Algebra and Twisted Algebra in Toroidal Target Space

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Abstract

Target space duality is reconsidered from the viewpoint of quantization in a space with nontrivial topology. An algebra of operators for the toroidal bosonic string is defined and its representations are constructed. It is shown that there exist an infinite number of inequivalent quantizations, which are parametrized by two parameters $0 \leq s, t < 1$. The spectrum exhibits the duality only when $s = t$ or $-t \pmod{1}$. A deformation of the algebra by a central extension is also introduced. It leads to a kind of twisted relation between the zero mode quantum number and the topological winding number.

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1 Introduction

Recently target space duality (T-duality) in string theory has attracted a lot of attention and is being investigated enthusiastically [1, 2, 3, 4, 5, 6]. Target space duality is an equivalence of two-dimensional sigma models which are defined in target spaces with different geometry and sometimes different topology. The simplest example of target space duality is found in toroidal bosonic string model which has a target space $S^1$ of radius $R$. Target space duality implies equivalence of two toroidal models; although one has a radius $R$ and the other one has a radius $1/(2\pi R)$, their spectra coincide.

On the other hand, several authors [7, 8, 9, 10, 11] have investigated the relation between geometry and quantum theory in a context different from string theory. They tried to construct quantum mechanics on a manifold with nontrivial topology by various methods. They extended the canonical formalism of quantization to include topological effects manifestly. Then they noticed that there are an infinite number of inequivalent quantizations on a space with nontrivial topology. Thus, although a beautiful understanding of target space duality is given [5, 6] in classical theory, a richer structure in quantum theory of target space duality is expected, which may have been missed before. Target space duality is also investigated from the path integral point of view [12, 13].

The aim of this letter is to explore the quantum aspect of target space duality from the viewpoint of quantization on a topologically nontrivial space. We reconsider the simplest model, the toroidal bosonic string. We define the algebra of quantum operators to describe the model while keeping its topological nature manifest. In this definition we will show the possibility of modifying the algebra; this modified algebra may be called an algebra with a central extension or a twisted algebra. Then representations of the algebra are constructed and it is shown that there are also an infinite number of inequivalent quantizations, which are parametrized by two parameters $s$ and $t$ ($0 \leq s, t < 1$). It is also shown that there is a target space duality only when $s = t$ or $-t$ (mod 1). Hence the concept of target space duality may become subtler and richer in quantum theory than in classical theory. This letter is an extension of a previous work [14], in which only one central extension
was considered. Here we take into consideration all possible central extensions.

2 Quantum algebra for the toroidal boson

In classical theory the toroidal boson model is described by a dynamical variable which is a map $X : S^1 \rightarrow T^n$, $\sigma \mapsto X^A(\sigma)$. For simplicity, we concentrate on the circular boson; $X : S^1 \rightarrow S^1, \sigma \mapsto X(\sigma)$. Moreover we assume that the radius of the target space is normalized to be of unit length. Eventually we will replace it with a parameter $R$ and we will consider the target space duality in this context.

To be a closed string, $X(\sigma)$ should satisfy the boundary condition $X(\sigma + 2\pi) = X(\sigma) + 2\pi n, (n \in \mathbb{Z})$. However the variable $X(\sigma)$ is not suitable to describe quantum theory since it is multi-valued. To make a concise description of the quantum theory, it is desirable to find a single-valued operator instead of $X(\sigma)$. Thus we introduce $\hat{V}(\sigma) = e^{i\hat{X}(\sigma)}$, which would be a unitary operator. Rigorously speaking, to make sense of the exponential of the local operator $\hat{X}(\sigma)$, we must treat the divergence accompanying it. However it is treated by the rather standard normal ordering procedure, so we will skip an argument to justify it.

$\hat{X}(\sigma)$ is assumed to satisfy the canonical commutation relation

$$[\hat{X}(\sigma), \hat{P}(\sigma')] = i\delta(\sigma - \sigma'),$$

where the delta function is defined on $S^1$. The geometric meaning of the above algebra will become clear if we introduce a unitary operator

$$\hat{U}_f = \exp\left[-i\int_0^{2\pi} f(\sigma)\hat{P}(\sigma)d\sigma\right]$$

for an arbitrary function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f(\sigma + 2\pi) - f(\sigma) = 2\pi m (m \in \mathbb{Z})$. Then it is easily seen that

$$\hat{U}_f^\dagger \hat{X}(\sigma) \hat{U}_f = \hat{X}(\sigma) + f(\sigma),$$

$$\hat{U}_f^\dagger \hat{V}(\sigma) \hat{U}_f = e^{if(\sigma)}\hat{V}(\sigma).$$

From these relations we may call $\hat{U}_f$ the deformation operator of the string configuration. $\hat{U}_f$ also generates a topology change of the string as a special case. Let us define a winding operator

$$\hat{W} = \exp\left[-i\int_0^{2\pi} \sigma\hat{P}(\sigma)d\sigma\right].$$
Then it satisfies
\[
\hat{W}^\dagger \hat{X}(\sigma) \hat{W} = \hat{X}(\sigma) + \sigma, \quad (2.6)
\]
\[
\hat{W}^\dagger \hat{V}(\sigma) \hat{W} = e^{i\sigma} \hat{V}(\sigma) \quad (2.7)
\]
as the name of the winding operator indicates. Additionally, we may define a translation operator
\[
\hat{T}_\lambda = \exp \left[ -i\lambda \int_0^{2\pi} \hat{P}(\sigma) d\sigma \right], \quad (2.8)
\]
which automatically satisfies the following;
\[
\hat{T}_\lambda^\dagger \hat{X}(\sigma) \hat{T}_\lambda = \hat{X}(\sigma) + \lambda, \quad (2.9)
\]
\[
\hat{T}_\lambda^\dagger \hat{V}(\sigma) \hat{T}_\lambda = e^{i\lambda} \hat{V}(\sigma). \quad (2.10)
\]
Using these operators $\hat{V}(\sigma)$ and $\hat{U}_f$, instead of $\hat{X}(\sigma)$ and $\hat{P}(\sigma)$, the fundamental algebra of the toroidal boson is written as
\[
[\hat{V}(\sigma), \hat{V}(\sigma')] = 0, \quad (2.11)
\]
\[
\hat{U}_f^\dagger \hat{V}(\sigma) \hat{U}_f = e^{i\hat{f}(\sigma)} \hat{V}(\sigma), \quad (2.12)
\]
\[
\hat{U}_f \hat{U}_g = e^{-ic(f,g)} \hat{U}_{f+g}, \quad (2.13)
\]
where the phase factor $e^{-ic(f,g)}$ is inserted for generality. This kind of generalization is often called a central extension. Such a generalization is possible because the deformation operator $\hat{U}_f$ acts on $\hat{V}(\sigma)$ by the adjoint action in (2.12). Hence such an extra phase factor does not spoil the associative action of deformations on the string configuration. Namely, we can deduce that
\[
\hat{U}_g^\dagger \hat{U}_f^\dagger \hat{V}(\sigma) \hat{U}_f \hat{U}_g = \hat{U}_{f+g}^\dagger \hat{V}(\sigma) \hat{U}_{f+g}. \quad (2.14)
\]
However, since $\hat{U}_f$ should satisfy the associativity $(\hat{U}_{f_1} \hat{U}_{f_2}) \hat{U}_{f_3} = \hat{U}_{f_1} (\hat{U}_{f_2} \hat{U}_{f_3})$, the phase $c(f,g)$ must satisfy a consistency condition
\[
c(f_1, f_2) + c(f_1 + f_2, f_3) = c(f_1, f_2 + f_3) + c(f_2, f_3) \quad \text{(mod } 2\pi\text{)}. \quad (2.15)
\]
Such a function $c$ is called a 2-cocycle. Possible 2-cocycles are already classified by Segal [15, 16]. To describe them we decompose the function $f_i(\sigma) \ (i = 1, 2)$ into three parts as
\[
f_i(\sigma) = m_i \sigma + \lambda_i + \hat{f}_i(\sigma), \quad (2.16)
\]
where we define each term by \(2\pi m_i = f_i(2\pi) - f_i(0)\), \(2\pi \lambda_i = \int_0^{2\pi} (f_i(\sigma) - m_i \sigma) d\sigma\) and \(\tilde{f}_i(\sigma) = f_i(\sigma) - m_i \sigma - \lambda_i\), respectively. Then the 2-cocycles of Segal are defined by

\[
c_d(f_1, f_2) = d \left\{ m_1 \lambda_2 - m_2 \lambda_1 + \frac{1}{4\pi} \int_0^{2\pi} \left( \frac{df_1}{d\sigma} \tilde{f}_2 - \frac{df_2}{d\sigma} \tilde{f}_1 \right) d\sigma \right\}.
\]

They are specified by an integer \(d\), which is called the rank of the central extension.

## 3 Representations with \(d = 0\)

To complete the quantization of the toroidal boson we will now construct representations of the fundamental algebra (2.11)–(2.13). We factorize \(\hat{V}(\sigma)\) and \(\hat{U}_f\) for convenience as

\[
\hat{V}(\sigma) = e^{i\hat{N} \sigma} \hat{V} e^{i\hat{x}(\sigma)},
\]

\[
\hat{U}_f = \hat{W}^m e^{-i\hat{P}} \exp \left[ -i \int_0^{2\pi} \tilde{f}(\sigma) \hat{\pi}(\sigma) d\sigma \right],
\]

where \(\hat{N}, \hat{x}, \hat{P}\) and \(\hat{\pi}\) are hermitian; \(\hat{V}\) and \(\hat{W}\) are unitary operators. Their geometric meaning is the following: \(\hat{N}\) denotes the winding number; \(\hat{V} = \exp[i/(2\pi) \int_0^{2\pi} (\hat{X}(\sigma) - \hat{N} \sigma) d\sigma]\) represents the zero mode which is a collective coordinate on the target space \(S^1\); \(\hat{x}(\sigma)\) denotes the oscillatory degrees of freedom satisfying \(\hat{x}(2\pi) = \hat{x}(0)\) and \(\int_0^{2\pi} \hat{x}(\sigma) d\sigma = 0\). \(\hat{W}\) is the winding operator already mentioned; \(\hat{P} = \int_0^{2\pi} \hat{P}(\sigma) d\sigma\) is the zero mode momentum; \(\hat{\pi}(\sigma)\) is the canonical momentum conjugate to the oscillator \(\hat{x}(\sigma)\).

According to the above decomposition of operators, the fundamental algebra (2.11)–(2.13) is rewritten as

\[
[\hat{N}, \hat{W}] = \hat{W},
\]

\[
[\hat{P}, \hat{V}] = \hat{V},
\]

\[
[\hat{x}(\sigma), \hat{\pi}(\sigma')] = i \left( \delta(\sigma - \sigma') - \frac{1}{2\pi} \right)
\]

and all the other commutators vanish when \(d = 0\). The case of nontrivial central extension \(d \neq 0\) will be considered in the next section.

Now representations of the algebra are easily constructed. First, (3.3) is represented by

\[
\hat{N} |n + s\rangle = (n + s) |n + s\rangle,
\]

\[
\hat{W} |n + s\rangle = |n + 1 + s\rangle,
\]
where \( n \) is an integer and \( s \) is an undetermined real number. The Hilbert space spanned by \( \{|n + s| n \in \mathbb{Z}\} \) is denoted by \( \mathcal{H}_s \). For each value of \( s \) \((0 \leq s < 1)\), the corresponding \( \mathcal{H}_s \) provides an inequivalent representation. The appearance of inequivalent representations is a phenomenon observed for quantization in a topologically nontrivial space [8, 10, 17].

Second, (3.4) is actually isomorphic to (3.3), so its representation is constructed in the same way;

\[
\hat{P} |p + t\rangle = (p + t) |p + t\rangle, \tag{3.8}
\]
\[
\hat{V} |p + t\rangle = |p + 1 + t\rangle, \tag{3.9}
\]
where \( p \) is an integer and \( t \) is a real parameter. Here the Hilbert space is denoted by \( \mathcal{H}_t \). The zero mode momentum is discrete, as expected for a toroidal target space, but it is shifted by a fraction \( t \) \((0 \leq t < 1)\). The inequivalent representations are again characterized by \( t \).

Third, the canonical commutation relation (3.5) is represented by the standard Fock representation; we make a Fourier expansion

\[
\hat{x}(\sigma) = \frac{1}{2\pi} \sum_{k \neq 0} \sqrt{\frac{\pi}{|k|}} (\hat{a}_k e^{ik\sigma} + \hat{a}_k^\dagger e^{-ik\sigma}), \tag{3.10}
\]
\[
\hat{\pi}(\sigma) = \frac{i}{2\pi} \sum_{k \neq 0} \sqrt{\pi|k|} (-\hat{a}_k e^{ik\sigma} + \hat{a}_k^\dagger e^{-ik\sigma}) \tag{3.11}
\]
and the oscillators \([\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}\) are represented on the Fock space \( \mathcal{F} \).

Combining the above parts, the fundamental algebra without the central extension is represented by the tensor product \( \mathcal{H}_s \otimes \mathcal{H}_t \otimes \mathcal{F} \). Thus it is concluded that there are an infinite number of inequivalent quantizations of the toroidal boson which are parametrized by \( 0 \leq s, t < 1 \).

Now we reintroduce the radius of the target space \( R \). Then we make replacement \( \hat{X}(\sigma) \rightarrow \frac{1}{R} \hat{X}(\sigma) \) and \( \hat{P}(\sigma) \rightarrow R \hat{P}(\sigma) \). Accordingly (3.1) and (3.2) are replaced by

\[
\hat{V}(\sigma) = e^{i\frac{1}{R} \hat{N}_s} \hat{V} e^{i\frac{1}{R} \hat{x}(\sigma)}, \tag{3.12}
\]
\[
\hat{U}_f = \hat{W}^m e^{-iRN\hat{P}} \exp \left[-iR \int_0^{2\pi} \tilde{f}(\sigma) \hat{\pi}(\sigma) d\sigma \right], \tag{3.13}
\]
namely, the substitution \( \hat{N} \rightarrow \frac{1}{R} \hat{N}, \hat{x}(\sigma) \rightarrow \frac{1}{R} \hat{x}(\sigma), \hat{P} \rightarrow R \hat{P} \) and \( \hat{\pi}(\sigma) \rightarrow R \hat{\pi}(\sigma) \) is made. The eigenvalues of \( \hat{N} \) and \( \hat{P} \) are rescaled;

\[
\hat{N} |n + s\rangle = R(n + s) |n + s\rangle, \tag{3.14}
\]
\[ \hat{P} \ket{p + t} = \frac{1}{R} (p + t) \ket{p + t}. \]  

(3.15)

In this case the Hamiltonian is given by

\[
\hat{H} = \frac{1}{2} \int_0^{2\pi} \left[ \hat{P}^2 (\sigma) + \left( \frac{d\hat{X}}{d\sigma} \right)^2 \right] d\sigma 
\]

\[ = \frac{1}{2} \left( \frac{1}{2\pi} \hat{P}^2 + 2\pi \hat{N}^2 \right) + \sum_{k \neq 0} |k| \hat{a}_k^\dagger \hat{a}_k, \]

(3.16)

where we have substitute \( \hat{X}(\sigma) = \hat{N} \sigma + \hat{x}(\sigma) \) and \( \hat{P}(\sigma) = \frac{1}{2\pi} \hat{P} + \hat{\pi}(\sigma) \). So excitations of zero modes and winding modes exhibit the spectrum

\[
\hat{H} \ket{n + s; p + t} = \frac{1}{2} \left[ \frac{1}{2\pi R^2} (p + t)^2 + 2\pi R^2 (n + s)^2 \right] \ket{n + s; p + t}.
\]

(3.17)

Hence the spectrum is invariant under the transformation \( R \to 1/(2\pi R) \) if and only if \( s = t \) or \( -t \) (mod 1). So the target space duality which is expected from classical geometry [6] becomes a rather subtle concept in quantum theory.

4 Representations with \( d \neq 0 \)

When the nonvanishing central extension (2.17) exists, both the fundamental algebra and its representations are modified. In our previous work [14] we considered only the case of \( d = 1 \). Here we consider all possible values of \( d \in \mathbb{Z} \).

At first we change the factorization of \( U_f \) from (3.2) to

\[
\hat{U}_f = e^{i\lambda \hat{W}} e^{-i\lambda \hat{\pi}} \exp \left[ -i \int_0^{2\pi} \tilde{f}(\sigma) \hat{\pi}(\sigma) d\sigma \right],
\]

(4.1)

Then the modified algebra (2.13) with the center (2.17) is satisfied if we add new commutation relations

\[
[ \hat{\pi}(\sigma), \hat{\pi}(\sigma') ] = \frac{i\lambda}{\pi} \delta'(\sigma - \sigma'),
\]

(3.3)–(3.5). The first one (4.2) implies that the zero mode momentum \( \hat{P} \) is decreased by \( 2d \) units when the winding number \( \hat{N} \) is increased by one unit under the operation of \( \hat{W} \). We may call this commutator a “twist”. Such an interrelation between the zero mode and winding number was unexpected and its dynamical
meaning is still obscure. The second equation (4.3) reminds us of the anomalous
commutator of the Schwinger model.

Representations of the above algebra are easy to construct but considerably dif-
ferent from the previous ones. The algebra (3.3) and (3.4) twisted with (4.2) is
represented by

\[
\hat{N} | n + s; p + t \rangle = (n + s) | n + s; p + t \rangle, \tag{4.4}
\]

\[
\hat{W} | n + s; p + t \rangle = | n + 1 + s; p - 2d + t \rangle, \tag{4.5}
\]

\[
\hat{P} | n + s; p + t \rangle = (p + t) | n + s; p + t \rangle, \tag{4.6}
\]

\[
\hat{V} | n + s; p + t \rangle = | n + s; p + 1 + t \rangle. \tag{4.7}
\]

The Hilbert space spanned by \{ | n + s; p + t \rangle | n, p \in \mathbb{Z} \} is denoted by \( T_{st}(d) \).

Taking the anomalous commutator (4.3) into account, the Fourier expansion of
\( \hat{x}(\sigma) \), \( \hat{\pi}(\sigma) \) into oscillators is changed. After a tedious calculation we obtain

\[
\hat{x}(\sigma) = \sum_{k \neq 0} \frac{1}{\sqrt{2|dk|}} (\hat{a}_k e^{ik\sigma} + \hat{a}^\dagger_k e^{-ik\sigma}), \tag{4.8}
\]

\[
\hat{\pi}(\sigma) = \frac{i}{2\pi} \left\{ \sum_{k=1}^{\infty} (d > 0) |d_{-1}^{<0}) \right\} \sqrt{2|dk|} (-\hat{a}_k e^{ik\sigma} + \hat{a}^\dagger_k e^{-ik\sigma}). \tag{4.9}
\]

Details of this calculation are shown in [18]. It should be noticed that only positive
\( k \) oscillators (right-moving boson) appear in the expansion of \( \hat{\pi} \) when \( d > 0 \), while
only negative \( k \) oscillators (left-moving boson) appear when \( d < 0 \). However both
positive and negative \( k \)'s appear in \( \hat{x} \). Let us denote the Fock space by \( \mathcal{F}(d) \) in which
half of the modes are separated according to the sign of \( d \).

It is concluded that the algebra with the the nonvanishing central extension is
represented by \( T_{st}(d) \otimes \mathcal{F}(d) \). The first implication of the central extension is the
twisted interrelation between the winding number and the zero mode. The second
one is the lack of half of the oscillator modes in \( \hat{\pi}(\sigma) \).

5 Conclusion

Here we shall give a brief summary of our method and results. We introduced the
zero mode and the winding number variables to describe the topological property of
the model in (3.1) and (3.2). We next defined the fundamental algebra (2.11)–(2.13)
and decomposed them into (3.3)–(3.5). Then we constructed its representations and showed the existence of inequivalent representations parametrized by $0 \leq s, t < 1$. The spectrum is shifted by $s$ and $t$ and target space duality remains only when $s = t$ or $-t \ (\text{mod} \ 1)$, as shown in (3.17). It was pointed that the algebra can be deformed by the central extension (2.17). The deformed algebra leads to the twisted interrelation between the zero mode quantum number $p$ and the topological number $n$ in (4.2) and (4.5) and also to the lack of half of the oscillators in $\hat{\pi}(\sigma)$, as observed in (4.9).

Our scheme depends on the abelian nature of the model; since our model is the $U(1)$ sigma model, it is possible to decompose its degrees of freedom in the rather simple way of (3.1) and (3.2). However it seems difficult to find suitable quantum variables to describe a nonlinear sigma model which has a general Riemannian manifold as a target space. Although in the context of classical geometry a neat consideration of target space duality is given to general nonlinear sigma models from the viewpoint of canonical transformations [4, 5], the implication of target space duality in the context of quantum theory should be investigated more carefully.

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**References**


