^{n} l_i - (L - L') \right) \frac{1}{(\sqrt{2\pi L'})^D} \exp \left\{ -\frac{(X^\mu - Z^\nu)^2}{2L'} \right\} \frac{1}{(\sqrt{2\pi (L - L'))}^D} \exp \left\{ -\frac{(Z^\mu - Y^\nu)^2}{2(L - L')} \right\},
\]  

(14)
where $X_0^\mu = X^\mu$, $X^\mu = Y^\mu$, and $X_n^\mu = Z^\mu$. Note that, integrating over $Z^\mu$ and $L'$, one gets $(n-1)Z_{RC}(n, L, X^\mu, Y^\mu)$. Let us concentrate on the $L'$ dependence of $G_{RC}^{(1)}[n, L, L', X^\mu, Y^\mu, Z^\mu]$, which is not subject to the overall renormalization or the renormalization of the cosmological constant. In order to get the correct $L'$ dependence of the continuum result (13),

$$
\int \prod_{i=1}^{n} \{dl_i \varphi_n(l_i)\} \frac{1}{\Gamma(Nn') \Gamma(N(n-n'))} L'^{Nn'} (L-L')^{N(n-n')} \sum_{n'=1}^{n-1} \delta \left( \sum_{i=1}^{n'} l_i - L' \right) \delta \left( \sum_{i=n'+1}^{n} l_i - (L-L') \right)
$$

(15)

should be independent of $L'$. Using the $\varphi_n(l)$ shown in eq. (10), the expression (15) reduces to

$$
f(n) e^{-g(n)L} \frac{1}{L'(L-L')} \sum_{n'=1}^{n-1} \frac{1}{\Gamma(Nn') \Gamma(N(n-n'))} L'^{Nn'} (L-L')^{1-x}
$$

(16)

For sufficiently large $n$ the summation over $n'$ can be replaced with the integral

$$
\sim \frac{n f(n) e^{-g(n)L}}{L'(L-L')} \int_{0}^{1} dx \frac{1}{\Gamma(x) \Gamma(1-x)} L'^{x} (L-L')^{1-x}
$$

(17)

which means that we cannot reproduce the correct $L'$ dependence for the Green's function (13) so long as we stick to reproducing the partition function using the definition (8).

Let us instead consider a class of measure which can be parametrized as

$$
\varphi_n(l) = \frac{\lambda^N}{\Gamma(N)} t^{N-1} e^{-\lambda t},
$$

(18)

where $N$ and $\lambda$ generally depend on $n$. When we take $N = 1/n$, it reduces to the one considered above. Eq. (15) then becomes

$$
\lambda^{Nn} e^{-\lambda L} \frac{1}{L'(L-L')} \sum_{n'=1}^{n-1} \frac{1}{\Gamma(Nn') \Gamma(N(n-n'))} L'^{Nn'} (L-L')^{N(n-n')}.
$$

(19)

When $Nn \gg 1$, using Stirling's formula, the summation in the above expression is given by

$$
\frac{N_n}{2\pi} e^{-N_n \ln{N_n} + N_n} \sum_{n'=1}^{n-1} \sqrt{x(1-x)} e^{-N_n f(x)},
$$

(20)

where $x = n'/n$ and

$$
f(x) = x \ln \frac{x}{L'} + (1-x) \ln \frac{(1-x)}{(L-L')}.
$$

(21)

When $\frac{1}{\sqrt{N_n}} \gg \frac{1}{n}$, the summation can be evaluated by integration in the large $n$ limit.

$$
\sum_{n'=1}^{n-1} \rightarrow n \int_{0}^{1} dx
$$

(22)
Using the saddle–point method, the expression (19) can be evaluated and yields
\[
\frac{\lambda^{N_n} e^{-\lambda L}}{L'(L - L')} \cdot \frac{N_n e^{-N_n \ln N_n + N_n N_n}}{2\pi} \cdot \sqrt{\frac{L'}{L}} \left(1 - \frac{L'}{L}\right) L^{N_n} \sqrt{\frac{2\pi}{N_n}} \frac{L'}{L} \left(1 - \frac{L'}{L}\right)
\]
\[
\simeq \frac{\lambda^{N_n} e^{-\lambda L}}{\Gamma(N_n)} e^{-\lambda L} L^{N_n - 1} \frac{n}{L}.
\] (23)

Thus the \(L'\) dependence disappears in the last expression, which means that the correct Green’s function can be reproduced as long as
\[
\frac{1}{n} \ll N \ll n
\] (24)
is satisfied.

Let us then reconsider the partition function for this class of measure (18) with the above condition (24). Eq.(7) can be given by
\[
Z_{RC}[n, L, X^{\mu}, Y^{\mu}] = \frac{\lambda^{N_n}}{\Gamma(N_n)} L^{N_n - 1} e^{-\lambda L} \cdot Z(L, X^{\mu}, Y^{\mu}).
\] (25)

If we take the \(\lambda \to \infty\) limit so that \(N_n/\lambda\) is kept to a constant, say to \(\bar{L}\), the prefactor of \(Z(L, X^{\mu}, Y^{\mu})\) in the above expression tends to \(\delta(\bar{L} - L)\). Therefore one finds that
\[
Z(\bar{L}, X^{\mu}, Y^{\mu}) = \lim_{n \to \infty} \int_{0}^{\infty} dL Z_{RC}(n, L, X^{\mu}, Y^{\mu}).
\] (26)

Similarly for the Green’s function, we have
\[
G^{(1)}(\bar{L}, L', X^{\mu}, Y^{\mu}, Z^{\mu}) = \lim_{n \to \infty} \frac{\bar{L}}{n} \int_{0}^{\infty} dL G^{(1)}_{RC}[n, L, L', X^{\mu}, Y^{\mu}, Z^{\mu}].
\] (27)

One can easily generalize these results to \(n\)–point Green’s functions.

Thus we have found a class of measure which reproduces not only the Green’s functions but also the partition function by employing a definition which is different from the one used in previous works. The problem of the definition (8) is that the \(L\) dependence is extracted for a fixed \(n\), which means that probing the \(L\) dependence corresponds to looking at configurations with different average link length. On the other hand, the point of the definition (26) is that the actual total length of the line segment is not fixed externally but that we are probing the \(\bar{L}\) dependence, which corresponds to looking at configurations with different \(n\) with equal average link length.

To summarize, we find that, in the one–dimensional case, employing the definition of the partition function used recently to claim that Regge calculus fails to reproduce the
KPZ formula, we cannot reproduce both the partition function and the Green’s functions simultaneously. By considering a class of measure parametrized as in (18) and the condition (24), that has to be satisfied in order to reproduce the Green’s functions correctly, we find that there is a definition of the partition function with which one can reproduce the correct result. We argue that this definition of the partition function is more natural than the one used in the previous studies. Thus the observations made in Refs. [7] are merely due to the wrong definition of the partition function and they do not mean immediately that Regge calculus is wrong. Also the fact that there is a successful measure at least for one–dimensional quantum Regge calculus is quite encouraging. Together with the observation made in Ref. [9] that the loop length distribution for the “baby loops” has been reproduced, we think that Regge calculus is still worth studying by investigating other measures, for example, before giving it up as a failure at this stage.

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References


