Relaxation dominated cosmological expansion

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Abstract

The behavior near the singularity of an isotropic, homogeneous cosmological model with a viscous fluid source is investigated. This turns out to be a relaxation dominated regime. Full extended irreversible thermodynamics is used, and comparison with results of the truncated theory is made. New singular behaviors are found and it is shown that a relaxation dominated inflationary epoch may exist for fluids with small heat capacity.

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There are many processes capable of producing important dissipative stresses in the early universe. This includes interactions between matter and radiation [1], quarks and gluons [2], different components of dark matter [3], and those mediated by massive particles. It happens also due to the decay of massive superstrings modes [4], gravitational particle production [5] [6] and phase transitions. Phenomenologically, these processes may be modeled in terms of a classical bulk viscosity. The dynamics of the viscous pressure term is ruled by a transport equation, and we use the expression given by the Extended Irreversible Thermodynamics theory [7] [8].

\[ \sigma + \tau \dot{\sigma} = -3\zeta H - \frac{\epsilon}{2} \tau \sigma \left( 3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} \right) \]  

(1)

where we are assuming a spatially flat Robertson-Walker metric, \( \sigma \) is the bulk viscous pressure, \( H = \dot{a}/a \) the Hubble variable, \( \tau \) is the relaxation time, \( \zeta \) is the bulk viscous coefficient, \( T \) is the temperature, and \( \epsilon = 0 \) corresponds to the truncated theory while \( \epsilon = 1 \) corresponds to the full theory. Equation (1) yields a causal and stable behavior due to the memory mechanism provided by the relaxation time. We assume that \( \zeta = \alpha \rho^m \), \( \tau = \zeta/\rho \) and \( T = \kappa \rho^r \), where \( \alpha > 0 \), \( m \), \( \kappa > 0 \) and \( r > 0 \) are constants [9].

We will examine in this Letter the evolution of the dissipative universe near a singularity, which is a relaxation dominated era. In effect, the Einstein equations are

\[ H^2 = \frac{1}{3} \rho \]  

(2)

\[ \dot{H} + 3H^2 = \frac{1}{2} (\rho - p - \sigma) \]  

(3)

where \( \rho \) is the energy density, \( p \) is the equilibrium pressure and we take the equation of state \( p = (\gamma - 1)\rho \) with a constant adiabatic index \( 0 \leq \gamma \leq 2 \). For a singularity in \( t_0 \), we will consider two types of leading behaviors when \( t \to t_0 \), \( H \sim 1/\Delta t^n \), where \( n > 0 \) is a constant, and \( H \sim 1/(\Delta t \ln \Delta t) \). For \( H \sim 1/\Delta t^n \), \( \sigma \propto H^2 \sim 1/(\Delta t)^{2n} \) for \( n \geq 1 \), and \( \sigma \propto \dot{H} \sim 1/(\Delta t)^{n+1} \) for \( n \leq 1 \). So, \( |\sigma/(\zeta H)| \sim \Delta t^{n(2m-1)} \) for \( n \geq 1 \), and \( |\sigma/(\zeta H)| \sim \Delta t^{2nm-1} \) for \( n \leq 1 \). Also \( \ddot{\tau}/\tau \sim \dot{\zeta}/\zeta \sim \dot{T}/T \sim 1/\Delta t \). Thus we find that the first term of the left hand side of (1) becomes much smaller than the rest of the terms provided that \( m > 1/2 \) and \( \tau \gg |\Delta t| \). Similar considerations are valid for the
second type of leading behavior. Then, using (1), (2) and (3), and neglecting the viscous term $\sigma$, we obtain

$$\ddot{H} - \epsilon(1 + r)\frac{H^2}{H} + 3 \left\{ \gamma + \frac{\epsilon}{2} \left[ 1 \times (1 + r) \gamma \right] \right\} H\dot{H} + \frac{9}{4} (\epsilon \gamma - 2) H^3 = 0$$

We investigate the solutions of this equation with one of these two types of behavior near the singularity, as the solutions of the non-approximated equation will have the same leading behavior for $\Delta t \to 0$. The study of these solutions for large time will be made elsewhere (see also [7]). As the relaxation regime has already been considered in the truncated theory [10], we will concentrate on the case $\epsilon = 1$. We also note that an equation with the same structure arises when considering the semiclassical Einstein equation with vacuum polarization terms near the singularity [11].

To verify the consistency of our approximation, we check first whether equation (4) has solutions of the form $a = a_0 \Delta t^\nu$. In effect there is a region of the parameter plane $(\gamma, r)$, such that two families of solutions of this form exist, with $\nu$ given by

$$\nu_\pm = \frac{1}{3(2 - \gamma)} \left\{ -1 + (1 - r) \gamma \pm \left[ (1 + (1 - r) \gamma)^2 - 4(2 - \gamma)(r - 1) \right]^{1/2} \right\}$$

This region is determined by the requirement $D > 0$, where $D$ is the discriminant in (5), and this condition can be satisfied for $\gamma < 2$. On the curve $D = 0$ there is only one family of solutions, and for $\gamma = 2$ there is also one family of solutions with $\nu_2 = 2(1 - r)/(3(3 - 2r))$.

To investigate further the two-parameter families of solutions of (4), it is convenient to make the change of variable $H = y^{-1/r}$, which turns this equation into

$$\ddot{y} + \frac{3}{2} \left( 1 + (1 - r) \gamma \right) y^{-\frac{3}{r}} \dot{y} + \frac{9}{4} (2 - \gamma) ry^{1 - \frac{3}{r}} = 0$$

In the case that $1 + (1 - r) \gamma \neq 0, r \neq 1$ we solve (6) applying a generalization of the transformation used in [12]

$$z = \frac{r}{r - 1} y^{\frac{r - 1}{r}}, \quad dz = \frac{3}{2} \left[ 1 + (1 - r) \gamma \right] y^{-\frac{1}{r}} dt$$
which linearizes (6)

\[ \frac{d^2z}{d\eta^2} + \frac{dz}{d\eta} + \beta z = 0 \]  

(7)

where \( \beta = \frac{(2-\gamma)(r-1)}{[1 + (1 - r)\gamma]^2} \). Thus, we obtain the general solution of (4) in parametric form

\[ H(\eta) = \left[ \frac{r-1}{r} z(\eta) \right]^{-\frac{1}{r-1}} \]

\[ \Delta t(\eta) = \frac{2}{3[1 + (1 - r)\gamma]} \int d\eta H(\eta)^{-1} \]  

(8)

This parametric form is very useful to obtain the singular behavior in the region \( D \leq 0 \), where no explicit solution is available. In this region \( r > 1 \), so that \( H \) diverges when \( z \to 0 \). Moreover, as this region corresponds to the oscillating or critically damped regime of the oscillator described by (7), when \( \gamma < 2 \), \( z(\eta) \propto \Delta \eta \) for \( \Delta \eta \to 0 \). Then we find \( H \sim C/\Delta t^{\frac{r}{r-1}} \) or

\[ a(t) \simeq a_0 \exp \left( K \Delta t^{\frac{r}{r-1}} \right), \quad \Delta t \to 0 \]  

(9)

where \( a_0 \) and \( K \) are arbitrary integration constants. This behavior exhibits a singularity at a finite value of the scale factor.

In the case that \( 1 + (1 - r)\gamma = 0 \) and \( \gamma < 2 \), equation (6) describes the motion of a point particle in a potential \( V(y) \propto y^{2(r-1)/r} \) and reduces to quadratures

\[ \Delta t = \frac{r}{\sqrt{2}} \int dH \frac{H^{-r-1}}{[E - GH^{2(r-1)}]^{1/2}} \]  

(10)

where \( E \) is an integration constant and \( G = 9(2 - \gamma)(1 + \gamma)^2/(8\gamma) \). This solution may be expressed in terms of the hypergeometric function, and we find that the singular behavior is also (9), where \( r = (1 + \gamma)/\gamma \).

On the other hand, for \( D > 0 \), explicit two-parameter families of solutions of (6) exist provided that the constrain [13]

\[ \frac{(2 - \gamma)r}{[1 + (1 - r)\gamma]^2} = \left( \frac{2 - 1}{r} \right)^{-2} \]  

(11)
is satisfied. They yield a representative sample of two-parameter families of solutions in this region, and have the form

\[ a(t) = a_0 \left| \Delta t \right|^{\frac{r-1}{r}} + K^{\nu_\gamma}, \quad r \neq 1 \] (12)

\[ a(t) = a_0 \left| \ln \Delta t \right| + K^{\frac{-2}{3}}, \quad r = 1 \] (13)

where \( \nu_\gamma = \frac{2(1 - 2r)}{3(1 + (1 - r)\gamma)} \). We find in this way four kinds of singular behaviors, as shown in the following table:

<table>
<thead>
<tr>
<th>( a(t) )</th>
<th>( \lim a(t) )</th>
<th>Parameter range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta t^{\nu_+} )</td>
<td>0</td>
<td>( 0 &lt; r &lt; 1 ) and ( r &gt; 3/2 )</td>
</tr>
<tr>
<td>( \Delta t^{\nu_-} )</td>
<td>( \infty )</td>
<td>( 0 &lt; r &lt; 1 ) and ( 1 &lt; r &lt; 3/2 )</td>
</tr>
<tr>
<td>( \left</td>
<td>\ln \Delta t \right</td>
<td>^{\nu_\gamma} )</td>
</tr>
<tr>
<td>( \exp(K\Delta t^{\nu_\gamma}) )</td>
<td>( a_0 )</td>
<td>( r &gt; 1 )</td>
</tr>
</tbody>
</table>

Finally, for \( \gamma = 2 \), equation (6) also reduces easily to quadratures

\[ \Delta t = r \int dH \frac{H^{-r-1}}{SH^{1-r} - E} \] (14)

where \( S = 3r(3 - 2r)/(2r - 2) \), and this solution be expressed in terms of the hypergeometric function. We find that their singular behavior is \( \Delta t^{\nu_2} \) for \( 0 < r < 1 \), \( \Delta t^{-2/3} \) for \( r = 1 \), and (9) for \( r > 1 \).

We have investigated the singular behavior of the full causal model for \( m > 1/2 \). Similarly to the truncated causal model, we find that it is dominated by the relaxation terms of the transport equation. However we have found that a greater variety of singular behaviors appear in the full model, and it is the dependence of the energy density on the temperature which determines the kind of singular behavior of two-parameter families of solutions.

Big-bang and explosive singularities already appeared in the truncated theory. Following the same steps as before and assuming \( H = \bar{\nu}/\Delta t \) we find \( \bar{\nu}_\pm = (1/3) \left[ \gamma \pm (\gamma^2 + 4)^{1/2} \right] \) (cf. [10]). We note that \( 0 < \nu_+ < \bar{\nu}_+ < 2/(3\gamma) \) for \( 0 < r < 1 \), while \( \nu_+ \geq 4/3 > \bar{\nu}_+ \) and \( \nu_+ > 2/(3\gamma) \) for \( r > 3/2 \). Thus we can state that for a fluid with large heat capacity, the rate of expansion of the scale factor after a Big-bang singularity is smaller than calculated in the truncated theory which is turn smaller than the perfect fluid rate. In this case, there are particle horizons. However, when the heat capacity is
small, the rate of expansion turns out much larger for a viscous fluid than a perfect fluid. Moreover, it turns out to be a relaxation dominated inflationary regime.

On the other hand, $\nu_\gamma < \bar{\nu}_\gamma < 0$ for $0 < r < r_c$ and $\bar{\nu}_\gamma < \nu_\gamma < 0$ for $r_c < r < 3/2$, where $r_c$ depends on $\gamma$. Thus, explosive singularities may be stronger or milder depending on the heat capacity being large or small. A finite scale factor singularity arises for $r > 1$, and a logarithmic explosive singularity occurs for $r = 1$. Neither of them appear in the truncated model, though finite scale singularities has been found in noncausal models for $m > 1/2$ [14] [15].
References


