Heat kernel for non–minimal operators on a Kähler manifold

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The heat kernel expansion for a general non–minimal operator on the spaces $C^\infty(\Lambda^k)$ and $C^\infty(\Lambda^{p,q})$ is studied. The coefficients of the heat kernel asymptotics for this operator are expressed in terms of the Seeley coefficients for the Hodge–de Rham Laplacian.

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I. INTRODUCTION

Let $M$ be a compact Reimannian manifold of dimension $m$ without boundary. If $M$ is equipped with integrable complex structure, one can split tangential indices into holomorphic and antiholomorphic ones and define space of differential forms $C^\infty(\Lambda^{p,q})$. The exterior differential $d$ can be also splitted in a sum $d = \partial + \bar{\partial}$ of anticommuting nilpotent operators: $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. If $M$ is a Kähler manifold, the corresponding “Laplacians” can be reduced to the Hodge–de Rham Laplacian:

$$\partial\partial^* + \partial^*\partial = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \frac{1}{2}\Delta = \frac{1}{2}(\delta d + d\delta).$$ (1)

Using these first order differential operators one can construct a (non–minimal) second order differential operator:

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\[ \mathcal{D} = g_1 \partial \partial^* + g_2 \partial^* \partial + g_3 \bar{\partial} \bar{\partial}^* + g_4 \bar{\partial}^* \bar{\partial} + g_5 \partial \bar{\partial}^* + g_5^* \bar{\partial} \partial^* \]  

(2)

with real constants \( g_1, \ldots, g_4 \) and a complex constant \( g_5 \). For some values of the constants this operator reduces to that considered previously in the paper [1], where one can find some motivations for studying non–minimal operators. Such operators appear naturally in quantum gauge theories after imposing gauge conditions [2–5].

For a self–adjoint second order operator \( L \) with non–negative eigenvalues \( \{ \lambda_{\nu} \} \) one can define the integrated heat kernel

\[ \text{Tr}(e^{-tL}) = \sum_{\nu} e^{-t\lambda_{\nu}}. \]  

(3)

As \( t \to 0^+ \), there is an asymptotic expansion of the form

\[ \text{Tr}(e^{-tL}) = \frac{1}{(4\pi)^{\frac{m}{2}}} \sum_{n=0}^{\infty} a_n(L)t^{\left(\frac{2n-m}{2}\right)}. \]  

(4)

In this paper we study the heat kernel expansion for the non–minimal operator \( \mathcal{D} \) (2) and relate the Seeley coefficients \( a_n(\mathcal{D}) \) to that for the Laplace operator \( a_n(\Delta) \). General expressions for \( a_n(\Delta) \), \( n = 0, 1, 2, 3 \) suitable for the spaces of differential forms can be found in the paper [6]. In particular cases this problem was solved in Refs. [1,3–5]. In a sense, we suggest an extension of the Theorem 1.2 of Ref. [1] for the case of complex geometry.

In the next section we study the heat kernel for \( \mathcal{D} \) acting on the space of \( k \)-forms, \( C^\infty(\Lambda^k) \). In the section 3 the case \( \mathcal{D} : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q}) \) is considered. This assumes some restrictions on the constants in (2), but a more detailed information can be obtained. In the Appendix we use complex projective space \( CP^2 \) as an example to check up some of the equations of previous sections.

**II. NON–MINIMAL OPERATORS ON \( K \)-FORMS**

In this section we consider the heat kernel for non-minimal operators acting on the space \( C^\infty(\Lambda^k) \) of \( k \)-forms. First let us discuss some properties of first order operators \( D_1 \) and \( D_2 \) which will be used later to build up a general non–minimal second order operator \( \mathcal{D} \).
Lemma 1. Let $D_1$ and $D_2$ be operators on $C^\infty(\Lambda)$ having the following properties:

(a) $D_1, D_2 : C^\infty(\Lambda^k) \to C^\infty(\Lambda^{k+1})$, (b) $D_1^2 = D_2^2 = 0$, (c) $D_1 D_2 + D_2 D_1 = D_1 D_2^* + D_2^* D_1 = 0$,

(d) $D_1 D_2^* + D_1^* D_1 = \alpha \Delta$, $D_2 D_2^* + D_2^* D_2 = \beta \Delta$, $\alpha, \beta \neq 0$. Then

1. $C^\infty(\Lambda^k) = \text{Ker}(\Delta) \oplus \text{im}(D_1) \oplus \text{im}(D_2) = \text{Ker}(\Delta) \oplus \text{im}(D_1) \oplus \text{im}(D_2)$,

2. $C^\infty(\Lambda^k) = \text{Ker}(\Delta) \oplus (D_1 D_2)_k \oplus (D_1 D_2^*)_k \oplus (D_2 D_2)_k \oplus (D_2 D_2^*)_k$.

3. The following mappings are isomorphisms:

$$
(D_1 D_2)_k \leftrightarrow (D_1 D_2^*)_k, \quad (D_1^* D_2)_k \leftrightarrow (D_1 D_2^*)_k,
$$

$$
(D_1 D_2^*)_k \leftrightarrow (D_1 D_2^*)_k, \quad (D_1 D_2)_k \leftrightarrow (D_1 D_2^*)_k,
$$

where operators $D$ act from right to left, and $D^*$ act from left to right. We introduced the notation $(AB)_k = \text{im}(A) \cap \text{im}(B) \cap C^\infty(\Lambda^k)$.

Proof. The proof of the first statement repeats standard proof [7] of the same property for operator $d$. The decomposition 2. can be obtained by repeating twice the decompositions 1. Last statement follows from the anticommutativity properties b) and c). □

Let us introduce the following notations:

$$
\Delta_k = \Delta|_{C^\infty(\Lambda^k)}, \quad f(t, D) = \text{Tr} \exp(-tD)
$$

$$
= \text{Tr} \exp(-t\Delta|_{(AB)_k}),
$$

$$
f_k(t) = f(t, D_1^*, D_2^*, k) \quad (5)
$$

$\beta_k$ denote Betti numbers.

Lemma 2. $f_k(t) = \sum_{l=0}^{k} (-1)^l (l+1)(f(t, \Delta_{k-l}) - \beta_{k-l})$.

Proof. First we observe that all spaces appearing in the second statement of Lemma 1 are eigenspaces of the Laplace operator. Hence,

$$
f(t, \Delta_k) = \beta_k + f(t, D_1, D_2, k) + f(t, D_1^*, D_2, k) + f(t, D_1, D_2^*, k) + f(t, D_1^*, D_2^*, k). \quad (6)
$$

By using commutativity of $D_1, D_2, D_1^*$ and $D_2^*$ with $\Delta$ and the last statement of Lemma 1 we obtain:
\[ f(t, D_1, D_2, k) = f(t, D_1^*, D_2, k - 1) = f(t, D_1, D_2^*, k - 1) = f(t, D_1^*, D_2^*, k - 2) = f_{k-2}(t) \]  
(7)

Now we can express \( f_k(t) \) from (6) and by repeated use of (7) obtain
\[ f_k(t) = -f_{k-1} + \sum_{l=0}^{k} (-1)^l (f(t, \Delta_{k-l}) - \beta_{k-l}). \]  
(8)

Now the statement of the Lemma follows by induction. \( \square \)

It is easy to see that the operators
\[ D_1 = x_1 \partial + y_1 \bar{\partial}, \quad D_2 = x_2 \partial + y_2 \bar{\partial} \]  
(9)
satisfy conditions of Lemma 1 provided the equation \( x_1^*x_2 + y_1^*y_2 = 0 \) holds for complex parameters \( x_1, x_2, y_1 \) and \( y_2 \). The constants \( \alpha \) and \( \beta \) are real and positive: \( \alpha = \frac{1}{2}(|x_1|^2 + |y_1|^2) \), \( \beta = \frac{1}{2}(|x_2|^2 + |y_2|^2) \). The non-minimal operator
\[ \mathcal{D} = aD_1D_1^* + bD_1^*D_1 + cD_2D_2^* + dD_2^*D_2 \]  
(10)

\[ = (a|x_1|^2 + c|x_2|^2)\partial\partial^* + (b|x_1|^2 + d|x_2|^2)\partial^*\partial + (a|y_1|^2 + c|y_2|^2)\bar{\partial}\bar{\partial}^* + (b|y_1|^2 + c|y_2|^2)\bar{\partial}^*\bar{\partial} \]
\[ + (a-b)x_1^*y_1 + (c-d)x_2^*y_2)\partial\bar{\partial} + ((a-b)y_1^*x_1 + (c-d)y_2^*x_2)\bar{\partial}\partial \]

with real constants \( a, b, c, d \) is the most general hermitian operator on \( C^\infty(\Lambda^k) \) which can be constructed using \( \partial, \bar{\partial}, \partial^* \) and \( \bar{\partial}^* \). This operator has the form (2).

The following Theorem represents the main result of this section.

**Theorem 1.** Let \( D_1 \) and \( D_2 \) satisfy conditions of Lemma 1. Then the coefficients \( a_n \) of the heat kernel expansion for the operator \( \mathcal{D} \) (10) have the form:

\[ a_n(\mathcal{D}|_{C^\infty(\Lambda^k)}) = (\alpha a + \beta c)^n + \frac{n}{2} \sum_{l=0}^{k-2} (-1)^{k-l}(k-l-1)a_n(\Delta_l) - \]
\[ -((\alpha a + \beta d)^n - \frac{n}{2} + (ab + bc)^n - \frac{n}{2}) \sum_{l=0}^{k-1} (-1)^{k-l}(k-l)a_n(\Delta_l) + \]
\[ + (ab + \beta d)^n - \frac{n}{2} \sum_{l=0}^{k} (-1)^{k-l}(k-l+1)a_n(\Delta_l). \]

**Proof.** By making use of Lemma 1 and equation (7) we find:
\[ f(t, D) = \beta_k + f(t, aD_1D_1^* + cD_2D_2^*, D_1, D_2, k) + f(t, D_1D_1^* + D_2^*D_2, D_1, D_2, k) + \\
+ f(t, bD_1D_1 + cD_2D_2^*, D_* - 2, k) + f(t, bD_1D_1 + dD_2D_2, D_1^*, D_2^*, k) \\
= \beta_k + f(t, (b\alpha + c\beta)\Delta, D_1, D_2, k) + f(t, (a\alpha + d\beta)\Delta, D_1, D_2^*, k) + \\
+ f(t, (b\alpha + c\beta)\Delta, D_1^*, D_2, k) + f(t, (a\alpha + d\beta)\Delta, D_1^*, D_2^*, k) \\
= \beta_k + f_{k-2}((a\alpha + c\beta)t) + f_{k-1}((a\alpha + d\beta)t) + f_{k-1}((b\alpha + c\beta)t) + f_k(b\alpha + d\beta)t \]

Now we can use Lemma 2 in order to express the last line in terms of the Laplace operators on \( C^\infty(\Lambda^k) \).

\[
\begin{align*}
    f(t, D) &= \beta_k + \sum_{l=0}^{k} (-1)^l (l + 1)(f((b\alpha + d\beta)t, \Delta_{k-l}) - \beta_{k-l} + \\
    &+ \sum_{l=0}^{k-1} (-1)^l[f((a\alpha + d\beta)t, \Delta_{k-l}) + f((b\alpha + c\beta)t, \Delta_{k-l-1}) - 2\beta_{k-l-1}] + \\
    &+ \sum_{l=0}^{k-2} (-1)^l (l + 1)(f((a\alpha + c\beta)t, \Delta_{k-l-2}) - \beta_{k-l-2}).
\end{align*}
\]

One can see that the contributions of the Betti numbers to different sums cancel each other.

By making asymptotic expansion of the last equation we arrive at the statement of this Theorem. \( \square \)

Note, that to ensure existence of all traces of exponentials for positive \( t \) one should take non-negative \( a, b, c, d \).

### III. NON-MINIMAL OPERATORS ON \((P, Q)\)-FORMS

This section is devoted to non–minimal operators on \( C^\infty(\Lambda^{p,q}) \). To ensure that \( D \) maps \( C^\infty(\Lambda^{p,q}) \) on itself we should choose

\[
    D_1 = \partial, \quad D_2 = \bar{\partial}.
\]

The following notations will be useful:

\[
(AB)_{p,q} = \text{im}(A) \cap \text{im}(B) \cap C^\infty(\Lambda^{p,q}),
\]

\[
\Delta_{p,q} = \Delta_{C^\infty(\Lambda^{p,q})}.
\]
Other notations are modified by replacing $k$ by $p, q$ in (5). $\beta_{p,q}$ will denote Hodge numbers.

Next Lemma replaces the Lemma 1.

**Lemma 3.** $C^\infty(\Lambda^{p,q}) = \text{Ker}(\Delta_{p,q}) \oplus (\partial\bar{\partial})_{p,q} \oplus (\bar{\partial}\partial^*)_{p,q} \oplus (\partial^*\bar{\partial})_{p,q} \oplus (\partial^*\bar{\partial}^*)_{p,q}$.

2. The following maps are isomorphisms:

\[
\begin{align*}
(\partial\bar{\partial})_{p,q} & \leftrightarrow (\partial^*\bar{\partial})_{p-1,q} & & (\partial^*\bar{\partial})_{p,q} & \leftrightarrow (\partial^*\bar{\partial}^*)_{p,q-1} \\
(\partial\bar{\partial}^*)_{p,q} & \leftrightarrow (\partial^*\bar{\partial}^*)_{p-1,q} & & (\partial\bar{\partial})_{p,q} & \leftrightarrow (\partial\bar{\partial}^*)_{p,q-1}
\end{align*}
\]

where the operators $\partial$ and $\bar{\partial}$ act from right to left, and the operators $\partial^*$ and $\bar{\partial}^*$ act from left to right.

The proof repeats that of Lemma 1.

**Lemma 4.** $f_{p,q}(t) = \sum_{k,l=0}^{p,q} (-1)^{k+l} (f(t, \Delta_{p-k,q-l}) - \beta_{p-k,q-l})$.

**Proof.** Lemma 3 gives the following identities:

\[
\begin{align*}
f(t, \Delta_{p,q}) & = \beta_{p,q} + f(t, \partial, \bar{\partial}, p, q) + f(t, \partial, \bar{\partial}^*, p, q) + f(t, \partial^*, \bar{\partial}, p, q) + f(t, \partial^*, \bar{\partial}^*, p, q), \\
f(t, \partial, \bar{\partial}, p, q) & = f(t, \partial^*, \bar{\partial}, p-1, q) = f(t, \partial, \bar{\partial}^*, p, q-1) = f(t, \partial^*, \bar{\partial}^*, p-1, q-1)
\end{align*}
\]

Thus we obtain:

\[
f_{p,q}(t) = f(t, \Delta_{p,q}) - \beta_{p,q} - f_{p-1,q}(t) - f_{p,q-1}(t) - f_{p-1,q-1}(t).
\] (13)

Repeated use of (13) gives the statement of Lemma 4. □

Now one can prove the following Theorem.

**Theorem 2.** Let $D = a\partial\partial^* + b\partial^*\partial + c\bar{\partial}\bar{\partial}^* + d\bar{\partial}^*\bar{\partial}$ on $C^\infty(\Lambda^{p,q})$. Then

\[
a_n(D) = \left(\frac{b+d}{2}\right)^{n-\frac{m}{2}} a_n(\Delta_{p,q}) + \left(\left(\frac{b+c}{2}\right)^{n-\frac{m}{2}} - \left(\frac{b+d}{2}\right)^{n-\frac{m}{2}}\right) a_n(\Delta_{p,q-1})
\]

\[
+ \left(\left(\frac{a+d}{2}\right)^{n-\frac{m}{2}} - \left(\frac{b+d}{2}\right)^{n-\frac{m}{2}}\right) a_n(\Delta_{p-1,q})
\]

\[
+ \left(\left(\frac{a+c}{2}\right)^{n-\frac{m}{2}} + \left(\frac{b+d}{2}\right)^{n-\frac{m}{2}} - \left(\frac{a+d}{2}\right)^{n-\frac{m}{2}} - \left(\frac{b+c}{2}\right)^{n-\frac{m}{2}}\right) p-1,q-1 \sum_{k,l=0}^{p,q-1} \left(-1\right)^{p+q-k-l} a_n(\Delta_{k,l}).
\]

As in the previous section we expressed the Seeley coefficients for a non–minimal operator in terms of the Seeley coefficients of the Laplacian.
IV. CONCLUSIONS

In this paper we expressed the heat kernel coefficients for non–minimal operators on $C^\infty(\Lambda^k)$ and $C^\infty(\Lambda^{p,q})$ in terms of the Seeley coefficients for the Laplace operators on the same spaces. Expressions for the heat kernel asymptotics applicable for Laplacian on differential forms can be found in the literature [6].

The fact that underlying manifold is Kählerian was used only to relate $\partial\partial^* + \partial^*\partial$ and $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ to $\Delta = \delta d + d\delta$. With some modifications our results can be extended to a general complex manifold. Another generalization could consist in adding an endomorphism $E$ to the operator $\mathcal{D}$.

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APPENDIX: CONSISTENCY CHECK. $\mathbb{C}P^2$

First let us construct the harmonic expansion for $(p,q)$ forms on $\mathbb{C}P^2 = SU(3)/SU(2) \times U(1)$. All needed material on structure and geometry of $\mathbb{C}P^2$, as well as a method of constructing of harmonic expansion, can extracted from ref. [8].

For any homogeneous space $G/H$ a field $\Phi$ belonging to an irreducible representation $D(H)$ of $H$ can be decomposed in a sum of harmonics corresponding to all representations of $G$ giving $D(H)$ after reduction to $H$. In the case of $\mathbb{C}P^2$ the representations $D(SU(2) \times U(1))$ corresponding to $(p,q)$ forms can be easily found:

\[
(p, q) = (0, 0) \quad D(SU(2) \times U(1)) = (1, 0)
\]

\[
(p, q) = (0, 1) \quad D(SU(2) \times U(1)) = (2, -1)
\]
\[(p, q) = (1, 0) \quad D(SU(2) \times U(1)) = (2, 1)\]
\[(p, q) = (0, 2) \quad D(SU(2) \times U(1)) = (1, -2)\]
\[(p, q) = (2, 0) \quad D(SU(2) \times U(1)) = (1, 2)\]
\[(p, q) = (1, 1) \quad D(SU(2) \times U(1)) = (1, 0) \oplus (3, 0) \quad (14)\]

The representations of \(SU(2) \times U(1)\) are labelled by dimension of \(SU(2)\) representation (first number) and the \(U(1)\) charge (second number). The representations for other values of \(p\) and \(q\) can be restored by using the duality transformation. Note that the representation in last line of (14) is reducible.

By repeating calculations of Ref. [8] one can find the \(SU(3)\) representations contributing to the harmonic expansion:

\[(p, q) = (0, 0) \quad D(SU(3)) = (m, m), \quad m = 0, 1, \ldots\]
\[(p, q) = (0, 1) \quad D(SU(3)) = (m, m) \oplus (n, n+3), n = 0, 1, \ldots \quad m = 1, 2\ldots\]
\[(p, q) = (1, 0) \quad D(SU(3)) = (m, m) \oplus (n+3, n), n = 0, 1, \ldots \quad m = 1, 2,\ldots\]
\[(p, q) = (0, 2) \quad D(SU(3)) = (n, n+3), \quad n = 0, 1, \ldots\]
\[(p, q) = (2, 0) \quad D(SU(3)) = (n+3, n), \quad n = 0, 1, \ldots\]
\[(p, q) = (1, 1) \quad D(SU(3)) = (n, n)_1 \oplus (n, n)_2 \oplus (0, 0) \oplus (m, m+3) \oplus (m+3, m)\]
\[m = 0, 1, \ldots, \quad n = 1, 2, \ldots \quad (15)\]

where the representations of \(SU(3)\) are labelled by their Dynkin indices. In the last line subscripts are introduced in order to distinguish between equivalent representations.

Let \(V^{p,q}(m, n)\) denotes the space of \((p, q)\)-forms transforming according to the representation \((m, n)\) of \(SU(3)\). One can easily prove that

\[\partial V^{0,0}(m, m) = V^{1,0}(m, m), \quad \bar{\partial} V^{0,0}(m, m) = V^{0,1}(m, m), \quad m \geq 1,\]
\[\partial V^{0,0}(0, 0) = \bar{\partial} V^{0,0}(0, 0) = \{0\},\]
\[\bar{\partial} V^{1,0}(m, m) = \partial V^{0,1}(m, m) = V^{1,1}(m, m)_1,\]
\[\partial V^{1,0}(m, m) = \bar{\partial} V^{0,1}(m, m) = \{0\},\]
\[ \partial V^{1.0}(n + 3, n) = V^{2.0}(n + 3, n), \quad \bar{\partial}V^{0.1}(n, n + 3) = V^{0.2}(n, n + 3), \]
\[ \bar{\partial}V^{1.0}(n + 3, n) = V^{1.1}(n + 3, n), \quad \partial V^{0.1}(n, n + 3) = V^{1.1}(n, n + 3), \]

and so on. Eigenvalues of the Laplace operator coincide with values of the quadratic Casimir operator \( C_2 \) of \( SU(3) \) in corresponding representation. Degeneracy of an eigenvalue \( C_2(m, n) \) is given by dimension \( d(m, n) \) of the representation \( (m, n) \). For the sake of completeness, we give here explicit expressions for \( C_2 \) and \( d \), though they will not be used in what follows.

\[ C_2(m, n) = \frac{1}{3}(m^2 + n^2 + mn + 3m + 3n) \]
\[ d(m, n) = \frac{1}{2}(m + 1)(n + 1)(m + n + 2) \]

The heat kernels become

\[ f(t, \Delta_{00}) = \sum_{m=0}^{\infty} d(m, m) \exp(-tC_2(m, m)) \]
\[ f(t, \Delta_{01}) = \sum_{m=1}^{\infty} d(m, m) \exp(-tC_2(m, m)) + \sum_{n=0}^{\infty} d(n, n + 3) \exp(-tC_2(n, n + 3)) \]

etc

With these formulae at hand one can check up Lemma 2 and Lemma 4.


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