UNIVERSITÄT LEIPZIG

Naturwissenschaftlich-Theoretisches

Zentrum

A Remark on Bound States in
Conical Spacetime

by

M. Bordag and N. Khusnutdinov

Universität Leipzig
Fakultät für Physik und Geowissenschaften
Institut für Theoretische Physik
Augustusplatz 10/11
04109 Leipzig
BRD

SCAN-9601161

CERN LIBRARIES, GENEVA

SC 9601

Preprint-Nr. 35/1995

Abstract

We consider the question about bound states of spin- $\frac{1}{2}$ and spin-0 particles in the vicinity of infinitely thin straight cosmic string owing to self interaction forces. It is shown that there are no bound states for charged particles, and that there exist bound states for neutral particles.

*On leave from Kazan State Pedagogical University, Department of Theoretical Physics, Mezhlauk str.1, 420021, Kazan, Russia

[†]e-mail:nail@ntp.ksu.ras.ru

A cosmic string is a solution of the Einstein equations with a locally flat spacetime everywhere except on the location of the string where it can be characterized by an angle deficit $\Delta = 8\pi G\mu/c^2$ where μ is mass per unit length of string [1, 2]. A testbody in the vicinity of the string experiences no direct gravitational force [2]. Taking into account the gravitational field of the testbody itself or a possible charge of it there arise forces between the testbody and the string. A interesting question is about possible bound states due to these forces. The main purpose of this letter is to show that there are bound states due to the gravitational forces and that there are no bound states due to the electromagnetic forces. The latter is in opposite to a recent paper [3].

The spacetime of an infinite thin cosmic string is described by the flat metric

$$ds^2 = c^2 dt^2 - d\rho^2 - \rho^2 d\varphi^2 - dz^2, \quad 0 \leq \varphi \leq 2\pi/\nu, \quad \nu \geq 1, \quad (1)$$

everywhere except on the line $\rho = 0$ accomplished by the condition $\psi(\varphi + 2\pi/\nu) = \psi(\varphi)$ for all fields ψ , where $\nu = (1 - 4G\mu/c^2)^{-1}$. Due to the flatness of the metric (1) a testparticle moves free. But taking into account its own gravitational field there results a small attractive force between the particle and the string [4]. It can be described by the potential energy

$$V_{grav} = -B \frac{Gm^2}{\rho}. \quad (2)$$

The classical motion allows closed orbits around the string [4]. Also, there are numerical calculations for the parameter B . It is always positive.

In the case of a charged particle, taking into account its electrostatic self energy there is a repulsive force [5, 6] between the particle and the string, which can be described by the potential energy

$$V_{elstat} = L \frac{e^2}{\rho}. \quad (3)$$

In general, the electrostatic self energy of a point particle is a infinite quantity. In the conical background it consists of an infinite, but constant part which is thrown away and the finite part (3) which depends on ρ . From a more general point of view this should be the static approximation to the 1-loop selfenergy of the particle in the given background. The parameter L has been calculated in [5]. It is always positive. The limiting values are $L \sim (\nu - 1)$ for a small angle deficit ($\nu - 1 \sim 10^{-6}$ for GUT strings, for instance) and $L \sim (\nu/\pi) \ln(2\nu/\pi)$ for $\nu \gg 1$ (for a supermassive cosmic string [6]).

This situation is in a closed analogy to the Aharonov-Bohm case where one considers a charged particle in the vicinity of a thin solenoid (i.e., a magnetic string). Also, in that case the spacetime is locally flat (there is no electromagnetic field outside the string in an appropriate gauge) and a repulsive force results from the topological nontriviality expressed by the condition $\psi(\varphi + 2\pi) = \psi(\varphi)e^{2\pi i\Phi}$ on the wave function ψ , where Φ is the flux. Let us note that in this case bound states

appear only when one considers an additional interaction. So, for instance, the interaction of the magnetic moment of the particle with the magnetic field inside the string which is attractive for the corresponding spin states leads to zero modes in the case of a gyromagnetic ratio $g = 2$ [7] and to bound states for $g > 2$ [8].

The force resulting from the potential V_{elstat} (3) is repulsive regardless to the sign of the charge. We consider the motion of a charged spin- $\frac{1}{2}$ particle. It can be described by the Dirac equation

$$(\gamma^i(x)\widetilde{\nabla}_i + m + U)\Psi = 0 , \quad (4)$$

where $\widetilde{\nabla}_i = \nabla_i + \Gamma_i$, $\Gamma_i = \frac{1}{2}\Sigma^{ab}\lambda_{(a)}^k\nabla_i\lambda_{(b)k}$, $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$.

The effect resulting from the additional forces V_{elstat} (3) and V_{grav} (2) can be taken into account phenomenologically by inserting the potential $U = Z/\rho$ into (4), where either $Z = L\alpha$ ($\alpha = e^2/\hbar c$) or $Z = -B\beta$ ($\beta = Gm^2/\hbar c$) (further we set $\hbar = G = c = 1$).

Concerning the electrostatic self force this is the key point where the authors of the paper [3] are in error. They added the potential as the zeroth component A_0 of a vector. This results in an attractive force for one of the particles and, consequently, yields the bound states proposed in that paper. This is clearly wrong. In fact, their procedure is rather equivalent to considering the motion of a charged particle in the field of a charged cosmic string (when redefining the constant L in (3) and substituting $1/\rho \rightarrow \ln \rho$).

In the following we consider the equation (4) with the potentials (3) and (2). In the first case we show that there are no bound states (although there is an attractive region in the effective potential) and in the second case we calculate the bound state levels. It is interesting to note there is an analogous situation when considering the motion of a charged particle in the field of a point charge in three dimensions with vector and scalar coupling [9].

We take the vierbein $\lambda_{(a)}^i$ in the following form [10]

$$\begin{aligned} \lambda_{(0)}^i &= (1, 0, 0, 0) , \\ \lambda_{(1)}^i &= (0, \cos \nu\varphi, -\sin \nu\varphi/\rho, 0) , \\ \lambda_{(2)}^i &= (0, \sin \nu\varphi, \cos \nu\varphi/\rho, 0) , \\ \lambda_{(3)}^i &= (0, 0, 0, 1) , \end{aligned} \quad (5)$$

and we shall use the standard representation of the γ matrixes. It is easy to verify that the three operators

$$\begin{aligned} \hat{H} &= \alpha^3\hat{p}_3 + \alpha^\rho(x)\hat{p}_\rho + \alpha^\varphi(x)\hat{p}_\varphi + \gamma^0(m + U) ; \quad \hat{H}\Psi = E\Psi , \\ \hat{J}_3 &= \hat{L}_3 + \Sigma_3/2 = \hat{p}_\varphi + \Sigma_3/2 ; \quad \hat{J}_3\Psi = \nu M\Psi , \\ \hat{p}_3 &= -i\partial_z ; \quad \hat{p}_3\Psi = p_3\Psi , \end{aligned} \quad (6)$$

commute each with other. Here

$$\hat{p}_\rho = -i\partial_\rho, \hat{p}_\varphi = -i\partial_\varphi + (\nu - 1)\Sigma_3/2 , \quad \Gamma_i = \delta_i^\varphi(\nu - 1)i\Sigma_3/2 ,$$

$$\begin{aligned}\alpha^\rho(x) &= \alpha^1 \cos \nu\varphi + \alpha^2 \sin \nu\varphi, \alpha^\varphi(x) = (-\alpha^1 \sin \nu\varphi + \alpha^2 \cos \nu\varphi)/\rho, \\ M &= l - 1/2, \quad l = 0, \pm 1, \dots, \quad -\infty < p_3 < +\infty.\end{aligned}$$

The fourth quantum number $s = \pm 1$ will be specified later.

Using equations (6) we can represent the wave function in the form

$$\Psi = e^{-iEt + ip_3 z + i\nu(M-1/2)\varphi} \frac{1}{\sqrt{\rho}} \begin{pmatrix} a(\rho) \\ -i\bar{b}(\rho) \exp(i\nu\varphi) \\ \bar{a}(\rho) \\ ib(\rho) \exp(i\nu\varphi) \end{pmatrix}, \quad (7)$$

where the four radial functions obey the equations

$$\begin{aligned}a' - \frac{\nu M}{\rho} a - p_3 \bar{b} + (E + m + U)b &= 0, \\ b' + \frac{\nu M}{\rho} b + p_3 \bar{b} - (E - m - U)a &= 0, \\ \bar{a}' - \frac{\nu M}{\rho} \bar{a} + p_3 b - (E - m - U)\bar{b} &= 0, \\ \bar{b}' + \frac{\nu M}{\rho} \bar{b} - p_3 a + (E + m + U)\bar{a} &= 0.\end{aligned} \quad (8)$$

This system of equations possesses the discrete symmetry

$$\bar{a} = \chi a, \quad \bar{b} = \chi b, \quad \chi = (E + s\sqrt{E^2 - p_3^2})/p_3, \quad s = \pm 1. \quad (9)$$

Thus, the quantum state is described by the four numbers (E, M, p_3, s) . Using the above symmetry (9) we have the following system for the radial functions a and b

$$\begin{aligned}\left\{ \frac{d^2}{d\rho^2} - \frac{\nu M(\nu M - 1)}{\rho^2} + (E^2 - p_3^2 - (m + U)^2) \right\} a + \frac{dU}{d\rho} b &= 0 \\ \left\{ \frac{d^2}{d\rho^2} - \frac{\nu M(\nu M + 1)}{\rho^2} + (E^2 - p_3^2 - (m + U)^2) \right\} b + \frac{dU}{d\rho} a &= 0.\end{aligned} \quad (10)$$

The new functions

$$Y_\mp = a + \frac{\nu M \mp \gamma}{Z} b, \quad \gamma = \sqrt{(\nu M)^2 + Z^2}, \quad (11)$$

obey the separate equations

$$Y''_\mp + (E^2 - p_3^2 - m^2 - \frac{2mZ}{\rho} - \frac{\gamma(\gamma \mp 1)}{\rho^2}) Y_\mp = 0. \quad (12)$$

These equations have the form of a Schrödinger equation for a particle with the energy $\mathcal{E} = (E^2 - p_3^2 - m^2)/2m$ and the effective potential

$$U_{eff} = \frac{Z}{\rho} + \frac{\gamma(\gamma \mp 1)}{2m\rho^2}. \quad (13)$$

This potential looks like a Coulomb potential with the coupling Z and the orbital momentum γ (in three dimensions). In the case of a charged particle the constant Z is positive. The effective potential (13) is attractive for $\rho < \gamma(1 - \gamma)/2mZ$ in the case when $\gamma < 1$ and for the upper sign (lowest orbital excitation). Let us suppose that there is a bound state solutions of equation (12). It can be expressed by Whittaker's functions [11]. We have to choose

$$Y_{\pm} = CW_{k,\mu}(2\lambda\rho) , \quad k = -\frac{mZ}{\lambda} , \quad \mu = \gamma \mp \frac{1}{2} \quad (14)$$

- the decreasing at infinite solution. From the behaviour at $\rho \rightarrow 0$

$$W_{k,\mu}(2\lambda\rho) \stackrel{\rho \rightarrow 0}{\sim} \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - k)}(2\lambda\rho)^{\mu+1/2} + \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - k)}(2\lambda\rho)^{-\mu+1/2} ,$$

in order to have a normalizable solution we can require that the coefficient before the increasing part vanishes

$$\frac{1}{2} + \mu - k = \frac{1}{2} + \gamma \mp \frac{1}{2} + \frac{mZ}{\lambda} = -N , \quad N = 0, 1, \dots \quad (15)$$

But this condition cannot be fulfilled because of

$$\min(\gamma) = \min \sqrt{(\nu M)^2 + Z^2} = \sqrt{\nu^2/4 + Z^2} > \nu/2 > 1/2 . \quad (16)$$

Consider the normalization integral of the complete solution :

$$\int \Psi^+ \Psi d^3x \propto \int_0^\infty d\rho \left\{ \left(1 + \frac{\nu M}{\gamma}\right) |Y_-|^2 + \left(1 - \frac{\nu M}{\gamma}\right) |Y_+|^2 \right\} . \quad (17)$$

Due to the behaviour at $\rho \rightarrow 0$ it can converge only when $2(\mu - 1/2) < 1$ for $\mu = \gamma \pm 1/2$ (for both signs) which is impossible due to $\gamma > 1/2$, or when $(1 - \nu M/\gamma) = 0$ which cannot be fulfilled too. Therefore no bound states are possible. Let's note that this fact is not entirely trivial. A potential (13) can have bound states in the general case and its absence is due to the specific dependence on the parameters we have in this case.

For a noncharged particle we have $Z = -B\beta < 0$ and there exist solutions of equations (15) with the energy spectrum

$$E = \pm \sqrt{p_3^2 + m^2} \left[1 - \frac{m^2}{p_3^2 + m^2} \frac{(B\beta)^2}{(N + \sqrt{(\nu M)^2 + (B\beta)^2})^2} \right]^{1/2} . \quad (18)$$

For particle moving in the plane $x^3 = \text{const}$, that is with $p_3 = 0$, the above spectrum coincides with the one for an electron in the Newtonian gravitation field [9].

In the end of this letter we briefly consider the above questions for spin-0 particle. The motion of this particle is described by the Klein - Gordon equation with scalar connection

$$\{g^{ik} \nabla_i \nabla_k + (m + U)^2\} \phi = 0 . \quad (19)$$

Using standard methods one can represent the solution in the form

$$\phi = e^{-iEt + ip_3 z + iM\nu\varphi} \frac{1}{\sqrt{\rho}} R(\rho) , \quad (20)$$

where $M = 0, \pm 1, \pm 2, \dots$ is the orbital quantum number ; $-\infty < p_3 < +\infty$ and the function $R(\rho)$ obeys the radial equation

$$R'' + \left(E^2 - p_3^2 - m^2 - \frac{2mZ}{\rho} - \frac{Z^2 + (M\nu)^2 - \frac{1}{4}}{\rho^2} \right) R = 0 . \quad (21)$$

Proceeding in the same manner as after eq.(12) we obtain a bound state solution of this equation with the condition

$$\frac{1}{2} + \sqrt{Z^2 + (M\nu)^2} + \frac{mZ}{\lambda} = -N , \quad N = 0, 1, \dots , \quad (22)$$

In the case of electrostatic self force ($Z > 0$) it does not have a solution although in this case again the effective potential has an attractive region (for $M = 0$). In the case of the gravitational self force ($Z < 0$) we obtain bound states whose energy is given by formula (18) with the replacement $N \rightarrow N + 1/2$.

We considered in this letter only the existence of bound states. Interesting questions are about possible cosmological implications, about considering cosmic strings with a finite radius in this context and about a better approximation to the electromagnetic self force which is given equation (3) in a static approximation only. Especially, one could speculate whether retardation effects could lead to an attractive force as they did in the case of the anomalous magnetic momentum of the electron.

The authors thank V Frolov and D Gal'tsov for useful discussions. One of us (NK) would like to thank the DAAD for support and the ITP of Leipzig University for hospitality.

References

- [1] Kibble T W B 1976 *J. Phys. A.* **9** 1387
- [2] Vilenkin A 1981 *Phys. Rev. D.* **23** 852
- [3] Bezerra V B and Araújo I G 1994 *Class. Quant. Grav.* **11** 1599
- [4] Gal'tsov D V 1990 *Fortschr. Phys.* **38** 945
- [5] Linet B 1986 *Phys. Rev. D.* **33** 1833
- [6] Khusnutdinov N R 1994 *Class. Quant. Grav.* **11** 1807
Khusnutdinov N R 1995 *Theor. Math. Phys.* **103** 339
- [7] Aharonov Y and Casher A 1979 *Phys. Rev. A.* **19** 2461

- [8] Bordag M and Voropaev S 1994 *Phys. Lett. B.* **333** 238
Bordag M and Voropaev S 1993 *J. Phys. A.* **26** 7637
- [9] Soff G, Müller B, Rafelsky J and Greiner W 1973 *Z. Naturforsch.* **28a** 1389
- [10] Skarzhinsky V D , Harari D D and Jasper U 1994 *Phys. Rev. D.* **49** 755
- [11] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
- [12] Garfinkle D 1985 *Phys. Rev. D.* **32** 1323