Abstract

The method of higher covariant derivative regularization of gauge theories is reviewed. The objections raised in the literature last years are discussed and the consistency of the method is proven. New approach to regularization of overlapping divergencies is developed.

1 Introduction.

In this paper we review the method of higher covariant derivative regularization of gauge theories [1, 2] supplemented by the additional Pauli-Villars (PV) type regularization as proposed in [3] (see also [4]). We analyze the objections raised in papers [5, 6, 7] and show that although indeed some minor modifications of the original scheme are needed, the general method is self-consistent, provides the gauge invariant regularization to all orders in perturbation theory and on the other hand may serve as a starting point for nonperturbative calculations.

The problem of invariant regularization is of extreme importance both from the point of view of practical calculations and for the general study of symmetry properties of renormalized quantum theory. It is widely believed that for anomaly free models an invariant regularization do exist although no general theorem was proven.

*E-mail: slavnov@class.mian.su
The method mostly used so far for calculations in gauge invariant models was the dimensional regularization \[9\]. However dimensional regularization is not applicable to chiral and supersymmetric models which are very important from the point of view of applications. Moreover there is no obvious generalization of this method to nonperturbative calculations, in particular dealing with topological aspects of a theory, as the dimensional regularization is formulated in terms of perturbative Feynmann diagrams.

The most natural nonperturbative regularization is provided probably by the lattice formulation. But this approach also meets some difficulties in treating topological and chiral models. It is also not very practical for weak coupling calculations due to appearance of new vertices and lack of Lorentz (rotational) invariance.

An alternative regularization scheme which may be implemented as a modification of classical Lagrangian and therefore has a nonperturbative meaning is the higher covariant derivative method. This method also has an advantage of being applicable to chiral and supersymmetric models, but due to a rather complicated structure of regularized Lagrangian it was used mainly for general proofs and not in practical calculations. However it seems that nowadays with the need in precision calculations of electroweak processes on one side and the big progress in computer facilities on the other side, this method may become a real practical tool. For that reason some tests of the procedure were carried out last years, which raised some controversy in the literature.

So we feel it is worthwhile to review the method and to discuss the problems raised in the process of it’s testing.

2 General idea of the method.

The most simple way to regularize the theory is to modify the propagators by introducing into Lagrangian higher derivative terms. However this procedure breaks gauge invariance. To preserve the symmetry it was proposed to modify the Yang-Mills(YM) Lagrangian by adding the terms containing higher covariant derivatives \[1, 2\], e.g.

\[
\mathcal{L}_{YM} \rightarrow \mathcal{L}_{n} = \frac{1}{8} \text{tr}\{\mathcal{F}^2 + \frac{1}{\Lambda^{2n}}(\nabla^n \mathcal{F})^2\}
\]

(1)

Here \(\mathcal{F}_{\mu\nu}\) is the usual curvature tensor and \(\nabla\) is the covariant derivative:

\[
\nabla_\alpha \mathcal{F}_{\mu\nu} = \partial_\alpha \mathcal{F}_{\mu\nu} + [A_\alpha, \mathcal{F}_{\mu\nu}]
\]

(2)

This regularization improves the ultraviolet behaviour of the YM field propagator, provided the gauge fixing term is chosen in the form:

\[
\frac{1}{4\alpha} \text{tr}\{f_n(\Box) \partial_\mu A_\mu\}^2
\]

(3)

where \(f_n\) is a polynomial of order \(\geq \frac{n}{2}\).

Using the Lagrangian (1) with the gauge fixing term (3) one easily sees that the divergency index of arbitrary diagram is equal to:

\[
\omega_n = 4 - 2n(I - 1) - E_A - (n + 1)E_C
\]

(4)

where \(I\) is the number of loops, \(E_A\) and \(E_C\) are the numbers of external gauge and ghost field lines correspondingly. Therefore for \(2n \geq 4\) we got the theory with a finite number
of divergent diagrams. Namely, the only divergent graphs are the one loop diagrams with
\( E_A = 2, 3, 4 \) and \( E_C = 0 \).

This procedure makes convergent all multiloop diagrams in arbitrary gauge theory, however the one loop diagrams require some additional regularization.

It was proposed in the paper [3] (see also [4]) that such a regularization may be
provided by a modified PV procedure.

The key observation was the following. In a higher covariant derivative gauge theory the remaining divergency must have a manifestly gauge invariant structure. The corresponding counterterm is:

\[
Z_{\text{tr}} \{ F_\mu F_{\mu} \} 
\]

(5)

It follows directly from Slavnov-Taylor identities [10, 11] and the fact that the ghost fields and vertex renormalizations in a higher covariant gauge theory are finite. It suggests that these divergencies may be regularized by adding some gauge invariant interaction providing analogous counterterms with the opposite sign.

The formal scheme looks as follows. The total contribution of one loop diagrams with external gauge field lines may be represented by:

\[
Z_\alpha[\mathcal{A}_\mu] = \int \exp \left\{ i \mathfrak{R}_n(\mathcal{A}_\mu, q_\mu) + \int \frac{1}{4\alpha} \text{tr} \left\{ f_n(\frac{\Box}{\Lambda^2}) \partial_\mu q_\mu \right\}^2 dx \right\} \det \mathcal{M}(\mathcal{A}) \prod_x Dq_\mu 
\]

(6)

where \( \mathfrak{R}_n(\mathcal{A}_\mu, q_\mu) = \frac{1}{2} \int \frac{\delta^2 S_n}{\delta \mathcal{A}_\mu(x) \delta \mathcal{A}_\mu(y)} q_\mu(x) q_\mu(y) dxdy \), \( S_n^A \) is the regularized action corresponding to the Lagrangian (1).

This functional is not invariant with respect to the gauge transformations of the fields \( \mathcal{A}_\mu \), but its divergent part is gauge invariant. We firstly demonstrate it for the special case of the Lorentz gauge \( \alpha = 0 \) and then consider the general case.

The functional \( Z_0 \) given by the equation:

\[
Z_0[\mathcal{A}] = \int e^{i \mathfrak{R}_n(\mathcal{A}_\mu, q_\mu)} \det \mathcal{M}(\mathcal{A}) \prod_x \delta(\partial_\mu q_\mu) Dq_\mu 
\]

(7)

may be transformed to the following form:

\[
\ln Z_0 = \ln Z_{\text{inv}} + \text{[finite part]} 
\]

(8)

where \( Z_{\text{inv}}[\mathcal{A}_\mu] \) is a manifestly gauge invariant functional. It can be done by passing in the eq.(7) to the covariant background gauge. To perform the transition we multiply eq.(7) by "unity":

\[
det F_n(\frac{\nabla^2}{\Lambda^2}) \det^2 \int \delta \left( F_n(\frac{\nabla^2}{\Lambda^2}) \nabla_\mu (q_\mu + \nabla_\mu u) - W(x) \right) \prod_x Du(x) = 1 
\]

(9)

where \( F_n(\frac{\nabla^2}{\Lambda^2}) \) is a polynomial of degree:

\[
\frac{n+1}{4} < \text{deg} F_n \leq \frac{n}{2} 
\]

(10)

\[
\lim_{\Lambda \to \infty} F_n(\frac{\nabla^2}{\Lambda^2}) = 1 
\]
and \( \nabla_\mu \) denotes a covariant derivative with respect to the field \( A_\mu \). Changing variables:

\[
q_\mu \rightarrow q_\mu - \nabla_\mu u
\]

we have:

\[
Z_0 = \int \exp \left\{ i \mathcal{R}_n (A_\mu; q_\mu - \nabla_\mu u) \right\} \det \mathcal{M} \det \nabla^2 \det F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \prod_x \delta (\partial_\mu (q_\mu - \nabla_\mu u)) \delta (F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\nu q_\nu - W(x)) Du Dq_\mu
\]

The functional \( Z_0 \) does not depend on \( W \), so we can integrate it over \( W \) with the weight \( \exp \left\{ \frac{i}{2\beta} \int W^2 (x) dx \right\} \). Integrating over \( W \) and \( u \) we get:

\[
Z_0 = \int \exp \left\{ i \mathcal{R}_n (A_\mu; q_\mu - \nabla_\mu u) \mathcal{M}^{-1} \partial_\rho q_\rho \right\} + \frac{1}{2\beta} \int \left[ F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\mu q_\mu \right]^2 dx \}
\cdot \det \nabla^2 \det F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \prod_x Dq_\mu (x)
\]

The functional:

\[
Z_{\text{inv}} [A] = \int e^{i \mathcal{R}_n (A_\mu; q_\mu - \nabla_\mu u) + \frac{1}{2\beta} \int \left[ F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\mu q_\mu \right]^2 dx} \det \nabla^2 \det F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \prod_x Dq_\mu (x)
\]

is invariant with respect to the gauge transformations of fields \( A_\mu \), as the exponent does not change under simultaneous transformations:

\[
\left\{ \begin{array}{l}
A_\mu \rightarrow A_\mu + [A_\mu, \epsilon] + \partial_\mu \epsilon \\
q_\mu \rightarrow q_\mu + [q_\mu, \epsilon]
\end{array} \right.
\]

At the same time it’s connected part differs from (13) only by finite terms. Indeed, since \( S_n^A \) is gauge invariant, we have:

\[
\int \frac{\delta^2 S_n^A}{\delta A_\mu^a (x) \delta A_\nu^c (y)} [\nabla_\mu \phi (x)]^a dx = g^{abc} \frac{\delta S_n^A}{\delta A_\nu^c (y)} \phi^c (y)
\]

The additional terms in the exponent of eq. (13) which are not present in (14) can be rewritten as follows:

\[
g^{abc} \int \frac{\delta S_n^A}{\delta A_\mu^a (x)} [q_\mu - \frac{1}{2} \nabla_\mu \mathcal{M}^{-1} \partial_\rho q_\rho]_x^b [\mathcal{M}^{-1} \partial_\sigma q_\sigma]_x^c
\]

Due to the fact that some derivatives in the eq.(17) act on the external fields \( A_\mu \), and the maximal number of derivatives acting on \( q_\mu \) is \( n + 1 \), the corresponding diagrams are not divergent if the condition (10) holds.

In the same way one can prove that the connected part of \( Z_\alpha \) differs from \( Z_0 \) by finite terms. Multiplying \( Z_0 \) by "unity":

\[
\det \mathcal{M} (A) \prod_x \delta (\partial_\mu (q_\mu + \nabla_\mu u) - W(x)) Du (x) = 1
\]
and integrating it over $W$ with the weight $\exp\left\{\frac{i}{2\alpha} \int [f_n \frac{\partial}{\partial x^2} W(x)]^2 dx \right\}$ one can prove it in complete analogy with the discussion above.

The transformation of $Z_\alpha$ described above makes the gauge invariance of its divergent part manifest. It also suggests a natural gauge invariant regularization of this functional. Under gauge transformations (15) the fields $q_\mu$ transform homogeneously. Therefore without breaking the gauge invariance one can add mass terms for the fields $q_\mu$. It allows to use for regularization of the functional (14) the PV procedure. The regularization looks as follows:

$$
Z_{\text{inv}}[A_\mu] \rightarrow Z_{\text{inv}}[A_\mu]det^{-1}F_n \left(\frac{\nabla^2}{\Lambda^2}\right) \prod_j \left[det_c(n^2 - M_j^2) \times det_n^{-c/2}Q_\beta(A_\mu; M_j^2; F_n)\right]
$$

(19)

where:

$$
det_n^{-1/2}Q_\beta(A_\mu; M^2; F_n) = \int \exp \left\{i[R_n(A_\mu, q_\mu) + \frac{1}{2\beta} \int [F_n \left(\frac{\nabla^2}{\Lambda^2}\right) \nabla_\mu q_\mu]^2 dx - \frac{M^2}{2} \int (q_\mu^2 dx) \right\} \prod_x Dq_\mu(x)
$$

(20)

and PV conditions hold:

$$
1 + \sum_j c_j = 0 \quad \text{(21)}
$$

$$
\sum_j c_j M_j^2 = 0 \quad \text{(22)}
$$

(In higher derivative theory the condition (21) is sufficient for the regularization of $det_n^{-1/2}Q_\beta$ as a single subtraction like

$$
\frac{1}{k^2 + \Lambda^{-2n}k^{2n+2}} - \frac{1}{k^2 + \Lambda^{-2n}k^{2n+2} + M^2}
$$

(23)

makes the integral convergent if $n \geq 1$.) Perturbative expansion of the expression (19) generates together with the loops formed by the original fields $q_\mu$ analogous loops of PV fields. Due to eq.(21,22) leading ultraviolet asymptotics cancel and the corresponding integrals are convergent. Regularizing terms are obviously gauge invariant.

To remove the regularization one should take the limit $\Lambda \rightarrow \infty, M \rightarrow \infty$. In this limit the PV fields decouple from the physical fields and contribute only to local counterterms. This is true for any $\beta \neq 0$. However the case $\beta = 0$ is singular and needs more careful study. This fact was overlooked in papers [3, 4] and if one applies directly the eqs. written in refs. [3, 4] for the Lorentz gauge $\beta = 0$ to calculations of one loop results one gets a wrong result. It was demonstrated explicitly in ref. [7] which lead the authors to the claim that the higher covariant derivative method is inconsistent. As was pointed out by M. Asorey and M. Falceto [8] the discrepancy does not mean inconsistency of the method but is due to the singular character of the Lorentz gauge. In this gauge in the limit $M \rightarrow \infty$ the regularizing fields do not decouple completely. It is most easily seen by rescaling the fields $q_\mu$ in the eq. (20): $q_\mu \rightarrow \frac{1}{M} q_\mu$. For $\beta \neq 0$ after this rescaling all the terms except for $q_\mu^2$ vanish in the limit $M \rightarrow 0$ and the integral over $q_\mu$ gives nonessential
constant. However for $\beta = 0$ this rescaling does not kill the gauge fixing term and the integration over $q_\mu$ produces additional factor which survives in the limit $M \to \infty$. As was shown in ref. [8] to get the correct result one has to subtract this factor. The analog of eqs. (19,20) for the case of covariant Lorentz gauge looks as follows:

$$Z_{\text{inv}} \to Z_{\text{inv}}^{det^{-1/2}\nabla^2} \prod_j \left[ det_n^{-c/2}Q_{\beta=0}(A_\mu; M_j^2) \times det^{c/2}(\nabla^2 - M_j^2) \right]$$  \hspace{1cm} (24)

where:

$$det_n^{-1/2}Q_{\beta=0}(A_\mu; M^2) = \int \exp \left\{ i [\mathcal{R}_n(A_\mu, q_\mu) - \frac{M^2}{2} \int (q_\mu^a)^2 dx] \right\} \prod_x \delta(\nabla_\mu q_\mu) Dq_\mu(x)$$  \hspace{1cm} (25)

As in the case $\beta \neq 0$ the regularizing terms are obviously gauge invariant.

Taking into account that as was proven above:

$$\ln Z_\alpha = \ln Z_{\text{inv}} + \text{[finite part]} \hspace{1cm} (26)$$

one would expect that a natural gauge invariant regularization of $Z_\alpha$ may be chosen in the form:

$$Z^n_{\Lambda, M^2}[J] = \int \exp \left\{ i \int [\mathcal{L}_n^\Lambda + J_\mu^a A_\mu^a + \frac{1}{4\alpha} \text{tr} \left\{ f_n \left( \frac{\Box}{\Lambda^2} \right) \partial^\mu A_\mu \right\}^2 dx] \right\} \det \mathcal{M}(A)$$

$$\cdot det^{-1} F_n \left( \frac{\nabla^2}{\Lambda^2} \right) \prod_j \left[ det^{c/2}(\nabla^2 - M_j^2) det_n^{-c/2}Q_{\beta}(A_\mu; M_j^2, F_n) \right] \prod_x D A_\mu(x)$$  \hspace{1cm} (27)

However the straightforward application of this equation is ambiguous. The point is that individual diagrams generated by the perturbative expansion of the functional (27) still may be divergent. The formal derivation given above refers to the sum of all diagrams of a given order in perturbation theory with fixed numbers of external lines. The statement that $\ln Z_\alpha$ differs of $\ln Z_{\text{inv}}$ by finite terms is true for the sum of the diagrams of a given order and not for individual diagrams. To give a precise meaning to the expression (27) we have to specify the procedure of summation of divergent diagrams. In fact the same problem exists for the usual PV regularization and in this case it is solved in the following way.

Let us denote the propagator of particles with the mass $M$ by $D^{ab}_M(p^2)$ and the vertex factor by $\Gamma^{ab}$. PV regularized expression for a loop with $n$ vertices looks as follows:

$$\int dp [D_{M_1^2}^{a_1 b_1}(p) \Gamma^{b_1 a_2} \ldots \cdot D_{M_n^2}^{a_n b_n}(p + k_{n-1}) \Gamma^{b_n a_1} +$$

$$+ \sum_{j=1}^J c_j [D_{M_j^2}^{a_1 b_1}(p) \Gamma^{b_1 a_2} \ldots \cdot D_{M_j^2}^{a_n b_n}(p + k_{n-1}) \Gamma^{b_n a_1}]]$$  \hspace{1cm} (28)

It corresponds to the sum of similar loop diagrams describing the propagation of particles with masses $M, M_1, \ldots, M_J$. Two rules are assumed: a) all internal momenta have to be arranged in the same way in each diagram; b) the sum is to be taken before integration.
Unfortunately this simple recipe does not work for the eq.(27). The diagrams generated by $\mathcal{L}_n^\Lambda$ and by the PV fields have a different structure. To make sense of this expression one has to introduce some preregularization (PR) procedure which makes the individual diagrams finite and the summation unambiguous. In ref.[12] the momentum cutoff procedure for all internal lines was used as a PR. The necessity of PR in higher covariant derivative PV regularization was also emphasized in ref.[7].

After the summation is performed the PR has to be removed. Of course one must show that this procedure does not break gauge invariance and after the PR is removed the functional $Z_{\Lambda,M^2}^n[J]$ satisfies Slavnov-Taylor identities.

The second objection is related to the problem of diagrams with divergent subgraphs, in particular with overlapping divergencies. Although the one loop diagrams with external $A_\mu$ lines generated by the eq. (27) are finite, the divergencies may arise when one integrate over $A_\mu$. For example the UV index of the diagram (Fig. 1) is equal to $\omega = 2 + 2n - 4\deg F_n \geq 2$ (here $q_\mu$ are the fields which represent $\det^{-1/2}Q_\beta(A_\mu; M^2; F_n)$).

For finite $M_j$ this diagram is divergent. It was pointed out in ref. [4] that this problem will not appear if the propagators of the fields $A_\mu$ decrease for large momenta faster than the propagators of the fields $q_\mu$. It can be achieved if the degree of covariant derivatives in the action is higher than in the PV determinants. (Another possibility was discussed in ref. [12].)

In the next section we shall give answers to both these questions. Firstly we demonstrate that by choosing a special form of a higher derivative term one can avoid the problem of overlapping divergencies completely. Secondly we present an unambiguous expression for regularized functional which does not require any additional preregularization apart from the usual PV prescription discussed above.

### 3 Unambiguous definition of regularized functional. Removing of overlapping divergencies.

Let us choose the regularized action in the following form:

$$S^\Lambda = S_{YM} + \frac{1}{4\Lambda^{10}} \int \left[ \frac{\delta S_0}{\delta A_\rho(x)} \right]^2 dx$$

where:

$$S_0 = \int (\nabla^2 F_{\mu\nu})^2 dx$$

As a gauge condition we choose the regularized $\alpha$-gauge:
Then the one loop functional can be written in the form:

\[
Z_\alpha[A_\mu] = \int \exp \left\{ \frac{\lambda}{2} \int \left[ \frac{\delta^2 S_{YM}}{\delta A^a_\mu(x) \delta A^b_\nu(y)} q^{a}_\mu(x) q^{b}_\nu(y) dxdy + \frac{\lambda}{4\Lambda^2} \int [\partial_\mu q^{a}_\mu]^2 dxdy \right] \right\}
\]

where

\[
\frac{1}{4\Lambda^2} \int \left[ \nabla^\rho \Box^2 \partial_\mu A_\mu \right]^2 dxdy + \frac{1}{4\alpha} \int [\partial_\mu A_\mu]^2 dxdy
\] (31)

is due to the gauge fixing term (31). It is convenient for us to redefine it up to the nonessential constant as follows:

\[
\frac{\delta S_0}{\delta A_{\rho}(z)} = 0
\] (33)

and the cross term in the second line of the exponent (32) is equal to zero. The additional determinant \( \text{det}^{1/2} [\nabla^2 + \alpha^{-1} \Lambda^2] \) is due to the gauge fixing term (31). It is convenient for us to redefine it up to the nonessential constant as follows:

\[
\text{det}^{1/2} [\nabla^2 + \alpha^{-1} \Lambda^2] \rightarrow \text{det}^{1/2} [\nabla^2 + \alpha^{-1} \Box^2 \Lambda^2]
\] (34)

The expression (32) is of course still formal, as it generates ultraviolet divergent Feynman diagrams. To make the following transformations rigorous we have to introduce some preregularization which makes the integral (32) meaningful. We assume that a finite lattice a la Wilson [13] is introduced, which makes all the integrals both ultraviolet and infrared convergent without breaking gauge invariance. Any other gauge invariant regularization could do the job as well, but we prefer to consider the lattice regularization as it is universal and has a nonperturbative meaning. We emphasize that this preregularization is needed only to make all the arguments which follow rigorous. At the end the preregularization will be removed and the final recipe will be formulated without any references to a particular preregularization procedure. Having this in mind we shall not write explicitly the corresponding lattice expressions and will keep the continuous notations assuming that:

\[
\int d^4x \rightarrow a^4 \sum_x ; \quad \partial_\mu \phi(x) \rightarrow \frac{\phi(x + a_\mu) - \phi(x)}{a};
\]

etc.

Differentiation of \( \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\rho \Box^2 \partial A(z) \right]^2 \) in the expression (32) will produce two kinds of vertices:

\[
q^a_\mu(x) q^b_\nu(y) \frac{\delta^2}{\delta A^a_\mu(x) \delta A^b_\nu(y)} \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\rho \Box^2 \partial A(z) \right] \times \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\rho \Box^2 \partial A(z) \right]
\] (35)

and

\[
q^a_\mu(x) \frac{\delta}{\delta A^a_\mu(x)} \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\rho \Box^2 \partial A(z) \right] \times \frac{\delta}{\delta A^b_\nu(y)} \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\rho \Box^2 \partial A(z) \right] q^b_\nu(y)
\] (36)

Note that due to the gauge invariance of \( S \):

\[
\frac{\delta S_0}{\delta A_\rho(z)} = 0
\] (33)

\[
\text{det}^{1/2} [\nabla^2 + \alpha^{-1} \Lambda^2] \rightarrow \text{det}^{1/2} [\nabla^2 + \alpha^{-1} \Box^2 \Lambda^2]
\] (34)
Obviously the vertices of the type (35) after removing a preregularization produce only convergent diagrams as the maximal number of derivatives acting on the fields $q_\mu$ is equal to 6, and the $q$-field propagators decrease at $k \to \infty$ as $k^{-12}$.

Analogously one can show that the substitution:

$$
\int q_\mu(x) \frac{\delta}{\delta A_\mu(x)} \left[ \nabla_\rho \square^2 \partial A(z) \right] dx \to \nabla_\rho \square^2 \partial_\mu q_\mu(z)
$$

affects only convergent diagrams.

Therefore being interested only in the ultraviolet divergent part of $Z_\alpha$ we can rewrite it in the form:

$$
Z_\alpha^{\text{div}}[A_\mu] = \int \exp\left\{ \frac{i}{4\Lambda^{10}} \int dz \left[ \int \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} q_\mu(x) dx + \nabla_\rho \square^2 \partial_\mu q_\mu(z) \right]^2 \right\} \times \det \mathcal{M}(A) \det^{1/2} \left[ \nabla^2 + \alpha^{-1} \square^{-4} \Lambda^{10} \right] \prod_x Dq
$$

The notation $Z_\alpha^{\text{div}}$ means that when the preregularization is removed (i.e. in the continuum limit) the difference between the functional $Z_\alpha$, defined by the eq.(32) and $Z_\alpha^{\text{div}}$ is ultraviolet finite. At the moment we consider the model on the lattice, so the expression (38) is both ultraviolet and infrared finite by itself.

Integration of eq.(38) over $q_\mu$ produces:

$$
det^{-1/2} \left[ \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} + \nabla_\rho \square^2 \partial_\mu \delta(x - z) \right]^2 =
$$

$$
det^{-1} \left[ \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} + \nabla_\rho \square^2 \partial_\mu \delta(x - z) \right]
$$

(39)

Hence we can rewrite the eq.(38) in the form:

$$
Z_\alpha^{\text{div}}[A_\mu] = det^{-1} \tilde{K} \det \mathcal{M}(A) det^{1/2} \left[ \nabla^2 + \alpha^{-1} \square^{-4} \Lambda^{10} \right]
$$

(40)

where:

$$
det^{-1} \tilde{K} = \int \exp\left\{ \frac{i}{2\Lambda^{10}} \int dx dz \tilde{q}_\mu(x) \left[ \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} + \nabla_\mu \square^2 \partial_\rho \delta(x - z) \right] q_\rho(y) \right\} \prod_x Dq D\tilde{q}
$$

(41)

and $\tilde{q}_\mu, q_\mu$ are nonhermitean commuting fields.

The eq.(40) has a very important property. When integrated over $A_\mu$ it does not produce in the limit $a \to 0$ any new divergent diagrams. Indeed, the maximal number of derivatives acting on the fields $A_\mu$ is equal to 5, and the $A_\mu$-propagator decreases at $k \to \infty$ as $k^{-12}$. Therefore any diagram which contains at least one internal $A_\mu$-line is convergent. This is in contrast with the expression (20) which, as was discussed in the previous section, being integrated over $A_\mu$ do produce ultraviolet divergencies. This observation solves the problem of overlapping divergencies in the higher covariant derivative regularization.
Now we are ready to derive an unambiguous expression for the regularized functional. Let us write the functional generated by the regularized action (29) and the gauge fixing term (31) in the form:

\[
Z_\alpha[J] = \int \exp\left\{ i S_{YM} + i \int \left[ \frac{1}{4\alpha} (\partial_\mu A_\mu)^2 + J_\mu A_\mu - \frac{1}{2} h^2_\rho \right] dx + \right. \\
\left. + \frac{i}{2\Lambda^5} \int h_\rho(z) \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\mu \Box^2 \partial A(z) \right] dz \right\} \left[ \text{det} K \cdot \text{det}^{-1} \hat{K} \right] \times \\
\times \text{det } M(A) \text{det}^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^4 \Lambda^{10} \right] \prod_x Dh_\rho DA_\mu
\]

(42)

Indeed, integrating over auxiliary fields \( h_\rho \) and taking into account the identity (33) we get the functional corresponding to the regularized action (29) with the gauge fixing term (31). Here \( \text{det} \hat{K} \) is defined by the equation similar to eq.(41):

\[
\text{det}^{-1} \hat{K} = \int \exp\left\{ \frac{i}{2} \int \int dxdz q_\mu(x) \left[ \frac{1}{\Lambda^5} \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} + \frac{1}{\Lambda^5} \nabla_\mu \Box^2 \partial \delta(x - z) + \right. \\
\left. + \frac{1}{\Lambda} (\nabla_\mu \nabla_\rho - \nabla^2 g_\mu\rho) \delta(x - z) + \frac{\nabla_\mu}{\Lambda} \partial \delta(x - z) \right] q_\rho(z) \right\} \prod_x Dq D\rho
\]

(43)

This expression differs from (41) by underbraced terms which are introduced to provide the infrared convergence of \( \text{det}^{-1} \hat{K} \) in the limit when the preregularization is removed. They do not influence the ultraviolet asymptotics, and the ultraviolet divergent parts of \( \text{det} \hat{K} \) and \( \text{det} K \) coincide.

Let us show that in the limit \( \alpha \to 0 \) \( \text{det} K \) exactly compensates the one loop divergencies generated by the integration of the exponent in eq.(42).

The free propagators generated by the exponent in the eq.(42) have the following UV behaviour: \( \hat{A}_\mu A_\nu \sim k^{-12} \); \( \hat{h}_\rho h_\sigma \sim k^{-10} \); \( \hat{A}_\mu h_\rho \sim k^{-6} \). One sees that as soon as a diagram includes at least one \( \hat{A}_\mu A_\nu \) or \( \hat{h}_\rho h_\sigma \) propagator it is convergent. The only divergent diagrams are one loop diagrams formed by the propagator \( \hat{h}_\rho \hat{A}_\mu \). The sum of these diagrams can be presented as follows:

\[
\int \exp\left\{ \frac{i}{2\Lambda^5} \int \int dxdzh_\rho(z) \frac{\delta}{\delta A_\mu(x)} \left[ \frac{\delta S_0}{\delta A_\rho(z)} + \nabla_\mu \Box^2 \partial A(z) \right] q_\mu(x) \right\} \prod_x Dh Dq
\]

(44)

and according to the discussion given above coincides up to the finite terms with \( \text{det}^{-1} \hat{K} \). The divergent diagrams generated by eq.s.(43) and (44) have the same structure and therefore to provide the existence of the limit \( \text{det}^{-1} \hat{K} \text{det} K \) when the preregularization is removed it is sufficient to impose the usual PV prescription.

The only divergent factors in the eq.(42) are:

\[
\text{det}^{-1} K \text{det } M(A) \text{det}^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^4 \Lambda^{10} \right]
\]

(45)

Let us show that in analogy with the heuristic arguments of the preceeding section this expression can be written in the form which makes it’s divergent part manifestly gauge invariant.
The expression (45) can be rewritten in the form:

\[
Z^\text{div}_\alpha[A_\mu] = \int \exp \left\{ \frac{i}{2 \Lambda^5} \int \int \overline{q}_\mu(x) \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} q_\rho(z) dx dz + \right. \\
+ \frac{1}{\Lambda} \int \left[ -\nabla_\mu \overline{q}_\mu W^a + \overline{q}_\rho (\nabla_\rho \nabla_\mu - \nabla^2 g_{\rho\mu}) q_\mu \right] dx \right\} \det \mathcal{M}(A) \times \\
\times \det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \prod_x \delta \left( \left( \frac{\Box}{\Lambda^4} + 1 \right) \partial_\mu q_\mu - W \right) D\overline{q} Dq D\Lambda (46)
\]

To pass to a covariant background gauge we multiply \( Z^\text{div}_\alpha \) by "unity":

\[
det \nabla^2 \det F \left( \frac{\nabla^2}{\Lambda^2} \right) \int \prod_x \delta \left( F \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\mu (q_\mu + \nabla_\mu u) - W(x) \right) Du(x) = 1
\]

(\text{where } F \left( \frac{\nabla^2}{\Lambda^2} \right) = \frac{\Box}{\Lambda^4} + 1) \) and make the change of variables:

\[
q_\mu \rightarrow q_\mu - \nabla_\mu u
\]

After these transformations one can integrate over \( u, W \) to get:

\[
Z^\text{div}_\alpha = \int \exp \left\{ \frac{i}{2 \Lambda^5} \int \int \overline{q}_\mu(x) \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} (q_\rho - \nabla_\rho u) dx dz + \right. \\
+ \frac{i}{2 \Lambda} \int \left[ -\nabla_\mu \overline{q}_\mu F \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\mu q_\rho + \overline{q}_\rho (\nabla_\mu \nabla_\rho - \nabla^2 g_{\rho\mu}) q_\rho \right] dx \right\} \times \\
\times \det \nabla^2 \det F \left( \frac{\nabla^2}{\Lambda^2} \right) \det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \prod_x D\overline{q} Dq D\Lambda (49)
\]

where:

\[
u = \mathcal{M}^{-1} \left[ \partial_\mu q_\mu - \left( \frac{\Box}{\Lambda^4} + 1 \right)^{-1} F \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\mu q_\mu \right]
\]

In the same way as it has been done in the section 2 one can show that the terms proportional to \( \nabla_\mu u \) in the limit when the lattice preregularization is removed \( (a \to 0) \) generate only convergent diagrams. Therefore apart from the factor \( \det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \) the divergent in the limit \( a \to 0 \) part of \( Z^\text{div}_\alpha \) has a manifestly gauge invariant form.

Although \( \det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \) is not manifestly gauge invariant, it can be regularized in a gauge invariant way. To make this determinant finite in the limit \( a \to 0 \), it is sufficient to multiply it by gauge invariant PV determinants:

\[
\det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \rightarrow \det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \prod_j \det^{5/2} (\nabla^2 - M_j^2) (50)
\]

Therefore to get the functional which is finite in the limit \( a \to 0 \) we can introduce a gauge invariant interaction of PV fields:

\[
I_{PV} = \int \exp \left\{ \frac{i}{2} \int \int \overline{B}_\mu(x) \left[ \frac{1}{\Lambda^5} \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} + \frac{\nabla_\mu F \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\rho \delta(x - z)}{\Lambda} \right] B_\rho(z) dx dz + \right. \\
+ \frac{i}{2} \int \prod_x DB DB \prod_j \det^{3/2} (\nabla^2 - M_j^2) \det^{-1} F \left( \frac{\nabla^2}{\Lambda^2} \right) (51)
\]
Here $\overline{B}_\mu, B_\mu$ are anticommuting PV fields, and conditions (21,22) hold. Integration over $\overline{B}_\mu, B_\mu$ will produce the factor:

$$I_{PV} = det \left[ \frac{1}{\Lambda^5} \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} + \frac{\nabla_\mu F(\nabla^2 \Lambda)}{\Lambda} \nabla_\rho \delta(x - z) + M g_{\mu \rho} \delta(x - z) \right] \times$$

$$\times \prod_j det^{3/2c_j} (\nabla^2 - M_j^2) det^{-1} F \left( \frac{\nabla^2}{\Lambda^2} \right)$$

(52)

which compensates the divergencies of the functional (49). Obviously in the limit $M \to \infty, \Lambda \to \infty$ all unphysical excitations decouple.

One sees that the sum of the diagrams generated by the functionals (52), (49) has a finite limit when $a \to 0$. Moreover in the limit $a \to 0$ the divergent diagrams generated by the functionals (52) and (49) have the same structure. So the auxiliary lattice preregularization can be omitted and to make the sum finite it is sufficient to use the standard PV prescription: the momenta in the similar diagrams have to be assigned in the same way. It allows to write an unambiguous finite expression for the regularized functional. As was shown above for a finite lattice preregularization we can replace in the functional (42) the factor:

$$det^{-1} \mathcal{K} det \mathcal{M}(A) det^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box - 4 \Lambda^4 \right]$$

(53)

by the expression (49), regularized by adding the gauge invariant PV terms (52). After that we can remove a preregularization. The limiting expression is finite provided the usual PV prescription is used. In other words one can forget about a preregularization at all and take the expression obtained in this way as a definition of the regularized functional.

Before writing the final result let us note that the nonlocal term in the eq.(49) proportional to $\mathcal{M}^{-1}$ does not contribute in the limit $\Lambda \to \infty$. For finite $\Lambda$ we have shown it produces finite diagrams, and in the limit $\Lambda \to \infty$ its contribution disappears. Being interested finally in the limit $\Lambda \to \infty$ we can omit this term in the eq.(49). It simplifies the expression for the regularized functional and make the effective action local. Omitting this term we break the gauge invariance for finite $\Lambda$. The Slavnov-Taylor identities will be violated by finite terms of order $O(\Lambda^{-1})$. These terms are harmless as they have no influence on the counterterms and disappear in the limit $\Lambda \to \infty$.

Having in mind these remarks we can write the unambiguous expression for the regularized functional which does not require any special preregularization procedure. It looks as follows:

$$Z_{\Lambda,M}[J] = \int \exp \left\{ i S_{YM} + \iota \int \frac{1}{2 \Lambda^5} \int \left[ \frac{\delta^2 S_0}{\delta A_\rho(z)} + \nabla_\rho \Box^2 \partial A(z) \right] h_\rho(z) dz + \right.$$  

$$+ \int \frac{1}{4\alpha} (\partial_\mu A_\mu)^2 + J_\mu A_\mu - \frac{1}{2} h^2(x) dx + \iota \int \int b_\mu(x) \left[ \frac{1}{\Lambda^5} \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} \right] +$$

$$+ \frac{\nabla_\mu}{\Lambda} \left( \frac{\Box^2}{\Lambda^4} + 1 \right) \partial_\rho \delta(x - z) + \frac{1}{2} (\nabla_\mu \nabla_\rho - \nabla^2 g_{\mu \rho}) \delta(x - z) b_\rho(z) dx dz \right\} \times$$

$$\times \exp \left\{ \frac{\iota}{2} \int \int \overline{q}\mu(x) \left[ \frac{1}{\Lambda^5} \frac{\delta^2 S_0}{\delta A_\mu(x) \delta A_\rho(z)} \right] + \frac{\nabla_\mu}{\Lambda} F \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_\rho \delta(x - z) +$$

$$+ \frac{\nabla_\mu}{\Lambda} \left( \frac{\Box^2}{\Lambda^4} + 1 \right) \partial_\rho \delta(x - z) + \frac{1}{2} (\nabla_\mu \nabla_\rho - \nabla^2 g_{\mu \rho}) \delta(x - z) b_\rho(z) dx dz \right\} \times$$

$$\times \prod_j det^{3/2c_j} (\nabla^2 - M_j^2) det^{-1} F \left( \frac{\nabla^2}{\Lambda^2} \right)$$
\[ + \frac{1}{\Lambda} (\nabla_{\mu} \nabla_{\rho} - \nabla^2 g_{\mu\rho}) \delta(x-z) q_\rho(z) dxdz + \frac{1}{2\Lambda} \int \int B_\mu(x) \left[ \frac{1}{\Lambda^4} \delta A_\mu(x) \delta A_\rho(z) \right] + \]
\[ + \nabla_{\mu} F \left( \frac{\nabla^2}{\Lambda^2} \right) \nabla_{\rho} \delta(x-z) B_\rho(z) dxdz + iM \int \int B_\mu B_\rho dx \right) \text{det}^{1/2} \left[ \nabla^2 + \alpha^{-1} \Box^{-4} \Lambda^{10} \right] \times \]
\[ \times \text{det} \nabla^2 \prod_{j} \text{det}^{3/2} \left( \nabla^2 - M_j^2 \right) \prod_{x} Dh_\mu D\overline{b}_\mu Db_\mu Da_\mu D\overline{q}_\mu Dq_\mu DB_\mu D\overline{B}_\mu D\overline{A}_\mu \]

(54)

This rather lengthy expression has in fact a simple meaning. The integral over the anticommuting fields \( \overline{b}_\mu, b_\rho \) subtract the divergent one loop diagrams which arise due to integration over \( A_\mu, h_\rho \). The integral over PV fields \( \overline{B}_\mu, B_\rho \) subtract analogous divergencies which arise due to integration over fields \( \overline{q}_\mu, q_\rho \). As the propagators of the fields \( A_\mu \) decrease for \( k \to \infty \) as \( k^{-12} \), no overlapping divergencies are present.

Let us remind how the expression (54) was obtained. We firstly transformed identically the preregularized functional which satisfied the correct Slavnov-Taylor identities. Then we multiplied it by a gauge invariant factor depending on PV fields. Obviously the resulting functional satisfies the same identities for any finite preregulator \( a \), and as the limit \( a \to 0 \) exists, in the limit \( a \to 0 \) as well. The only procedure which could break the gauge invariance was omitting of the nonlocal term \( \sim M^{-1} \). But as we discussed above it has no influence on the final result.

The functional (54) has unambiguous meaning as all the diagrams generated by the expansion of eq.(54) are finite, provided the momenta of similar diagrams are assigned in the same way. Therefore as we have already discussed we can take it as a definition of regularized functional forgetting completely about a preregularization.

4 Discussion.

In this paper we showed that contrary to the statement of the authors [7] the higher covariant derivative regularization supplemented by the additional PV type regularization of one loop diagrams do provide a consistent regularization of QCD and other gauge invariant models. It can be used as a practical method of calculations in theories where dimensional regularization is not applicable, and may also serve as a starting point for nonperturbative approaches. The formulation of the method given in the present paper avoids the problem of overlapping divergencies and gives unambiguous method of calculations which do not require any additional preregularization.

References


13


