Unitarity of rational $N = 2$ superconformal theories

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Abstract

We demonstrate that all rational models of the $N = 2$ super Virasoro algebra are unitary. Our arguments are based on three different methods: we determine Zhu’s algebra $A(H_0)$ (for which we give a physically motivated derivation) explicitly for certain theories, we analyse the modular properties of some of the vacuum characters, and we use the coset realisation of the algebra in terms of $su(2)$ and two free fermions. Some of our arguments generalise to the Kazama-Suzuki models indicating that all rational $N = 2$ supersymmetric models might be unitary.

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1 Introduction

Among the various conformal field theories, the supersymmetric field theories play a special rôle as they are important for the construction of realistic string theories which involve fermions. There exist different classes of superconformal field theories which are parametrised by \(N\), the number of fermionic (Grassmann) variables of the underlying space. For realistic string theories with \(N = 1\) space-time supersymmetry, the world-sheet conformal field theory is believed to require \(N = 2\) supersymmetry.

In contrast to the \(N = 1\) super Virasoro algebra which is rather similar to the non-supersymmetric \((N = 0)\) algebra, the \(N = 2\) algebra seems to be structurally different. For example the Neveu-Schwarz and Ramond sector of the \(N = 2\) algebra are connected by the spectral flow [39], and the embedding structure of its Verma modules is much more complicated [9, 10]. In this paper another special feature of the \(N = 2\) superconformal field theory is analysed in detail: the property that all rational theories are unitary. Here we call a theory rational if it has only finitely many irreducible highest weight representations, and if the highest weight space of each of them is finite dimensional. We shall use three different methods to analyse this problem which we briefly describe in turn.

It was shown by Zhu [42] that a theory is rational in this sense if a certain quotient \(A(H_0)\) of the vacuum representation \(H_0\) is finite-dimensional. This space also forms an associative algebra, and the irreducible representations of this algebra, the so-called Zhu algebra, are in one-to-one correspondence with the irreducible representations of the meromorphic conformal field theory \(H_0\). For the case of the \(N = 2\) superconformal theory, the algebra has always the structure of a finitely generated quotient of a polynomial algebra in two variables, and this implies that \(A(H_0)\) is finite dimensional for every rational theory.

In this paper we give a physically motivated definition for \(A(H_0)\). We then show, using the embedding diagrams of the vacuum representations of the \(N = 2\) algebra [9, 10], that \(A(H_0)\) is infinite dimensional for a certain class of non-unitary theories, thereby proving that these theories are not rational. In addition, we also calculate \(A(H_0)\) explicitly for a few special values of the central charge. We find that \(A(H_0)\) is indeed infinite dimensional for the non-unitary cases we consider (and finite dimensional in the unitary cases).

In order to be able to determine the dimension of \(A(H_0)\) for arbitrary central charge one would need to know all vacuum Verma module embedding diagrams and explicit formulae for certain singular vectors. The embedding diagrams are known [9, 10], but sufficiently simple explicit formulae for the singular vectors do not exist so far in general.

It was also shown by Zhu [42] that the space of torus amplitudes (which is invariant under the modular group) is finite dimensional for a rational superconformal field theory\(^2\). For such theories, this implies in particular that the orbit of the vacuum

\(^2\)Here we use again that for the case of the \(N = 2\) theory, \(A(H_0)\) is finite dimensional for every rational theory.
character under the modular group is a finite dimensional vector space. If this is not the case, on the other hand, the theory cannot be rational. We determine the vacuum characters using the embedding diagrams, and analyse the action of the modular group on it. We then show that the relevant space is infinite dimensional for \( c \geq 3 \), and for the class of non-unitary theories with \( c < 3 \) which was already analysed by the previous method. As a non-trivial check, we also show that this space is finite dimensional in the unitary minimal cases, where \( c < 3 \).

The only cases which remain can be analysed using the coset realisation of the \( N = 2 \) super Virasoro algebra (see e.g. [30])

\[
\frac{su(2)_k \oplus (\mathfrak{fe})^2}{u(1)},
\]

which is known to preserve unitarity [20]. Because of this property, a non-unitary (rational) \( N = 2 \) theory must correspond to non-integer level for the \( su(2)_k \). The only remaining cases correspond to admissible level \( k \not\in \mathbb{N} \) for which \( su(2)_k \) always has at least one (admissible) representation whose highest weight space is infinite dimensional. Following a simple counting argument due to Ahn et al. [1] we then show that this gives rise to infinitely many inequivalent representations of the \( N = 2 \) theory, thus proving that the theory cannot be rational. The reasoning should be contrasted with the situation for \( N = 0 \) and \( N = 1 \), where the corresponding counting argument does not work: there the admissible representations of \( su(2)_k \) give rise to the non-unitary minimal models [27, 31].

The last method is well amenable to generalisation. Apart from some mathematical subtleties which we discuss, it can also be applied to the large class of Kazama-Suzuki models, and we therefore formulate it in this setup. As this class already provides most of the known \( N = 2 \) models, our arguments seem to indicate that actually all rational \( N = 2 \) superconformal field theories might be unitary.

All three methods rely to varying degrees on the (conjectured) embedding diagrams for the vacuum representations of the \( N = 2 \) super Virasoro algebra which we shall discuss in some detail. For example we use the embedding diagrams to obtain a formula for the vacuum character, whose modular properties we analyse. In the calculation of \( A(\mathcal{H}_0) \) we conclude from the embedding diagrams that there exist no further relations, and finally, we check that the coset (1) actually realises the \( N = 2 \) algebra by comparing the coset vacuum character with the one obtained from the embedding diagrams.

The paper is organised as follows. In section 2 we fix our notations and describe the embedding diagrams for the vacuum Verma modules of the \( N = 2 \) super Virasoro algebra following Dörrrzapf [9, 10]. In §3 we give a physically motivated derivation for \( A(\mathcal{H}_0) \), and calculate it for certain cases. In §4, we use the coset realisation of the algebra (1) to analyse the theories which correspond to admissible level. Furthermore, we indicate how the arguments generalise to the Kazama-Suzuki \( N = 2 \) superconformal theories. Finally, we remark in §5 how the construction can be even further generalised and give some prospective remarks.
In appendix A we derive the vacuum character using the embedding diagrams of §2 as well as the coset realisation. In appendix B, we analyse the modular properties of these characters, thereby showing that certain classes cannot be rational.

## 2 Preliminaries and embedding diagrams

Let us first fix some notations and conventions. Throughout this paper we will consider the Neveu-Schwarz sector of the $N = 2$ super Virasoro algebra which is the infinite dimensional Lie super algebra with basis $L_n, T_n, G_r^\pm, C (n, r + \frac{1}{2} \in \mathbb{Z})$ and (anti)-commutation relations given by

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, G_r^\pm] &= (\frac{1}{2}m - r)G_{m+r}^\pm \\
[L_m, T_n] &= -nT_{m+n} \\
[T_m, T_n] &= \frac{1}{3}Cm\delta_{m+n,0} \\
[T_m, G_r^\pm] &= \pm G_{m+r}^\pm \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r - s)T_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} \\
[L_m, C] &= [T_n, C] = [G_r^+, C] = 0 \\
\{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0
\end{align*}
\]

for all $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$.

We denote the Verma module generated from a highest weight state $|h, q, c>\,$ with $L_0$ eigenvalue $h$, $T_0$ eigenvalue $q$ and central charge $C = cI$ by $\mathcal{V}_{h, q, c}$. An element of a highest weight representation of the $N = 2$ super Virasoro algebra which is not proportional to the highest weight itself will be called a `singular vector’ if it is annihilated by all positive modes $L_n, T_n, G_r^\pm$ ($n, r + \frac{1}{2} \in \mathbb{N} := \{1, 2 \ldots \}$) and is an eigenvector of $L_0$ and $T_0$. A singular vector is called uncharged if its $T_0$ eigenvalue is equal to the $T_0$ eigenvalue of the highest weight state and charged otherwise. The character $\chi_{\mathcal{V}}$ of a highest weight representation $\mathcal{V}$ is defined by

\[
\chi_{\mathcal{V}}(q) := q^{h-c/24} \sum_{n \in \frac{1}{2} \mathbb{N} - \frac{1}{2}} \dim(\mathcal{V}_n)q^n,
\]

where $\mathcal{V}_n$ is the $L_0$ eigenspace with eigenvalue $h + n$, $c$ is the central charge and $h$ the conformal dimension of $\mathcal{V}$. The character of the Verma module $\mathcal{V}_{h, q, c}$ is for example given by

\[
\chi_{\mathcal{V}_{h, q, c}} = q^{h-c/24} \prod_{n=1}^\infty \frac{1 + q^{n+\frac{1}{2}}}{(1 - q^n)^2}
\]

and is called the ‘generic’ character.
We call a meromorphic conformal field theory (MCFT) [19] rational if it possesses only finitely many irreducible MCFT representations\(^3\), and if the highest weight space of each of them is finite dimensional. We should stress that this definition of rationality differs from the definition used in the mathematical literature, where it is not assumed that the highest weight spaces are finite dimensional, but where in addition all representations are required to be completely reducible. It was shown by Dong et al. [12] that the mathematical definition of rationality implies the one used in this paper. On the other hand the converse is not true as there exists a counterexample [17].

For bosonic rational theories it has been shown by Zhu [42] that the space of torus amplitudes which is invariant under the natural action of the modular group is finite dimensional\(^4\). The generalisation to the fermionic case has been studied in [24, Satz 1.4.6]. If all representations are completely reducible, the space of torus amplitudes is generated by the (finitely) many characters of the irreducible representations. In this case, the central charge and the conformal dimensions of the highest weight states are all rational numbers [2, 40].

We parametrise the central charge \(c\) as

\[
c(p, p') = 3\left(1 - \frac{2p'}{p}\right),
\]

where \(p\) and \(p'\) will be chosen positive for \(c < 3\). The well-known series of unitary minimal models then corresponds to the central charges being given as \(c(p, 1)\), where \(p \geq 2\) [5, 7]. Finally, let us denote by \(\chi_{p,p'}\) the vacuum character of the model with central charge \(c(p, p')\), i.e. the character of the irreducible quotient of \(\mathcal{V}_{0,0,c(p,p')}\).

One of the main points realised in [8, 9] is that there can be up to two linearly independent uncharged singular vectors at the same level. Indeed, this happens for example for the Verma modules related to the unitary minimal models of the \(N = 2\) super Virasoro algebra\(^5\). In [9] a complete list of all embedding diagrams of the \(N = 2\) super Virasoro algebra has been conjectured\(^6\).

In contrast to the case of the Virasoro algebra it is not directly clear how to define embedding diagrams for the Verma modules of the \(N = 2\) super Virasoro algebra. This is due to the fermionic nature of the \(N = 2\) algebra: suppose that there is a singular vector \(\psi_{n,p} = \theta_{n,p}|h, q, c\rangle\) of energy \(h + n\) and charge \(q + p\) in \(\mathcal{V}_{h,q,c}\) and that \(\psi'_{n',p'} = \theta'_{n',p'}|h + n, q + p, c\rangle\) is singular in \(\mathcal{V}_{h+n,q+p,c}\). Then \(\theta_{n',p'}\theta_{n,p}|h, q, c\rangle\) might be identically zero in \(\mathcal{V}_{h,q,c}\).

Our definition of embedding diagrams of Verma modules of the \(N = 2\) super Virasoro algebra follows [9] and includes only those Verma modules which are actually

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\(^3\)A MCFT representation is a representation which is compatible with the vacuum representation, i.e. the null-fields of the MCFT act trivially. We do not assume that the graded components of a MCFT representation are finite dimensional.

\(^4\)Actually, Zhu showed that this is true if \(A(\mathcal{H}_0)\) is finite dimensional. In the case under consideration, \(A(\mathcal{H}_0)\) is a finitely generated quotient of the polynomial algebra in two variables (as we shall show in section 3), and thus, the theory is rational if and only if \(A(\mathcal{H}_0)\) is finite dimensional.

\(^5\)The embedding diagrams conjectured in [11, 32, 34] for the unitary case are not correct.

\(^6\)However some of them are still not correct [10].
embedded in the original Verma module. To be more specific, the embedding diagram of a Verma module of the $N = 2$ super Virasoro algebra shows the highest weight vector and all non-trivial singular vectors contained in it up to proportionality. These vectors are connected by a line to a singular vector if there exists an operator mapping the singular vector of lower level (or the highest weight vector) onto the singular vector of higher level. As in the case of the embedding diagrams of the Virasoro algebra we shall omit lines between two vectors if these vectors are already indirectly connected.

We also want to include in the embedding diagrams information about the type of the singular vectors. To this end we use the following notation: the highest weight vector is denoted by a square and the singular vectors by circles. These circles are filled for singular vectors corresponding to Kac-determinant formula vanishings and unfilled for descendant singular vectors (for the explicit form of the Kac-determinant see [5, eq. (6)]). Furthermore, uncharged singular vectors which have no singular descendants of positive or negative charge, respectively, are denoted by surrounding triangles pointing to the left or right, respectively. (These singular vectors are of type $\Delta(1,0)$ or $\Delta(0,1)$ in the notation of [9]).

It has been shown in [9] that all singular vectors in $V_{h,q,c}$ have charge 0 or $\pm 1$. Therefore we indicate the charge of a singular vector relative to the highest weight vector by drawing the uncharged vectors vertically underneath the highest weight vector, the $-1$ charged vectors in a strip to the left of the highest weight vector and finally the $+1$ charged singular vectors in a strip to the right of the highest weight vector.

Let us now consider all embedding diagrams of the Verma modules $V_{0,0,c(1,p')}$ with $p, p' > 0$. For these values of the central charge there exist three types of embedding diagrams corresponding to $p = 1$, $p' \notin \mathbb{Q}$ or $p = 1$, $p' \in \mathbb{N}$ or $2 \leq p \in \mathbb{N}, p' \in \mathbb{N}, (p, p') = 1$, whose respective embedding diagrams are shown in Fig. 2.1, Fig. 2.2 and Fig. 2.3. [10]².

![Fig. 2.1: Embedding diagram for $V_{0,0,c(1,p')}$ with $p' \notin \mathbb{Q}$](image)

The Verma modules $V_{0,0,c(1,p')}$ with $p' \notin \mathbb{Q}$ contain only two singular vectors, namely $G_{-\frac{1}{2}}^{\pm}, \Omega$, where $\Omega = \vert 0, 0, c \rangle$ is the vacuum vector (c.f. Fig. 2.1). The Verma modules corresponding to the embedding diagram in Fig. 2.2 and Fig. 2.3 contain infinitely many singular vectors whose conformal dimensions are given by $pn(p'n \pm 1)$ ($n = 0, 1, 2, \ldots$).

²Note that in [9] the embedding diagram of type $III_0^0 A^+ B^-, III_0^0 A^- B^+$ (corresponding in our notation to $V_{0,0,c(p,p')}$ with coprime $p, p' \in \mathbb{N}$ and $p, p' \geq 2$) is not correct. The correct embedding diagram is the same as the embedding diagram for the case $III_0^0 A^+ B^+, III_0^0 A^- B^+$ (corresponding in our notation to $V_{0,0,c(p,1)}$ with $2 \leq p \in \mathbb{N}$).[10].
for the uncharged singular vectors and by $pn(p'n + 1) + p'n + \frac{1}{2}$ and $pn(p'n - 1) - p'n - \frac{1}{2}$ ($n = 0, 1, 2, \ldots$) for the charged singular vectors (for the cases corresponding to Fig. 2.2 set $p = 1$). The main difference between these two diagrams is that in Fig. 2.2 all singular vectors are descendants of the two singular vectors $G_{-\frac{1}{2}}^\pm \Omega$, which is not true in Fig. 2.3.

The line from the first uncharged singular vector to the two dimensional space of singular vectors at level $p'(p + 1)$ in Fig. 2.3 deserves comment: it means that the first uncharged singular vector at level $p'(p - 1)$ has exactly one descendent uncharged singular vector at level $p'(p + 1)$ which is contained in the two-dimensional space of singular vectors at this level. The module generated from this descendent singular vector contains singular vectors both of charge $0$ and $\pm 1$.

Let us end with two comments on the embedding diagrams of the vacuum Verma modules for $c \geq 3$ which we will need in the appendix (for details see [9]). For $c = 3$
there are infinitely many singular vectors which are all embedded in the two generic singular vectors $G^\pm_\frac{3}{2}$. For $c > 3$ all embedding diagrams terminate, i.e. there are only finitely many singular vectors contained in the vacuum Verma modules.

3 Zhu’s algebra

In this section we shall first give a physically motivated derivation of Zhu’s algebra; we shall then use this formulation to determine $A(\mathcal{H}_0)$ for a certain class of theories of the $N = 2$ super Virasoro algebra, and for some special cases.

A conceptually interesting way to determine all irreducible representations of a (bosonic) meromorphic conformal field theory is the method introduced by Zhu [42], whereby one associates an associative algebra, usually denoted by $A(\mathcal{H}_0)$, to the vacuum representation $\mathcal{H}_0$ of a conformal field theory. It was shown by Zhu that the irreducible representations of this associative algebra are in one-to-one correspondence with the irreducible representations of the meromorphic field theory $\mathcal{H}_0$. To define this algebra, a certain product structure was introduced by means of some rather complicated formulae, and it was not clear, how this construction could be understood from the more traditional point of view of conformal field theory. Here we shall give a different derivation for this algebra, from which it will be immediate that all representations of $\mathcal{H}_0$ have to be representations of $A(\mathcal{H}_0)$ (this derivation follows in spirit [14] and [41]); to show the converse direction, a similar argument as the one given in [42] would be sufficient. Another virtue of our derivation is that the Neveu-Schwarz fermionic case (which has by now been independently worked out by Kac and Wang in [29]) can essentially be treated on the same footing.

To fix notation, let us denote the modes of a holomorphic field $S(z)$ of conformal weight $h$ by

$$S(z) = \sum_{l \in \mathbb{Z}} S_{-l} z^{-h}.$$  

(2)

Given two representations of the chiral symmetry algebra $A$, $\mathcal{H}_1$ and $\mathcal{H}_2$, and two points $z_1, z_2 \in \mathbb{C}$ in the complex plane, the fusion tensor product can be defined by the following construction [15]. First we consider the product space $(\mathcal{H}_1 \otimes \mathcal{H}_2)$ on which two different actions of the chiral algebra are given by the two comultiplication formulae [16]

$$\Delta_{z_1, z_2}(S_n) = \Delta_{z_1, z_2}(S_n) = \sum_{m=1-h}^{n} \binom{n + h - 1}{m + h - 1} z_1^{n-m} (S_n \otimes 1)$$

$$+ \varepsilon_1 \sum_{l=1-h}^{n} \binom{n + h - 1}{l + h - 1} z_2^{n-l} (1 \otimes S_l),$$

(3)

$$\Delta_{z_1, z_2}(S_{-n}) = \sum_{m=1-h}^{\infty} \binom{n + m - 1}{n - h} (-1)^{m+h-1} z_1^{-(n+m)} (S_m \otimes 1)$$

$$- \varepsilon_1 \sum_{l=1-h}^{\infty} \binom{n + m - 1}{l - h} (-1)^{m+h-1} z_2^{-(n+m)} (1 \otimes S_{-l}).$$

8
where in (3) we have $n \geq 1 - h$, in (4), (5) $n \geq h$, and $\varepsilon_1$ is ±1 according to whether the left-hand vector in the tensor product and the field $S$ are both fermionic or not.\footnote{The second formula differs from the one given in [16] by a different $\varepsilon$ factor. There the two comultiplication formulae were evaluated on different branches; this is corrected here.}

The fusion tensor product is then defined as the quotient of the product space by all left-hand vectors in the tensor product and the cycled product:

$$
\Delta_{z_1,z_2}(S_{-n}) = \sum_{m=n}^{\infty} \left( \frac{m-h}{n-h} \right) (-z_1)^{m-n} (S_m \otimes 1)
$$

$$
+ \varepsilon_1 \sum_{l=1-h}^{\infty} \left( \frac{n+l-1}{n-h} \right) (-1)^{l+n-h-1} z_2^{-l+n} (1 \otimes S_l),
$$

where in (3) we have $n \geq 1 - h$, in (4), (5) $n \geq h$, and $\varepsilon_1$ is ±1 according to whether the left-hand vector in the tensor product and the field $S$ are both fermionic or not.

The fusion product is then defined as the quotient of the product space by all relations which come from the equality of $\Delta_{z_1,z_2}$ and $\Delta_{z_1,z_2}$

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)_f := (\mathcal{H}_1 \otimes \mathcal{H}_2) / (\Delta_{z_1,z_2} - \Delta_{z_1,z_2}).$$

It has been shown for a number of examples that this definition reproduces the known restrictions for the fusion rules [15, 16].

To analyse the possible representations of the meromorphic field theory $\mathcal{H}_0$, let us consider the fusion product of a given representation $\mathcal{H}$ at $z_2 = 0$ with the vacuum representation $\mathcal{H}_0$ at $z_1 = z$. We shall be interested in the quotient of the fusion product by all states of the form

$$\Delta_{z,0}(A_\pm)(\mathcal{H}_0 \otimes \mathcal{H})_f,$$

where $A_\pm$ is the algebra generated by all negative modes. (In the conventional approach to fusion in terms of 3-point functions, all such states vanish if there is a highest weight vector at infinity.) Using the comultiplication $\Delta_{z,0}$, it is clear that we can identify this quotient space with a certain subspace of

$$(\mathcal{H}_0 \otimes \mathcal{H})_f / \Delta_{z,0}(A_\pm)(\mathcal{H}_0 \otimes \mathcal{H})_f \subset (\mathcal{H}_0 \otimes \mathcal{H}(0)),$$

where $\mathcal{H}(0)$ is the highest weight space of the representation $\mathcal{H}$. The idea is now to analyse this quotient space for the universal highest weight representation $\mathcal{H} = \mathcal{H}_{\text{univ}}$, i.e. to use no property of $\psi \in \mathcal{H}_{\text{univ}}$, other than that it is a highest weight state. We can then identify this quotient space with a certain quotient of the vacuum representation $\mathcal{H}_0$, thus defining $A(\mathcal{H}_0)$,

$$
(A(\mathcal{H}_0) \otimes \mathcal{H}_{\text{univ}}(0)) = (\mathcal{H}_0 \otimes \mathcal{H}_{\text{univ}})_f / \Delta_{z,0}(A_\pm)(\mathcal{H}_0 \otimes \mathcal{H}_{\text{univ}})_f.
$$

In order to do this analysis without using any information about $\psi$, we have to find a formula for

$$(1 \otimes S_0)(\mathcal{H}_0 \otimes \psi) \mod \Delta_{z,0}(A_\pm)(\mathcal{H}_0 \otimes \mathcal{H}_{\text{univ}}),$$

where

$$
\Delta_{z,0}(S_m \otimes 1) \equiv (-z)^m (S_m \otimes 1).
$$
in terms of modes acting on the left-hand factor in the tensor product, where \( S \) is any bosonic field, \( \textit{i.e.} \) \( S \) has integral conformal dimension \( h \).

The crucial ingredient we shall be using is the observation

\[
\tilde{\Delta}_{0,z}(S_{-h}) = \Delta_{z,0}(e^{zL_{-1}} S_{-h} e^{-zL_{-1}}) \in \Delta_{z,0}(\mathcal{A}_-). \tag{11}
\]

Hence we have (for \( h \geq 2 \))

\[
0 \cong \tilde{\Delta}_{0,z}(S_{-h}) + \sum_{t=1-h}^{\infty} z^{-(h+t)} \Delta_{z,0}(S_t)
\]

\[
= (S_{-h} \otimes 1) + \sum_{t=1-h}^{\infty} \left( \frac{h + l - 1}{h - h} \right) (-1)^t + h - 1 (z)^{-t} \Delta_{z,0} (1 \otimes S_t)
\]

\[
+ \sum_{t=1-h}^{\infty} \left\{ z^{-(h+t)} (1 \otimes S_t) + z^{-(h+t)} \sum_{m=1-h}^{t} \left( \frac{l + h - 1}{m + h - 1} \right) z^{l-m} (S_m \otimes 1) \right\}
\]

\[
= (S_{-h} \otimes 1) - z^{-h} (1 \otimes S_0) + (1 \otimes \mathcal{A}_+)
\]

\[
+ \sum_{l=1-h}^{\infty} \sum_{m=1-h}^{l} z^{-(h+m)} \left( \frac{l + h - 1}{m + h - 1} \right) (S_m \otimes 1),
\]

where \( \mathcal{A}_+ \) is the algebra generated by the positive modes, and \( \cong \) denotes equality up to terms in the quotient. Evaluated on \((\mathcal{H}_0 \otimes \psi)\), where \( \psi \) is a highest weight, we then have

\[
(1 \otimes S_0) \cong z^h (S_{-h} \otimes 1) + \sum_{l=1-h}^{\infty} \sum_{m=1-h}^{l} z^{-m} \left( \frac{l + h - 1}{m + h - 1} \right) (S_m \otimes 1). \tag{12}
\]

In particular, we can use this result to obtain a formula for the action of \( \Delta_{z,0}(S_0) \) on \((\psi^0 \otimes \psi)\) modulo vectors in the quotient, where \( \psi^0 \in A(\mathcal{H}_0) \). (It is clear that \( \Delta_{z,0}(S_0) \) is well-defined on the quotient.) We calculate

\[
\Delta_{z,0}(S_0) \cong z^h (S_{-h} \otimes 1) + (S_0 \otimes 1)
\]

\[
+ \sum_{m=1-h}^{\infty} z^{-m} (S_m \otimes 1) \left\{ \left( \frac{h - 1}{m + h - 1} \right) + \sum_{l=m}^{\infty} \left( \frac{l + h - 1}{m + h - 1} \right) \right\}
\]

\[
= \sum_{m=0}^{h} z^m \left( \frac{h}{m} \right) (S_{-m} \otimes 1),
\]

where we have used the identity

\[
\sum_{k=0}^{l} \binom{a + k}{k} = \binom{a + l + 1}{l} \tag{13}
\]

(see \textit{e.g.} [21, p. 174]) to rewrite the sum in curly brackets. This reproduces precisely the product formula of Zhu for \( z = 1 \) [42]

\[
S \ast \psi^0 = \sum_{m=0}^{h} \left( \frac{h}{m} \right) S_{m-h} \psi^0. \tag{14}
\]
If $S$ is the field corresponding to a state in the subspace of the vacuum representation $\mathcal{H}_0$ by which we quotient to obtain $A(\mathcal{H}_0)$, then its zero mode vanishes by definition on all highest weight states. This implies that the product structure defined by the action of $\Delta_{\Omega}(S_0)$ gives rise to a well-defined product on $A(\mathcal{H}_0)$.

We have now achieved our first goal, namely to express the zero modes of the holomorphic fields on a highest weight state in terms of modes acting in the vacuum representation, modulo terms which vanish if there is a highest weight vector at infinity. In the next step we want to derive the relations by which the vacuum representation has to be divided in order to give $A(\mathcal{H}_0)$. In particular, we shall see that we can express all states of the form $(S_{-n} \otimes \mathbb{I})(\mathcal{H}_0 \otimes \psi)$ with $n > h$ by corresponding states with $n \leq h$. (Again this can be done without using any property of $\psi$ other than that it is a highest weight vector.)

We shall do the calculation for the bosonic case first, i.e., for $h \in \mathbb{N}$; we shall explain later, what modifications arise in the fermionic case. As before we have

$$0 \cong \Delta_{\Omega,-z}(S_{-n}) + \sum_{l=1-h}^{n-1} \left( \frac{n+l-1}{n-h} \right)(-1)^{n-h-n} \Delta_{\Omega}(S_l)$$

$$= (S_{-n} \otimes \mathbb{I}) - \left( \frac{n-1}{n-h} \right)(-1)^{n-h-n}(\mathbb{I} \otimes S_0) + (\mathbb{I} \otimes \mathcal{A}_+)}$$

$$+ \sum_{l=1-h}^{n-1} \left( \frac{n+l-1}{n-h} \right)(-1)^{n-h-n} \sum_{m=1-h}^{l} \left( \frac{l+h-1}{m+h-1} \right)z^{l-m}(S_m \otimes \mathbb{I}).$$

Using (12) we can rewrite the $(\mathbb{I} \otimes S_0)$ term, and find after a short calculation

$$(S_{-n} \otimes \mathbb{I}) \cong \left( \frac{n-1}{n-h} \right)(-1)^{n-h-n} \left\{ z^{n-h}(S_{-h} \otimes \mathbb{I}) + \sum_{m=1-h}^{n-1} z^{-(m+n)}C_m(S_m \otimes \mathbb{I}) \right\},$$

where

$$C_m = \sum_{l=1}^{n-1} \left( \frac{l+h-1}{m+h-1} \right)\left( 1 - \frac{(n+l-1)! (h-1)!}{(n-1)! (l+h-1)!} \right).$$

For completeness we should also give the result for $h = 1$, where the analysis simplifies to

$$(T_{-n} \otimes \mathbb{I}) \cong -(z)^{-n}(\mathbb{I} \otimes T_0) \cong (-z)^{-n+1}(T_{-1} \otimes \mathbb{I}).$$

Taking $n = h + 1$, we note that (15) and (17) become

$$0 \cong \sum_{m=1}^{h+1} \left( \frac{h}{h+1-m} \right)z^{n-h-1}(S_{-m} \otimes \mathbb{I}),$$

which, for $z = 1$, just reproduces the formula of Zhu [42]. Here we have used

$$\sum_{l=1}^{m} \left( \frac{h-1-l}{h-1-m} \right) = \left( \frac{h}{h+1-m} \right),$$

(19)
From our definition of $A(H_0)$ it is clear that every highest weight representation of $H_0$ gives rise to a representation of $A(H_0)$ with respect to the product structure induced by $\Delta_{z,0}(S_0)$. As our space is at most as large as the space of Zhu, it is then clear that our definition has to agree with the one of Zhu.

The fermionic case is slightly simpler, as there are no zero modes, and thus there is no relation corresponding to (12). We therefore only have to calculate for $n \geq h$

$$ (S_{-n} \otimes 1) = \Delta_{0,-z}(S_{-n}) - \varepsilon_1 \sum_{l=1-h}^{\infty} \left( \frac{n+l-1}{n-h} \right) (-1)^{l+h-1} (-z)^{-(n+l)} (1 \otimes S_l) $$

$$ \cong \varepsilon_1 (-1)^{h-n} \sum_{l=1-h}^{-\frac{1}{2}} \left( \frac{n+l-1}{n-h} \right) z^{-(n+l)} (1 \otimes S_l). $$

For $l = 1 - h, \ldots, -\frac{1}{2}$, we have furthermore

$$ \varepsilon_1 (1 \otimes S_l) = \Delta_{z,0}(S_l) - \sum_{m=1-h}^{l} \left( \frac{l+h-1}{m+h-1} \right) z^{-m} (S_m \otimes 1). \quad (20) $$

Hence we find for $n \geq h$

$$ (S_{-n} \otimes 1) \cong (-1)^{h-n+1} \sum_{m=1-h}^{-\frac{1}{2}} z^{-(n+m)} D_m (S_m \otimes 1), \quad (21) $$

where

$$ D_m = \sum_{l=m}^{-\frac{1}{2}} \left( \frac{l+h-1}{m+h-1} \right) \left( \frac{n+l-1}{n-h} \right). \quad (22) $$

Using (13) this formula simplifies for $n = h$ to

$$ 0 \cong \sum_{m=0}^{h-\frac{1}{2}} \left( \frac{h-1}{m} \right) z^{-m} (S_{m-h} \otimes 1), \quad (23) $$

which, for $z = 1$, just reproduces the formula of Kac and Wang [29]. By the same reasoning as before, it is then clear that our definition agrees with the one of Kac and Wang.

In the case of the $N = 2$ algebra, the only fermionic fields are $G^\pm$ of conformal weight $h = 3/2$. Because of (21), all negative modes of $G_{-m}^\pm$ with $m \geq 3/2$ can be eliminated in the quotient. On the other hand, $G_{-\frac{1}{2}}^\pm$, vanishes on the vacuum for $m = \frac{1}{2}$, and thus all $G^\pm$ modes can be removed. Furthermore, using (15) and (17), all (negative) modes of $L$ and $T$ can be eliminated, except for $T_{-1}$ and $L_{-2}, L_{-1}$. On the other hand, $L_{-1}$ vanishes on the vacuum, and thus can be removed by commuting it through to the right. The space $A(H_0)$ is therefore a certain quotient space of the space generated by $L_{-2}$ and $T_{-1}$. 


We can then equally well describe \( A(\mathcal{H}_0) \) as a quotient space of the space generated by \( h = L_{-2} + L_{-1} \) and \( q = T_{-1} \); this formulation has the advantage that the two generators commute, and that they can be directly identified with the eigenvalues of the highest weight with respect to \( L_0 \) and \( T_0 \) (by (12)), since

\[
(\mathbb{1} \otimes L_0) \cong (h \otimes \mathbb{1}), \quad \quad (\mathbb{1} \otimes T_0) \cong (q \otimes \mathbb{1}).
\]  

(24)

Thus \( A(\mathcal{H}_0) \) is a quotient space of the space of polynomials in \( h \) and \( q \).

Generically, this space is infinite dimensional, and to obtain some restrictions, we have to use singular vectors in the vacuum representation. It is clear that all bosonic descendants of singular vectors do not give new information for the quotient, as we can always replace negative modes of \( L \) and \( T \) by \( h, q, L_{-1} \) and some non-negative modes. Apart from the \( L_{-1} \) contributions, these give only restriction which contain, as a factor, restrictions from the original singular vector. The \( L_{-1} \) contributions simply correspond to an infinitesimal shift in the insertion point \( z \), and thus do not give new restrictions either. For fermionic descendants, a similar argument implies that the only descendants of potential interest are

\[
G^+_{-\frac{1}{2}} \mathcal{N}, \quad \quad G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \mathcal{N},
\]

(25)

where \( \mathcal{N} \) is a singular vector. It is clear that all three are trivial for the first generic singular vectors of the vacuum representation, \( G^+_{-\frac{1}{2}} \Omega = 0 \), but in general, they need not be trivial.

It follows from the embedding structure for the cases corresponding to Fig. 2.1 and Fig. 2.2 that all singular vectors are descendants of the generic singular vectors of the vacuum representation. We can thus conclude that \( A(\mathcal{H}_0) \) is isomorphic to a polynomial ring in two independent variables, and thus, in particular, infinite dimensional. This shows that the corresponding theories are not rational.

In the case of the diagram of Fig. 2.3, there exists an additional independent bosonic singular vector \( \mathcal{N}^9 \). The relations of the generic singular vectors have already been taken into account in the above derivation, and \( A(\mathcal{H}_0) \) is therefore only finite dimensional (and the corresponding theory rational) if \( \mathcal{N} \) gives rise to two independent relations. For a bosonic singular vector \( \mathcal{N} \), only the third descendant in (25) can contribute, as the other two have odd fermion number and thus are equivalent to zero in the quotient.

We conclude from this that the potential rational models all have a non-trivial bosonic singular vector in the vacuum representation. Whether the model is actually rational depends then on whether the \( G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \) descendent gives an independent relation or not. We have calculated the relations coming from \( \mathcal{N} \) and the \( G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \) descendent for a few examples explicitly. The first bosonic singular vector is given

\footnote{To avoid confusion, we should point out that \( L_{-1} \Omega \) is a descendent of the two generic singular vectors. We should also note that we implicitly assume here, that the vacuum representation does not possess any subsingular vectors which might give additional relations.}
in each case as

\[ \mathcal{N}_{c=1} = (2L_{-2} - 3T_{-1}T_{-1}) \Omega, \]
\[ \mathcal{N}_{c=3/2} = (10T_{-3} - 3L_{-3} + 3G^{+}_{-3/2}G^{-}_{-3/2} - 12L_{-2}T_{-1} + 8T_{-1}T_{-1}T_{-1}) \Omega, \]
\[ \mathcal{N}_{c=-6} = (-10T_{-3} - 6L_{-3} + 6G^{+}_{-3/2}G^{-}_{-3/2} + 6L_{-2}T_{-1} + T_{-1}T_{-1}T_{-1}) \Omega, \]
\[ \mathcal{N}_{c=-1} = (42T_{-4} + 24L_{-4} + 2T_{-2}T_{-2} - 84T_{-3}T_{-1} - 6G^{+}_{-3/2}G^{-}_{-5/2} + 6G^{+}_{5/2}G^{-}_{-3/2} - 32L_{-2}L_{-2} - 36L_{-3}T_{-1} + 36T_{-1}G^{+}_{-3/2}G^{-}_{-3/2} + 12L_{-2}T_{-1}T_{-1} + 9T_{-1}T_{-1}T_{-1}) \Omega, \]
\[ \mathcal{N}_{c=-12} = (-240L_{-5} + 360G^{+}_{-3/2}G^{-}_{-7/2} + 120G^{+}_{-5/2}G^{-}_{-5/2} + 840L_{-3}T_{-3} + 360G^{+}_{7/2}G^{-}_{-3/2} + 600L_{-3}L_{-2} + 120L_{-3}T_{-2} + 180L_{-4}T_{-1} - 60T_{-1}G^{+}_{-3/2}G^{-}_{-5/2} + 60T_{-1}G^{+}_{5/2}G^{-}_{-3/2} - 600L_{-2}G^{+}_{3/2}G^{-}_{3/2} - 300L_{-2}L_{-2}T_{-1} + 60L_{-3}T_{-1}T_{-1} - 60T_{-1}T_{-1}T_{-1} + 30T_{-2}T_{-2}T_{-1} - 60L_{-2}T_{-1}T_{-1}T_{-1} + 180T_{-3}T_{-1}T_{-1} - 60T_{-1}T_{-1}G^{+}_{-3/2}G^{-}_{3/2} - 3T_{-1}T_{-1}T_{-1}T_{-1} - 1992T_{-5}) \Omega. \]

The singular vector and its descendent give rise to the polynomial relations (in \( h \) and \( q \)) \( p_1 \), and \( p_2 \), respectively. The algebra \( A(H_0) \) is then given by

\[ A(H_0) = \mathbb{C}[h, q]/ \langle p_1(h, q), p_2(h, q) \rangle. \]

Our results for the five cases above are contained in Table 3.1.

<table>
<thead>
<tr>
<th>( c )</th>
<th>( p_1(h, q) )</th>
<th>( p_2(h, q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((2h - 3q^2))</td>
<td>((-2h + q)(1 + 3q))</td>
</tr>
<tr>
<td>(3/2)</td>
<td>(q(1 - 12h + 8q^2))</td>
<td>((2h - q)(1 - 2h + 5q + 8q^2))</td>
</tr>
<tr>
<td>(-6)</td>
<td>(q(2 + 6h + q^2))</td>
<td>((2h - q)(2 + 6h + q^2))</td>
</tr>
<tr>
<td>(-1)</td>
<td>((-4h + 3q^2)(1 + 8h + 3q^2))</td>
<td>((2h - q)(2 + 3q)(1 + 8h + 3q^2))</td>
</tr>
<tr>
<td>(-12)</td>
<td>(q(4 + 10h + q^2)(6 + 10h + q^2))</td>
<td>((-2h + q)(4 + 10h + q^2)(6 + 10h + q^2))</td>
</tr>
</tbody>
</table>

Table 3.1: Polynomials determining \( A(H_0) \) for certain values of \( c \)

We note that for the unitary models \( c = 1 \) and \( c = \frac{3}{2} \), \( A(H_0) \) is finite-dimensional, as the two relations are independent. In the other three cases, however, the two relations contain a common factor, and thus \( A(H_0) \) is infinite dimensional.

### 4 The coset argument

We have shown in the last section that the \( N = 2 \) super Virasoro algebra is not rational for certain non-unitary cases. In this section we will analyse the remaining cases with \( c < 3 \). The analysis for \( c \geq 3 \), using the modular properties of the vacuum character, is contained in the appendix B.

We shall use the coset realisation (1) of the \( N = 2 \) super Virasoro algebra to show that certain admissible representations of \( \widehat{su(2)}_k \) give rise to infinitely many irreducible
representations in the non-unitary cases. The basic idea of this argument is due to Ahn et al. [1]. We shall present the argument in the more general setting of the Kazama-Suzuki models as the counting argument generalises.

Recall that the Kazama-Suzuki models can be constructed from hermitian symmetric spaces [30]. More precisely, it has been shown in ref. [30] that if $G/H$ is a hermitian symmetric space, the coset

$$\frac{\hat{\mathfrak{g}}_k \oplus (\mathfrak{fer})^{2n}}{\hat{\mathfrak{h}}_k},$$

(26)

where $n = \text{rank}(G) = \text{rank}(H)$, contains the $N = 2$ super Virasoro algebra and the explicit form of the $N = 2$ super Virasoro generators $T, G^\pm, L$ in terms of the $2n$ free fermions and the $\hat{\mathfrak{g}}_k$ currents has been given. A complete list of hermitian symmetric space can, for example, be found in ref. [30, Table 1]. Note that for all hermitian symmetric spaces $\text{rank}(G) = \text{rank}(H), \mathfrak{g}$ is simple and that $\mathfrak{h}$ is of the form $\mathfrak{h} = u(1) \oplus \mathfrak{h}_1$, where $\mathfrak{h}_1$ is semisimple. The case of the $N = 2$ super Virasoro algebra corresponds to $\mathfrak{g} = su(2)$ and $\mathfrak{h} = u(1)$.

Before proceeding, we should note that it is in general rather difficult to determine the actual coset algebra of a given coset. In particular, even if the coset algebra is correctly identified for generic level (which is usually a tractable problem, for example by comparing generic characters), it is a priori not clear that this identification remains correct at arbitrary level. However, for the case of $\mathfrak{g} = su(2), \mathfrak{h} = u(1)$, the character calculations of appendix A show that the coset is indeed the $N = 2$ super Virasoro algebra for arbitrary level $k$.\textsuperscript{10}

Using the explicit form of the $\hat{u}(1)$ current $T$ and the Virasoro field $L$ in the coset [30, eq. (4.5)] it is easy to obtain a formula for the eigenvalues $q$ and $h$ of $T_0$ and $L_0$, respectively, acting on the subspace of the $\hat{\mathfrak{g}}_k$ highest weight space with $\mathfrak{h}$-weight $\lambda$ of a $\hat{\mathfrak{g}}_k \oplus (\mathfrak{fer})^{2n}$ highest weight representation

$$h = \frac{C_2(\mathfrak{g})}{2(k + g)} - \frac{(\lambda, \lambda + 2\rho_\mathfrak{h})}{2(k + g)}$$

$$q = -\frac{2}{k + g}(\rho_\mathfrak{g} - \rho_\mathfrak{h}, \lambda).$$

(27)

Here $C_2(\mathfrak{g})$ denotes the second order Casimir of the $\mathfrak{g}$ representation on the $\hat{\mathfrak{g}}_k$ highest weight space, $g$ the dual Coxeter number of $\mathfrak{g}$, and $\rho_\mathfrak{g}$ and $\rho_\mathfrak{h}$ are half the sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. For the case of $\mathfrak{g} = su(2)$, the formulae become

$$h = \frac{j(j + 1)}{(k + 2)} - \frac{m^2}{(k + 2)}$$

$$q = -\frac{2m}{k + 2},$$

(28)

where $j$ and $m$ label the spin and the magnetic quantum number of the corresponding $su(2)$ representation. If for admissible $k \not\in \mathbb{N}$ the admissible representations of $\hat{su}(2)_k$ are MCFT representations, it follows directly from these formulae that there are

\textsuperscript{10}This argument relies on the conjectured embedding diagrams of the $N = 2$ algebra.
infinite many highest weight states, as was already observed by Ahn et al. [1].
(For more details see below.) This implies then directly that the corresponding
theory is not rational.
It has now been shown that the admissible representations of \( \widehat{su(2)} \) are indeed MCFT
representations [13, Corollary 2.11]. For general \( \mathfrak{g} \), the corresponding result is not
yet known, but we believe it to be true.
Assuming this for the general case, the argument can be generalised as follows: we
note that \( \mathfrak{h} \) has one simple root less than \( \mathfrak{g} \) which we denote by \( \alpha \), and, that the
Dynkin index of \( \alpha \) is 1. For an admissible but non-integer level \( k \) there always
exists an admissible representation of \( \widehat{\mathfrak{g}_k} \) whose Dynkin label corresponding to the
fundamental weight dual to \( \alpha \) is fractional ([28, Theorem 2.1 (c)] see also [35, p.
236]). This representation has in particular an infinite dimensional highest weight
space. Let \( \Lambda \) be the \( \mathfrak{g} \)-weight of the highest weight vector \( v_\Lambda \) of this representation,
and denote by \( \Lambda_\mathfrak{h} \) the \( \mathfrak{h} \)-weight of \( v_\Lambda \). Furthermore, let \( E_\alpha \) be the step operator
(corresp onding to \( \alpha \)). Then the vectors \( E_\alpha^n v_\Lambda \) are highest weight vectors of \( \mathfrak{h} \), and
their \( \mathfrak{h} \)-weights are given as \( \lambda_n = \Lambda_\mathfrak{h} + n\mu \), where \( \mu \) is a non-zero \( \mathfrak{h} \)-weight and
\( n + 1 \in \mathbb{N} \). This implies that the expression \( (\lambda_n, \lambda_n + 2\rho_\mathfrak{h}) \) is unbounded for \( n \rightarrow \infty
\) and hence, that there are infinitely many different values for the conformal weight
\( h \) in (27). Thus the coset (26) has infinitely many inequivalent representations.
To relate the arguments for \( \mathfrak{g} = \mathfrak{su}(2) \) to the results of §3, let us parametrise the
admissible level as \( k = p/p'-2 \), where \( p, p' \) are coprime positive integers and \( p \geq 2 \).
The admissible representations are given by the \( \mathfrak{su}(2)_k \) weights [27, p. 4958]
\[
\lambda_{n,l} = (k - n + l(k+2))\Lambda_0 + (n - l(k+2))\Lambda_1,
\]
where \( n \) and \( l \) run through \( n = 0, \ldots, p-2 \), \( l = 0, \ldots, p'-1 \), and \( \Lambda_0, \Lambda_1 \) are the
fundamental weights of \( \mathfrak{su}(2) \). We note that the spin \( j \) of the \( \mathfrak{su}(2) \) representation on
the highest weight space of the \( \mathfrak{su}(2)_k \) representation corresponding to \( \lambda_{n,l} \) is given
by \( j = \frac{1}{2}(n - l(k+2)) \). In particular, for \( k \in \mathbb{N} \) which corresponds to the unitary
case, the spin \( j \) is always half-integral. If \( k \not\in \mathbb{N} \), the admissible representations
(corresp onding to the weights \( \lambda_{n,l} \) with \( l \neq 0 \) have an infinite dimensional highest
weight space as \( 2j + 1 \not\in \mathbb{N} \). These representations give rise to infinitely many MCFT
representations of the coset algebra, thus showing that it cannot be rational. Indeed,
eq. (28) implies that
\[
h + \frac{k + 2}{4}q^2 = \frac{j(j + 1)}{k + 2},
\]
where \( j \) is the spin of the \( \mathfrak{su}(2) \) representation. This equation gives precisely the
common factors in Table 3.1 for \( c = -6, -1 \) and \(-12 \), i.e. \( k + 2 = p/p' = \frac{2}{3}, \frac{3}{2} \) and \( \frac{2}{10} \),
where the spin \( j \) corresponds to the admissible representations of \( \mathfrak{su}(2)_k \) with infinite
dimensional highest weight space which are given by weights \( \lambda_{n,l} \) with \( l \neq 0 \). The
additional discrete representations of \( A(\mathcal{H}_0) \) correspond to the \( \mathfrak{su}(2)_k \) representations
with the weights \( \lambda_{n,0} (n = 0, \ldots, p - 2) \): for example in the second case \( c = -1, \)
\( (h, q) = (\frac{1}{3}, -\frac{2}{3}) \) comes from \( n = 1 \) and \( (h, q) = (0, 0) \) from \( n = 0 \).
We know that all irreducible representations satisfying the conditions given by the polynomials in Table 3.1 are MCFT representations, so in particular there exists a continuum of MCFT representations in the non-unitary cases. The above argument, however, only shows that those representations satisfying (28) can be obtained from the coset construction. Furthermore, it is clear that only countably many representations of the coset MCFT can be constructed from the admissible $su(2)_k$ representations. It therefore seems that the remaining representations cannot be constructed using the coset realisation.

Finally, let us mention that our findings are in perfect agreement with the results obtained in ref. [4]. The authors of loc. cit. have investigated the representation theory of several exceptional $N = 2$ super $\mathcal{W}$-algebras from a completely different point of view. The only rational models they found were unitary and even contained in the unitary minimal series of the $N = 2$ super Virasoro algebra.

### 5 Conclusion

In this paper we have analysed systematically the question whether the rational theories of the $N = 2$ superconformal algebra are always unitary. We have used three independent arguments to exclude the existence of rational non-unitary theories. Where possible we have checked that the different methods lead to consistent conclusions.

One of the methods is based on the coset realisation of the $N = 2$ algebra, and we have already indicated in section 4 how this argument can be generalised to the Kazama-Suzuki models. Apart from the (aforementioned) problem that the admissible representations are not yet known to be MCFT representations in general, this argument does not exclude that the theories corresponding to non-admissible affine theories are rational. However, we expect that there should be ‘fewer’ singular vectors than in the admissible case, and thus that the corresponding theories should also not be rational. It should be possible to settle both of these problems as soon as the representation theory of the Kac-Moody algebras at non-integer level is understood in detail.

The coset argument is even more general, as it does not involve the fermions. Indeed, ignoring the fermions where applicable, it can be applied to cosets of the form

$$\frac{\hat{g}^{(1)}_{k_1} \oplus \cdots \oplus \hat{g}^{(n)}_{k_n}}{\hat{h}^{(1)}_{l_1} \oplus \cdots \oplus \hat{h}^{(m)}_{l_m}},$$

where the $\hat{g}^{(i)}_{k_i}$ and $\hat{h}^{(j)}_{l_j}$ are simple affine Kac-Moody algebras at admissible level, and the sum of the numbers of simple roots of $\hat{g}^{(i)}_{k_i}$ with $k_i \not\in \mathbb{N}$, is bigger than the corresponding sum for the denominator. For the arguments to work for general (admissible) $k_i$, we also have to assume that one of the $\hat{g}^{(i)}$ with $k_i \not\in \mathbb{N}$ contains a simple root of Dynkin index 1 which is not contained in the denominator.
In particular, the arguments apply to the diagonal coset

\[ \frac{\hat{g}_{k_1} \oplus \hat{g}_{k_2}}{\hat{g}_{k_1 + k_2}} \]

indicating that they only give rise to rational models if at least one of the levels \( k_1 \) or \( k_2 \) is a positive integer (cf. the conjecture in ref. [3, p. 2421]). This is for example the case for the coset realisation of the (non-unitary) minimal models of the Virasoro algebra where one has \( g = su(2) \) and \( k_2 = 1 \). There are certain purely bosonic coset MCFTs which are of the general form (31), e.g. those corresponding to the unifying \( W \)-algebras associated to the unitary series of the Casimir \( W \)-algebras \( W_{A_n} \) or \( W_{D_n} \) [3, Table 7]. Our arguments confirm in these cases the conjecture of ref. [3, p. 2422] that all rational models of the unifying \( W \)-algebras are also rational models of the Casimir \( W \)-algebras (which are, in these cases, all unitary).

Let us close by mentioning some open problems. It would be interesting to know under which conditions all representations of a coset MCFT \( \hat{g}/\hat{h} \) can be obtained from MCFT representations of \( \hat{g} \) — as we have seen in section 4, this is not the case for certain non-rational \( N = 2 \) models, where there exists a continuum of representations. In the same spirit it would be interesting to describe all representations of the \( N = 2 \) super Virasoro algebra that can be obtained from the admissible \( su(2) \) representations for given \( k \) and to investigate whether they define quasi-rational theories in the sense of [36]. It would also be important to have a more general criterion for determining whether a coset MFCT is rational or not. Finally, in order to complete the arguments for the general case, it would be necessary to have a better understanding of the representation theory of affine Kac-Moody algebras, in particular at admissible level.

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### A Calculation of vacuum characters

In this appendix we want to calculate the vacuum characters of the \( N = 2 \) super Virasoro algebra from the embedding diagrams of §2 and from the coset realisation described in §4.

Let us first calculate the vacuum characters of the \( N = 2 \) super Virasoro algebra in the cases corresponding to the embedding diagrams in Fig. 2.1-2.3. In order to be
able to determine the characters from the embedding diagrams we have to assume that there do not exist subsingular vectors in the vacuum representation, i.e. vectors which are not singular in the vacuum Verma module but become singular in the quotient of the vacuum Verma module by its maximal proper submodule\textsuperscript{11}. We also want to assume that the character of a submodule of a Verma module generated from a level \( n \) charged singular vector is given by \( q^n/(1+q^{n-n'}) \) where \( n' < n \) is the level of the uncharged singular vector (or highest weight vector) connected by a line to the charged singular vector in the embedding diagram. This means for example that the character of the submodule generated from \( G_{-\frac{1}{2}}^{\pm}\Omega \) or \( G_{-\frac{3}{2}}^{\pm}\Omega \) is just given by \( q^n/(1+q^{\frac{n}{2}}) \), which is obvious in this case.

In the case of the embedding diagram shown in Fig. 2.1 and Fig. 2.2 all singular vectors are embedded in the two submodules generated from \( G_{-\frac{1}{2}}^{\pm}\Omega \). Moreover, the embedding diagram implies that the intersection of the two submodules generated by \( G_{-\frac{1}{2}}^{1}\Omega \) and \( G_{-\frac{3}{2}}^{1}\Omega \) is trivial. Therefore, the vacuum character of the \( N = 2 \) super Virasoro algebra with \( c(1,p') = 3(1-2p') \) (\( p' \notin \mathbb{Q} \) or \( p' \in \mathbb{N} \)) is given by

\[
\chi_{1,p'}(q) = q^{-c(1,p')/24} \prod_{n=1}^{\infty} \left( \frac{1 + q^{n+\frac{3}{2}}}{1 - q^n} \right)^2 \left( 1 - 2q^{\frac{n}{2}} \right). \tag{32}
\]

The case corresponding to the embedding diagram in Fig. 2.3 is more interesting. Here we have to subtract and add successively the characters of the modules generated by the corresponding singular vectors. Using the two assumptions above we obtain that the vacuum character of the \( N = 2 \) super Virasoro algebra with \( c(p,p') = 3(1-2p'/p) \); \( p,p' \in \mathbb{N} ; (p,p') = 1; p \geq 2 \) is given by

\[
\chi_{p,p'}(q) = q^{-c(p,p')/24} \prod_{n=1}^{\infty} \left( \frac{1 + q^{n+\frac{3}{2}}}{1 - q^n} \right)^2 \times \left( 1 - \sum_{n=0}^{\infty} q^{p'(n+1)(p(n+1)-1)} + 2q^{p[np'(n+1)+p'n+\frac{1}{2}]} \frac{q^{pn[pn-1]+p'n+\frac{1}{2}}}{1 + q^{pn+\frac{1}{2}}} 
\right.
\left. + \sum_{n=1}^{\infty} q^{p'pn+pn} + 2q^{pn[pn-1]+p'n+\frac{1}{2}} \frac{q^{pn+\frac{1}{2}}}{1 + q^{pn-\frac{1}{2}}} \right). \tag{33}
\]

In particular, for \( p' = 1 \) the above formula gives the vacuum character of the unitary minimal model with central charge \( c = 3(1-\frac{2}{p}) \) \textsuperscript{12}.

Finally, note that the last expression for the vacuum character \( \chi_{p,p'} \) can be rewritten as

\[
\chi_{p,p'}(q) = q^{-c(p,p')/24} \left( \prod_{n=1}^{\infty} \left( \frac{1 + q^{n+\frac{1}{2}}}{1 - q^{n}} \right)^2 \sum_{n \in \mathbb{Z}} q^{p'pn+pn+1} \right) \frac{1 - q^{pn+\frac{1}{2}}}{1 + q^{pn+\frac{1}{2}}} \tag{33}
\]

\textsuperscript{11}In [18] certain representations of the \( N = 2 \) superconformal algebra have been found which possess subsingular vectors. However, these representations are rather special and do not include the vacuum representation.

\textsuperscript{12}Although the multiplicities in the embedding diagrams of ref. [11, 34, 32] are not correct, the authors of loc.cit. obtained the correct characters in the unitary case.
In the second part of this appendix we use the coset realisation of the $N = 2$ super Virasoro algebra (1) for the calculation of the vacuum characters $\chi_{p,p'}$. Recall that the central charge of the $N = 2$ algebra is given as $c = \frac{3k}{k + 2}$ ($k \neq 0, -2$), and that the vacuum representation is given by the space of all uncharged $u(1)$ highest weight states in the vacuum representation of $su(2)_k \oplus \mathfrak{e}r^2$.

The character $\chi_{p,p'}$ is therefore the $u(1)$ uncharged part of $\chi^{su(2)}_{k} \chi^{e_{rr}^2}$ divided by the $u(1)$ character

$$\chi_{p,p'}(q) = \frac{1}{\chi^{u(1)}(q)} \text{Res}_z \left( \frac{1}{z} \chi^{su(2)}_k(q, z) \chi^{e_{rr}^2}(q, z) \right),$$  

(34)

where $\chi^{u(1)}(q) = 1/\eta(q) = q^{-\frac{1}{k+2}}/\tilde{\eta}(q)$ and $\eta(q) = e^{\frac{\pi i}{k+2}} \tilde{\eta}(q) = e^{\frac{\pi i}{k+2}} \prod_{n=1}^{\infty} (1 - q^n)$. (Here we have used the $su(2)$ characters $\chi^{su(2)}_k(q, z) = \text{tr} q^{L_0} z^{2L_0}$ which also take the zero mode of $J^3$ into account.)

There exist two types of embedding diagrams for $su(2)_k$ vacuum representations with level $k > -2$ (corresponding to $c < 3$) [33, Lemma 4.1]: either the level $k$ can be written as $k = p/p' - 2$, where $p, p' \in \mathbb{N}$, $(p, p') = 1$ and $p \geq 2$, or the vacuum representation is generic, i.e. all singular vectors are descendants of the level zero singular vector. In the former case the representation is admissible and the vacuum character is given by [27]

$$\chi^{su(2)}_k(q, z) = \frac{\partial_{p',pp'}(\tau, z/p') - \partial_{-p',pp'}(\tau, z/p')}{\partial_{1,2}(\tau, z) - \partial_{-1,2}(\tau, z)},$$  

(35)

where

$$\partial_{k,\lambda}(\tau, z) := \sum_{n \in \mathbb{Z} + \frac{1}{2k}} q^{k n^2} z^{2kn}.$$

(Note that our $z$ corresponds to $q^{z/2}$ in [27].) In the latter case the vacuum character is generic, and is given by

$$\chi^{su(2)}_k(q, z) = \frac{q^{-\frac{1}{2k}}}{\prod_{n=1}^{\infty} (1 - q^n)(1 - q^n z^2)(1 - q^n z^{-2})}.$$  

Let us first consider the generic (i.e. not admissible) case with $c < 3$. To evaluate the above residue (34) we want to use the following expression for the fermionic character

$$\chi^{e_{rr}^2}(q, z) = q^{-\frac{1}{2k}} \prod_{n \geq 1} \left( 1 + q^{n-\frac{1}{2}} z^2 \right) \left( 1 + q^{n-\frac{1}{2}} z^{-2} \right) = \frac{1}{\eta(q)} \sum_{m \in \mathbb{Z}} q^{\frac{1}{2} m^2} z^{2m}$$  

(36)

which follows from the product formula for the $\vartheta_3$ function (see e.g. [23, p. 164])

$$\vartheta_3(\tau/2, z) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2} n^2} z^{2n} = \prod_{n=1}^{\infty} (1 - q^n)(1 - z^2 q^{n-1/2})(1 - z^{-2} q^{n-1/2}).$$  

(37)
Furthermore, we shall also use the following identity for the denominator of the $su(2)$ character (see e.g. [26, p. 262 eq. (5.26)])

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n z^2)(1-q^n z^{-2})} = \frac{1}{\eta(q)^2} \sum_{l \in \mathbb{Z}} \phi_l(q) z^{2l},$$

(38)

where

$$\phi_l(q) = \sum_{r=0}^{\infty}(-1)^r q^{lr+\frac{1}{2}r(r+1)}.$$

An important property of $\phi_l(q)$ is that $\phi_{-l}(q) = q^l \phi_l(q)$.

The vacuum character of the $N = 2$ model is then (up to the $q^{-c(1,p')/24}$ term)

$$-\text{Res}_z \left\{ \frac{1}{\eta(q)^3} \sum_{m,l \in \mathbb{Z}} q^{\frac{3m}{2}} \phi_l(q) \left( z^{2l} z^{-2m} (z^1 - z^{-1}) \right) \right\},$$

where $p' \in \mathbb{N}$ or $p' \notin \mathbb{Q}$. The evaluation of the residue gives

$$\frac{1}{\eta(q)^3} \sum_{l \in \mathbb{Z}} \left(q^{\frac{1}{2}l^2} \phi_l(q) - q^{\frac{1}{2}l(l+1)^2} \phi_l(q)\right) = \frac{1}{\eta(q)^3} \sum_{l \in \mathbb{Z}} q^{\frac{1}{2}l^2} \left( \phi_l(q) - q^{-\frac{1}{2}l^2 - \frac{1}{2}l} \phi_{-l}(q) \right)$$

$$= \frac{1}{\eta(q)^3} \sum_{l \in \mathbb{Z}} q^{\frac{1}{2}l^2} \phi_l(q) (1 - q^{\frac{1}{2}})$$

$$= \frac{1}{\eta(q)^3} (1 - q^{\frac{1}{2}}) \sum_{l \in \mathbb{Z}} \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(r+2lr+p')}$$

$$= \frac{1}{\eta(q)^3} (1 - q^{\frac{1}{2}}) \sum_{l \in \mathbb{Z}} \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r} q^{4l^2},$$

where $\hat{l} = l + r$. We can now do the sums over $\hat{l}$ and $r$, and obtain the $N = 2$ vacuum character

$$\chi_{1,p'}(q) = q^{-c(1,p')/24} \left( \prod_{n=1}^{\infty} \frac{1 + q^{n-1/2}}{(1 - q^n)^2} \right) \left( 1 - 2 \frac{q^{1/2}}{1 + q^{1/2}} \right),$$

where we have used that $\frac{1 - q^{1/2}}{1 + q^{1/2}} = 1 - 2 \frac{q^{1/2}}{1 + q^{1/2}}$. This is indeed the generic $N = 2$ vacuum character, where the only null vectors are $G_{\frac{1}{2}}^\pm \Omega$ (c.f. eq. (32)).

Finally, consider the case where the $su(2)_k$ vacuum character is admissible, i.e. $k = p/p' - 2$ with $p, p' \in \mathbb{N}, (p, p') = 1$ and $p \geq 2$. In this case we find, using (35) and the well-known denominator formula,

$$\frac{\chi_{su(2)}(q, z)}{\chi_{su(1)}(q)} = \frac{q^{-c(p,p')/24}}{\prod_{n=1}^{\infty}(1-q^n z^2)(1-q^n z^{-2})} \sum_{n \in \mathbb{Z}} q^{np(1+pm)} \frac{z^{2pm+1} - z^{-2pm-1}}{z^2 - 1}.$$s

Using (34), the $N = 2$ vacuum character is then (up to the $q^{-c(p,p')/24}$ term which we suppress for the moment) the residue

$$-\text{Res}_z \left\{ \frac{1}{\eta(q)^3} \sum_{n,m \in \mathbb{Z}} q^{np(1+pm)} q^{\frac{1}{2}m^2} \phi_m(q) z^{2l} z^{-2m} (z^{2pm+1} - z^{-2pm-1}) \right\} = (*).$$

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The $z$-dependent part is $\text{Res}_z \left( z^{-1+2(l+m-pn)} - z^{-1+2(pm+m+l)} \right)$, whose residue is easily obtained. By expressing $m$ in terms of $n$ and $l$, the sum then becomes

\[
(*) = \frac{1}{\eta(q)^3} \sum_{n,l \in \mathbb{Z}} \left\{ q^{np'(1+pn)} \phi_l(q) q^{\frac{1}{2}(l-pn)^2} - q^{np'(1+pn)} \phi_l(q) q^{\frac{1}{2}(l+pm+1)^2} \right\}
\]

\[
= \frac{1}{\eta(q)^3} \sum_{n,l \in \mathbb{Z}} \left[ q^{np'(1+pn)} q^{\frac{1}{2}(l^2-2lpn+p^2n^2)} \phi_l(q) - q^{np'(1+pn)} q^{\frac{1}{2}(l^2+p^2n^2+1+2l+2pm+2pm)} \phi_l(q) \right]
\]

\[
= \frac{1}{\eta(q)^3} \sum_{n,l \in \mathbb{Z}} \left[ q^{np'(1+pn)} q^{\frac{1}{2}(l^2+p^2n^2-2lpn)} \left( \phi_l(q) - q^{\frac{1}{2}(l^2+2lpn)} \phi_{-l}(q) \right) \right]
\]

\[
= \frac{1}{\eta(q)^3} \sum_{n,l \in \mathbb{Z}} \left[ q^{np'(1+pn)} q^{\frac{1}{2}(l^2+p^2n^2-2lpn)} \phi_l(q) \left( 1 - q^{pn+\frac{l}{2}} \right) \right],
\]

where we have replaced $l$ by $-l$ in the second sum of the penultimate line, and used the previously mentioned symmetry of $\phi_l$ in the last equation. Next we use the explicit expression for $\phi_l$ to obtain

\[
(*) = \frac{1}{\eta(q)^3} \sum_{n,l \in \mathbb{Z}} q^{np'(1+pn)} \left( 1 - q^{pn+\frac{l}{2}} \right) \sum_{l \in \mathbb{Z}} \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}(r^2+l^2+p^2n^2+2lr-2lpn)}.
\]

The last exponent of $q$ can be rewritten as

\[
\frac{1}{2} \left( r^2 + r + l^2 + p^2n^2 + 2lr - 2lpn \right) = \frac{1}{2} \left( \hat{l}^2 + r + 2r pn \right),
\]

where $\hat{l} = l - pn + r$. We then replace the sum over $l$ by a sum over $\hat{l}$ which gives $\sum_{l \in \mathbb{Z}} q^{\hat{l}^2} = \eta(q) \prod_{m=1}^{\infty} \frac{1 + q^{m^2-\frac{l}{2}}}{1 + q^{m^2}}$. The sum over $r$ is the geometric series $\sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}(r^2+2pm)} = 1/(1 + q^{pn+\frac{l}{2}})$, and we thus arrive at (compare for example [38, 25, 1])

\[
\chi_{p,p'}(q) = q^{\frac{-c(p,p')}{24}} \left( \prod_{m=1}^{\infty} \frac{(1 + q^{m^2-\frac{l}{2}})^2}{(1 - q^n)^2} \right) \sum_{n \in \mathbb{Z}} q^{nn^2 + pn^2} \frac{1 - q^{pn+\frac{l}{2}}}{1 + q^{pn+\frac{l}{2}}}.
\]

This expression equals the one derived from the embedding diagram (33).

**B  Modular properties of the vacuum characters**

As already mentioned in §2, it was shown in [42] that for bosonic rational conformal field theories the space of torus amplitudes which is invariant under the natural action of the modular group is finite dimensional.

We expect therefore that the dimension of the space spanned by the functions $\chi|_{\Delta}(\tau) = \chi(A\tau)$ ($A \in \text{SL}(2,\mathbb{Z})$), where $\chi$ is the vacuum character, is finite if and
only if the $N = 2$ super Virasoro algebra is rational for the corresponding value of $c$. We shall show, using the following two lemmas, that the dimension of this space is infinite for $c \geq 3$ and for the non-unitary models corresponding to the embedding diagrams in Fig. 2.1 and Fig. 2.2, i.e. $c = c(p, p')$ with $p = 1, p' \not\in \mathbb{Q}$ or $p = 1, p' \in \mathbb{N}$. On the other hand, it is finite for the unitary models with $c = (p, 1)$ and $p' = 1, p \geq 3$. (For $p' = 1, p = 2$ the dimension is clearly 1 since $\chi_{1,2} = 1$.)

Lemma B.1 For $k = p - 2 \in \mathbb{N}$ the vacuum characters $\chi_k(\tau) := \chi_{p,1}(q)$ $(q = e^{2\pi i \tau})$ of the $N = 2$ super Virasoro algebra are modular functions on $\Gamma(24k(k + 2))$. More explicitly, they are given by

$$\chi_k(\tau) = \frac{1}{\eta^2(\tau)} \sum_{m \bmod 2k} \Theta_{L,\mu}(\frac{1}{2(k+2)}, \frac{3}{2k}) \vartheta_{m(k+2),k(k+2)}(\tau/2),$$

where the $\vartheta_{\lambda,k} = \sum_{n \in \mathbb{Z}} q^{\frac{|2kn + \lambda|^2}{4k}}$ are Riemann-Jacobi theta functions and the $\Theta_{L,\mu}$ are Hecke indefinite modular forms (of weight one) associated to the lattice $L = \mathbb{Z} \oplus \mathbb{Z}$ and the quadratic form $Q(\gamma) = 2(k + 2)\gamma_1^2 - 2k\gamma_2^2$.

Proof. We first recall the definition of a Hecke indefinite modular form (see [22] or [26, pp. 254] for more details). Let $L \subset \mathbb{R}^2$ be a lattice of rank two and $Q : L \rightarrow \mathbb{R}$ an indefinite quadratic form such that $Q(x) = 0, x \in L$ implies $x = 0$. Denote by $L^\perp$ the lattice dual to $L$, $L^\perp = \{ x \in \mathbb{R}^2 | B(x, y) \in \mathbb{Z} \text{ for } y \in L \}$, where $B(\gamma, \gamma') = \frac{1}{2}(Q(\gamma + \gamma') - Q(\gamma) - Q(\gamma'))$ is the bilinear form associated to $Q$. Let $G_0$ be the subgroup of the identity component of the orthogonal group of $(B, \mathbb{R}^2)$ which preserves $L$ and fixes all elements of $L^\perp / L$. Fix a factorisation $Q(\gamma) = l_1(\gamma)l_2(\gamma)$ where $l_1$ and $l_2$ are real linear and set $\text{sign}(\gamma) = \text{sign}(l_1(\gamma))$. Then

$$\Theta_{L,\mu}(\tau) := \sum_{\gamma \in L^\perp, \mu(\gamma, \gamma) > 0 \atop \gamma \bmod \mathfrak{c}_0} \text{sign}(\gamma) q^{Q(\gamma)/2}$$

is called a Hecke indefinite modular form associated to $\mu$ and $L$. It is a modular form of weight one on $\Gamma(N)$, where $N \in \mathbb{N}$ satisfies $NQ(\gamma) \in 2\mathbb{Z}$ for all $\gamma \in L^\perp$. The case we are interested in has been studied in [26, pp. 256]. We have $L = \mathbb{Z} \oplus \mathbb{Z}$ and

$$Q(\gamma) = 2(k + 2)\gamma_1^2 - 2k\gamma_2^2 = l_1(\gamma)l_2(\gamma),$$

where $l_1(\gamma) = \sqrt{2(k + 2)\gamma_1^2 - 2k\gamma_2^2}$ and $l_2(\gamma) = \sqrt{2(k + 2)\gamma_1^2 - 2k\gamma_2^2}$ so that $Q(\gamma) = 0$ for $\gamma \in L$ implies $\gamma = 0$. Then $B$ is given by

$$B(\gamma, \gamma') = 2(k + 2)\gamma_1\gamma'_1 - 2k\gamma_2\gamma'_2,$$

implying that $L^\perp$ equals $\frac{1}{2(k+2)} \mathbb{Z} \oplus \frac{1}{2k} \mathbb{Z}$. We observe that $A$, given by

$$A(\gamma_1, \gamma_2) = ((k + 1)\gamma_1 + k\gamma_2, (k + 2)\gamma_1 + (k + 1)\gamma_2),$$

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satisfies \( Q(\gamma) = Q(A\gamma) \), and that the group generated by \( A \) is the identity component of the orthogonal group of \( (B, \mathbb{R}^2) \) which leaves \( L \) invariant. Furthermore, \( A^2 \) generates \( G_0 \). Hence the functions \( \Theta_{L, (\frac{1}{2(k+2)}, \frac{m}{2k})} (m \in \mathbb{Z}) \) are modular forms of weight one on \( \Gamma(4k(k + 2)) \).

Moreover, one can show that the Hecke indefinite modular forms \( \Theta_{L, (\frac{1}{2(k+2)}, \frac{m}{2k})} \) are given by [26, p. 258]

\[
\Theta_{L, (\frac{1}{2(k+2)}, \frac{m}{2k})}(\tau) = \left( \sum_{s \geq n \geq 0} - \sum_{s < 0, n < 0} \right) (-1)^s q^{(k+2)(\frac{s}{2} + n + \frac{1}{2(k+2)})^2 - k(\frac{s}{2} + \frac{m}{2k})^2} \\
- \left( \sum_{s \geq 0, n > 0} - \sum_{s < 0, n \leq 0} \right) (-1)^s q^{(k+2)(\frac{s}{2} + n - \frac{1}{2(k+2)})^2 - k(\frac{s}{2} + \frac{m}{2k})^2}.
\]

We thus find

\[
\frac{1}{\eta^3} \sum_{m \bmod 2k} \Theta_{L, (\frac{1}{2(k+2)}, \frac{m}{2k})}(\tau) \vartheta_{m(k+2), k(k+2)}(\tau/2) = \frac{1}{\eta^3} \sum_{m' \in \mathbb{Z}} \left( \sum_{s \geq n \geq 0} - \sum_{s < 0, n < 0} \right) (-1)^s q^{(k+2)(\frac{s}{2} + n + \frac{1}{2(k+2)})^2 - k(\frac{s}{2} + \frac{m'}{2k})^2 + \frac{k+2}{2k} m'^2} \\
- \frac{1}{\eta^3} \sum_{m' \in \mathbb{Z}} \left( \sum_{s \geq 0, n > 0} - \sum_{s < 0, n \leq 0} \right) (-1)^s q^{(k+2)(\frac{s}{2} + n - \frac{1}{2(k+2)})^2 - k(\frac{s}{2} + \frac{m'}{2k})^2 + \frac{k+2}{2k} m'^2} \\
= q^{-\frac{3}{4k+4k+2}} \prod_{n=1}^{\infty} \frac{(1 + q^{n+1/2})^2}{(1 - q^n)^2} \times \\
\left( \sum_{s \geq n \geq 0} - \sum_{s < 0, n < 0} \right) (-1)^s q^{(k+2)(\frac{n}{2} + n + \frac{1}{2(k+2)})^2 + (k+2)n^2 + n} \\
- \left( \sum_{s \geq 0, n > 0} - \sum_{s < 0, n \leq 0} \right) (-1)^s q^{(k+2)(\frac{n}{2} + n - \frac{1}{2(k+2)})^2 + (k+2)n^2 - n} \\
= q^{-\frac{e^{(k+2,1,1)}}{2k+2}} \prod_{n=1}^{\infty} \frac{(1 + q^{n+1/2})^2}{(1 - q^n)^2} \sum_{n \in \mathbb{Z}} q^{(k+2)n^2 + n} \frac{1 - q^{(k+2)n+1/2}}{1 + q^{(k+2)n+1/2}},
\]

where we have used the definition of the Riemann-Jacobi theta functions \( \vartheta_{m,k} = \vartheta_{m+2kz,k} \) and \( \Theta_{L,\mu} = \Theta_{L,\mu + (0,z)} \) in the second equality, and the well-known product formula for the \( \vartheta_3 \) function (37) for \( \tau/2 \) and \( z = 1 \) in the third. The last expression equals (33) for \( p = k + 2, p' = 1 \).

The property that the \( \chi_k \) are modular functions on \( \Gamma(24k(k+2)) \) follows now directly from the well-known modular properties of the Riemann-Jacobi theta functions and the \( \eta \) function.

\( \square \)

It is then clear that the space spanned by \( \chi_{p,1}|_A (A \in \text{SL}(2, \mathbb{Z})) \) is finite dimensional, as \( \Gamma(24p(p-2)) \) has finite index in \( \text{SL}(2, \mathbb{Z}) \) for \( p \geq 3 \).

In order to prove that the space spanned by the functions \( \chi_{1,p'}(A\tau) (A \in \text{SL}(2, \mathbb{Z})) \) is infinite dimensional, we need the following lemma:
Lemma B.2 Let $f : \mathbb{C} \to \mathbb{C}$ be a function of the form $f(\tau) = q^\alpha P(q)/Q(q)$, where $\alpha \in \mathbb{Q}$, $P$ and $Q$ polynomials and $q = e^{2\pi i \tau}$. Then for any $N > 0$ and $k \in \mathbb{Z}$, the space spanned by the functions $f|_{k,A} (A \in \Gamma(N))$ is infinite dimensional if $f$ is not constant. Here $f|_{k,A}$ is defined as

$$f|_{k,A}(\tau) = (c\tau + d)^{-k} f(A\tau),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A\tau = \frac{a\tau + b}{c\tau + d}$.

Proof. Assume that the space spanned by the $f|_{k,A}$ ($A \in \text{SL}(2,\mathbb{Z})$) is $n$ dimensional, where $n < \infty$, and that $f$ is not constant. Let $\tau$ be $\tau/s$ where $s$ is the denominator of $\alpha = \frac{r}{s}$, $\bar{q} = e^{2\pi i \bar{\tau}}$, and let $\bar{P}$, $\bar{Q}$ be the polynomials given by

$$\bar{P}(\bar{q}) = q^\alpha P(q), \quad \bar{Q}(\bar{q}) = Q(q) \quad \text{for } \alpha \geq 0$$
$$\bar{P}(\bar{q}) = P(q), \quad \bar{Q}(\bar{q}) = q^{-\alpha}Q(q) \quad \text{for } \alpha < 0.$$ Then there exist $n$ matrices $A_i$ $(i = 1, \ldots, n)$ such that the functions $f|_{k,A_i}$ $(i = 1, \ldots, n)$ are linearly dependent over $\mathbb{C}$. (Without loss of generality we can assume that $A_i^{-1}A_j$ are not of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ as we are interested in a basis over $\mathbb{C}$).

Hence the polynomials $\bar{P}(\bar{q}_j) \prod_{i \neq j} \bar{Q}(\bar{q}_i)$ $(j = 1, \ldots, n)$ with $\bar{q}_i = e^{2\pi i A_i \bar{\tau}}$ are linearly dependent over $\mathbb{C}[\bar{\tau}]$, and thus the $\bar{q}_i$ are algebraically dependent over $\mathbb{C}[\bar{\tau}]$. Applying $A_i^{-1}$ to $\bar{\tau}$ we can assume that $A_1$ is the identity. Looking at the asymptotic behaviour of the $\bar{q}_i$ for $\tau \to -i\infty$ we observe that there cannot be a term containing $\bar{q}_i$. By induction on $n$ we find that the $\bar{q}_i$ are algebraically independent. This gives the desired contradiction. \qed

The proof of lemma B.2 is due to J. Nekovar [37].

The last lemma proves that the space spanned by the functions $\chi_{1,p'}(A\tau)$ $(A \in \Gamma(48))$ is infinite dimensional since the function $\frac{n((\tau+1)/2)^2}{n(\tau)^4} \chi_{1,p}(\tau)$ satisfies the assumptions of the lemma and $\frac{n((\tau+1)/2)^2}{n(\tau)^4}$ is invariant under the $|_{-1,A}$ action for $A \in \Gamma(48)$. Therefore, the space spanned by the functions $\chi_{1,p'}(A\tau)$ $(A \in \text{SL}(2,\mathbb{Z}))$ is infinite dimensional. We also expect that the dimension of the corresponding space is infinite for $c = c(p,p')$ with coprime integers $p', p \geq 2$.

Finally the embedding diagrams of the vacuum Verma modules for $c \geq 3$ (c.f. the end of §2) imply that the corresponding vacuum characters are given by the product of the generic Verma module character and a rational function of $q^{\frac{1}{2}}$. We can therefore again apply lemma B.2 to conclude that the space obtained from the $\text{SL}(2,\mathbb{Z})$ action on such a vacuum character is infinite dimensional. This shows that all theories with $c \geq 3$ are not rational.
References


