Canonical quantization of the relativistic particle in static spacetimes

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Abstract

We perform the canonical quantization of a relativistic spinless particle moving in a curved and static spacetime. We show that the classical theory already describes at the same time both particle and antiparticle. The analyses involves time-depending constraints and we are able to construct the two-particle Hilbert space. The requirement of a static spacetime is necessary in order to have a well defined Schrödinger equation and to avoid problems with vacuum instabilities. The severe ordering ambiguities we found are in essence the same ones of the well known non-relativistic case.

PACS: 0350, 0365, 0420
Many works have been devoted in the last 20 years to the study of the quantization of relativistic particles (See, for instance, ref. [1] and references therein). The canonical quantization of a relativistic spinless particle in the presence of background fields is a problem of especial interest. For the case of electromagnetic background fields, canonical quantization was performed for a large class of fields, mainly for that ones where no pair creation occurs, see [4] for further references. As to gravitational field case, in spite of the BRST quantization was done some time ago [2], the canonical quantization is still lacking. The great difficulties are the severe ordering ambiguities inherent to the classical–quantum transition and the presence of time-depending constraints. The quantization by Hamiltonian reduction of a relativistic particle moving on some group manifolds was recently considered in [3].

The purpose of this work is to present the canonical quantization of a spinless relativistic particle moving in a static (pseudo)riemannian manifold of dimension $D$. The canonical analysis involves time-depending constraints and as we will see below, the classical theory already describes both particle and antiparticles at the same time, in spite of the original lagrangian is a single particle one. We construct the two-particle Hilbert space $\mathcal{H}$, which mimics some properties of its Minkowskian counterpart [5]. The severe ordering ambiguities that we found are in essence the same ones of the well known non-relativistic case, and we will consider the most general hermitean and invariant Hamiltonian operator. We start the analysis with the lagrangian formulation of a spinless relativistic particle moving in a static manifold, and in order to perform the canonical quantization we go to hamiltonian formulation by following standard steps of the theory of constrained systems [1,6].

First, we shall explain precisely what means a static manifold. We call a riemannian manifold static if [7]: (a) There is a timelike Killing vector field, and (b) There is a family of spacelike surfaces orthogonal to the Killing vector everywhere. These requirements are equivalent to, in an appropriate coordinate system where $x^0$ is timelike, the following restrictions on the metric $g_{\mu\nu}$ of the manifold

(a) $g_{\mu\nu}(x)$ is independent of $x^0$, 

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(b) $g_{0j}(x) = 0$.  \hspace{1cm} (1)

It is assumed hereafter that Greek indices run over $(0, D-1)$, and Roman ones over $(1, D-1)$. We adopt the conventions of ref. [1], and in particular the metric has signature $(1, -1, \ldots, -1)$. We assume also that $g = |\det\{g_{\mu\nu}\}|$. It may seem that the conditions (1) are too restrictive, and we remind that physically relevant examples as the exterior regions of Schwarzschild and Reissner-Nordström solutions and the Rindler metric obey (1). These restrictions are used also in many of the path-integral approaches to the problem, mainly when initial value problems are treated [8]. When the space-time obeys (1), the quantity $\sqrt{g_{00}(x)} = \left(\sqrt{g^{00}(x)}\right)^{-1}$ is called lapse function, because it measures the distance between the spacelike surfaces $x^0$ and $x^0 + dx^0$ constants. We will discuss latter the necessity and the actual role of such restrictions.

We start with the action of a relativistic spinless particle in a static manifold,

$$S = -m \int ds = -m \int L d\tau,$$

$$L = -m \sqrt{g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta}, \quad \dot{x}^\alpha = \frac{dx^\alpha}{d\tau}. \hspace{1cm} (2)$$

The action (2) is invariant under the reparameterizations $\tau \rightarrow f(\tau)$ with $\dot{f} > 0$. Due to this the lagrangian $L$ is singular, and we get primary constraints when going to the hamiltonian formalism. To see it, let us introduce the canonical momenta

$$\pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}, \hspace{1cm} (3)$$

and one can easily check that the constraint $g^{\mu\nu} \pi_\mu \pi_\nu = m^2$ holds, which we write for convenience in the equivalent form

$$\Phi^{(1)} = \sqrt{g_{00}} \sqrt{m^2 - g^{ij} \pi_i \pi_j} - |\pi_0| = 0. \hspace{1cm} (4)$$

The situation here is similar to the flat space case [1]. From (3) we can express only the velocities $\dot{x}^i$ and the sign of $\dot{x}^0$ by means of the variables $\pi_i$, $\lambda = |\dot{x}^0|$, and $\xi = -\text{sign} \pi_0$,

$$\dot{x}_i = -\frac{\pi_i \sqrt{g_{00} \lambda}}{\sqrt{m^2 - g^{ij} \pi_i \pi_j}}, \quad \text{sign } \dot{x}^0 = \xi. \hspace{1cm} (5)$$
The modulus of $\dot{x}^0$, $\lambda$, can not be expressed by means of (3). The hamiltonian $H^{(1)}$ can now be construct calculating $\pi_{\mu} \dot{x}^\mu - L$, and we obtain

$$H^{(1)} = \lambda \Phi^{(1)}. \quad (6)$$

One can follow Dirac procedure and verify that no more constraints arise and that $\lambda$ remains undetermined. The model involves only one first class constraint, and to continue the analysis one needs to choose a gauge fixing condition [1,6]. We use

$$\Phi^G = x^0 - \xi \tau. \quad (7)$$

From the condition of conservation of this gauge choice in $\tau$ one can determine $\lambda$. To avoid $\tau$-depending constraints, we make the canonical transformation that leads $x^{0'} = x^0 - \xi \tau$ and leaves all the other canonical variables unchanged. In the new variables the gauge fixing is given by $\Phi^G = x^{0'} = 0$. One can check that such canonical transformation is defined by the generating function $W = \pi'_{\mu} x^\mu + \tau |\pi_0|$, and that the hamiltonian transforms as

$$H^{(1)'} = H^{(1)} + \frac{\partial W}{\partial \tau} = \sqrt{g_{00}} \sqrt{m^2 - g_{ij} \pi_i \pi_j } + (\lambda - 1) \Phi^{(1)}. \quad (8)$$

The constraints $\Phi = (\Phi^G, \Phi^{(1)})$ form a set of second class ones and are of special form [1], and we can use them to eliminate the variables $x^0$ and $|\pi_0|$. We can check also that the Dirac brackets of the physical variables $x^i, \pi_i, \dot{\xi}$ with respect to the constraints $\Phi$ reduce to the ordinary Poisson ones. The restriction of (8) on the constraint surface gives the physical hamiltonian $H$, which describe the dynamics of the physical variables,

$$H = \sqrt{g_{00}} \sqrt{m^2 - g_{ij} \pi_i \pi_j },$$

$$\dot{x}^i = \{ x^i, H \}, \quad \dot{\pi}_i = \{ \pi_i, H \}, \quad \dot{\xi} = 0. \quad (9)$$

As in the flat space case, the variable $\xi$, that assumes the values $\pm 1$, is a constant of motion. We can interpret the variable $\xi$ by introducing an external electromagnetic field.
The situation is analogous to the flat space case [1], and by introducing a magnetic field and comparing the trajectories one concludes that $\xi = 1$ and $\xi = -1$ correspond respectively to the trajectories of particles and antiparticles. The canonical quantization will confirm such conclusion.

Now we can proceed with the quantization of the system described by (9). The only non vanishing commutator for the Schrödinger operators $\hat{x}^i, \hat{\pi}_i$, and $\hat{\xi}$ is

$$\left[ \hat{x}^j, \hat{\pi}_k \right] = i\delta^j_k.$$  \hspace{1cm} (10)

By analogy with the classical theory, let us assume that the operator $\hat{\xi}$ has eigenvalues $\xi = \pm 1$. We introduce the Hilbert space $\mathcal{H}$, whose elements $\Psi$ are complex two-components columns

$$\Psi(x^\mu) = \begin{pmatrix} \psi_+(x^\mu) \\ \psi_-(x^\mu) \end{pmatrix},$$  \hspace{1cm} (11)

and the invariant inner product is given by

$$\langle \Psi^1, \Psi^2 \rangle = \int d^D x \sqrt{g} \Psi^{1\dagger} \Psi^2.$$  \hspace{1cm} (12)

We choose for our Schrödinger operators the following representation

$$\hat{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{x}^k = x^k \mathbf{I},$$

$$\hat{\pi}_k = -ig^{-\frac{1}{4}} \partial_k g^{\frac{1}{4}} \mathbf{I},$$  \hspace{1cm} (13)

where $\mathbf{I}$ is the unit $2 \times 2$ matrix. All these operators are hermitean with respect to the inner product (12) (We assume for simplicity that $\psi_+$ and $\psi_-$ have compact support).

The dynamics of the physical states $\Psi_{\text{ph}} \in \mathcal{H}$ are described by the Schrödinger equation

$$i \frac{\partial}{\partial \tau} \Psi_{\text{ph}} = \hat{H} \Psi_{\text{ph}},$$  \hspace{1cm} (14)

where $\hat{H}$ is the quantum counterpart of the hamiltonian $H$ in (9). It is clear that the determination of $\hat{H}$ is plagued with ordering ambiguities. It is more convenient to our
purposes to introduce the physical time $x^0$ in (14). We can do it by exploring (7), and we have

$$i \frac{\partial}{\partial x^0} \Psi_{ph} = \xi \hat{H} \Psi_{ph}.$$  

(15)

These last equations deserve some comments. It is here that for the first time the necessity of the restriction to a static spacetime arises. It is the existence of a timelike Killing vector that makes possible to write (14) as (15). Also, in order to have a well-defined Cauchy problem to the equation (15) we need to have spacelike surfaces orthogonal to the timelike Killing vector. In fact we need also that this surfaces be Cauchy surfaces, or what is equivalent that the space-time be globally hyperbolic [7].

For our purposes, it is more convenient to consider now the second order equation obtained by applying $ig^{00} \frac{\partial}{\partial x^0}$ to (15),

$$K \Psi_{ph} = (g^{00} \partial^2_0 + \tilde{H}^2) \Psi_{ph} = 0,$$  

(16)

where $\tilde{H}^2 = g^{00} \hat{H}$. Each component of the state vector $\Psi_{ph}$ will obey (16). Now, it turns out that it is more easy to write down a general form for $\tilde{H}^2$ from (16), than to do for $\hat{H}$ from (15). There are examples in the literature where some ordering ambiguities can be solved by iterating the relevant operators [9]. It is known [10] that to define a time-invariant scalar product for the solutions of (16) the operator $K$ must be hermitean with respect to (12). Also, since the state vectors $\Psi_{ph}$ are assumed to be scalars under coordinate transformations, the equation (16) must be covariant. These two conditions, the classical expression for $H$, and the requirement that only terms up to $\hbar^2$ order should be present in the Hamiltonian $\tilde{H}^2$ lead to the following general expression: $\tilde{H}^2 = \hat{\pi}^2 + m^2 + \lambda R$, where $\hat{\pi}^2 = g^{-\frac{1}{2}} \hat{\pi} \sqrt{g} g^{ij} \hat{\pi}_j g^{-\frac{1}{2}}$, $R$ is the scalar of curvature, and $\lambda$ is a real number. The equation (16) becomes

$$K \Psi_{ph} = \left( \frac{1}{\sqrt{g}} \partial_{\mu} \sqrt{g} g^{\mu \nu} \partial_{\nu} + m^2 + \lambda R \right) \Psi_{ph} = 0,$$  

(17)

which is the standard generalization of the Klein-Gordon equation in a curved manifold. There is a vast literature about the choice of the constant $\lambda$, but we do not pay attention.
to it because its value does not affect our analysis and results. The time independent scalar product for the $\Psi_{\text{ph}}$ in this case is given by

$$\langle \Psi_{\text{ph}}^1, \Psi_{\text{ph}}^2 \rangle_{\text{ph}} = \int d^{D-1}x \sqrt{\bar{g}} \sqrt{|g^{00}|} \left( \Psi_{\text{ph}}^{1\dagger} \partial_0 \Psi_{\text{ph}}^2 - (\partial_0 \Psi_{\text{ph}}^{1\dagger}) \Psi_{\text{ph}}^2 \right),$$  \hspace{1cm} (18)

where $d^{D-1}x \sqrt{\bar{g}}$ is the invariant volume element in one of the spacelike surfaces $x^0 = \text{const}$, and as in the flat space case (18) is not positive defined.

Since the role of the operator $\hat{\xi}$ is realized only in the Schrödinger equation ($\hat{\xi}^2 = \mathbb{I}$), it would be of great interesting to continue with the analysis of (15). To this purpose we will restrict ourselves to ultrastatic spacetimes [7], what means that besides of the restrictions (1), we have also $\sqrt{g^{00}} = 1$. This restriction is convenient to avoid other ordering ambiguities with the lapse function and it does not affect the role played by the operators $\hat{\xi}$. With such a metric the distance between the two surfaces labeled by $x^0 = \text{and } x^0 + dx^0$ is independent of $x^i$. In this case we define formally $\hat{H} = \sqrt{\bar{\pi}^2 + m^2 + \lambda R}$, to proceed with the analysis of (15). The variable $x^0$ is perfectly separated from the $x^i$ in (15), what allows us to write its solutions $\Psi_{\text{ph}}$ as

$$\Psi_{\text{ph}}(x^\mu) = \begin{pmatrix} e^{-i\omega x^0} f(x^i) \\ e^{i\omega x^0} f(x^i) \end{pmatrix}, \hspace{1cm} \sqrt{\bar{\pi}^2 + m^2 + \lambda R} f(x^i) = \omega f(x^i).$$ \hspace{1cm} (19)

Now it is clear that the eigenstates of $\hat{\xi}$, $\Psi_{\text{ph}}^+ = e^{-i\omega x^0} f(x^i) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Psi_{\text{ph}}^- = e^{i\omega x^0} f(x^i) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, are eigenstates of the operator $i\partial_0$ with eigenvalues $\omega$ and $-\omega$ respectively. Written in a coordinate free way we have

$$i \mathcal{L}_v \Psi_{\text{ph}}^\pm = \pm \omega \Psi_{\text{ph}}^\pm,$$ \hspace{1cm} (20)

where $\mathcal{L}_v$ stands for the Lie derivative along the timelike Killing vector $v$. One recognize (20) as the covariant separation in parts of positive and negative frequencies of the wave function $\Psi_{\text{ph}}$ [11], what confirms in the quantum dynamics the classical interpretation that $\xi$ distinguishes between particles and antiparticles. We know also that (20) guarantees that the vacuum is stable [11], avoiding pair creation and annihilation, in agreement with the
fact that $\xi$ is a constant of motion. In a general spacetime without a timelike Killing vector, the separation (20) is not possible, and we will have inequivalent vacua connected by non-trivial Boguliubov transformations. In such case, we could not use the gauge choice (7), and analysis would become extremely more complicated.

It is interesting to note also that in an ultrastatic spacetime one can define a positive defined time-invariant inner product for the $\Psi_{ph}$ as

$$\langle \Psi_{ph}^1, \Psi_{ph}^2 \rangle_{ph} = \int d^{D-1}x \sqrt{g} \Psi_{ph}^1 \Psi_{ph}^2,$$  

and we can check that all the physical operators are hermitean with respect to (21).

As the conclusion, we stress that in a static spacetime is possible to perform the canonical quantization of the spinless relativistic particle getting the Hilbert space $\mathcal{H}$ describing a two-particle quantum mechanics. Both particle and antiparticle are already present at the classical level, corresponding to trajectories labeled by the two possible values of $\xi$.

The author wishes to thank CNPq for the financial support.
REFERENCES


