Abstract

We show how classical spacetime emerges from quantum gravity through the study of a quantum FRW cosmological model coupled to a free massive scalar field using a new asymptotic expansion method of the Wheeler-DeWitt equation. It is shown that the coherent states of the nonadiabatic basis of a particular generalized invariant give rise to the quantum back reaction of the matter field proportional to the classical energy and the Einstein-Hamilton-Jacobi with the matter field becomes equivalent to the classical Einstein equation.

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In recent years, apart from the attempt to endow canonical gravity with a quantum theory of gravity by overcoming many conceptual and technical problems, canonical quantum gravity for a gravity and a matter field, mostly based on the Wheeler-DeWitt (WD) equation, has also been used to derive a semiclassical gravity. The main idea of this scheme is that from a supposed quantum gravity written formally as $\hat{G}_{\mu\nu} = 8\pi\kappa\hat{T}_{\mu\nu}$, in which $\kappa$ denotes the gravitation constant and both the gravity and matter field are quantized at the same time, one could derive a semiclassical gravity written also formally as $G_{\mu\nu} = 8\pi\kappa\langle\hat{T}_{\mu\nu}\rangle$, in which the spacetime emerges as a classical background in some sense but the matter field maintains the status of quantum variables. The final step would go from the semiclassical gravity down to the classical Einstein gravity $G_{\mu\nu} = 8\pi\kappa T_{\mu\nu}$, in which both the gravity and matter field are treated as classical variables.

Following the above scheme the WKB approximations or Born-Oppenheimer type of expansions have been employed to the WD equation [1]. However, in this approach to the semiclassical gravity one obtains only the vacuous Einstein equation in the form of the Einstein-Hamilton-Jacobi equation together with a time-dependent functional Schrödinger equation for the matter field. In order to include a quantum back reaction of the matter field into the Einstein-Hamilton-Jacobi equation one should keep the expectation value of the Hamiltonian operator of the matter field by hand in the Born-Oppenheimer expansion. Beside this methodological difficulty, there are still gaps to be filled satisfactorily [2]. Firstly, there is an arbitrariness in the separation of the WD equation into the gravitational field equation which should reduce to the Einstein-Hamilton-Jacobi equation in the end and the matter field equation which should describe the Schrödinger equation for the matter field in a curved spacetime with some gravitational quantum corrections. Secondly, even after a cosmological time is introduced from the Einstein-Hamilton-Jacobi equation and the Schrödinger equation is derived, the form of quantum states of the matter field should still have the WKB form in order to give rise to the classical Einstein gravity equated with the matter field in the form of the Einstein-Hamilton-Jacobi equation.

Recently a new asymptotic expansion method was suggested to derive consistently a
semiclassical gravity from canonical quantum gravity [3]. According to the new asymptotic expansion method the WD equation can be separated into a gravitational field equation and a time-dependent Schrödinger equation for the matter field. The matter field obeys purely a quantum equation equivalent to the time-dependent functional Schrödinger equation with higher order gravitational quantum corrections, and the gravitational field equation in due turn reduces to the Einstein-Hamilton-Jacobi equation equated with the quantum back reaction of the matter field. The new asymptotic expansion method differs from the others in that we keep the quantum status of the gravitational and matter field equations and two coupled equations are asymptotically correct in the limit of the Planck mass squared approaching to an infinity.

The main purpose of this letter is to show how classical spacetime emerges from quantum gravity by applying the new asymptotic expansion method to a quantum Friedmann-Robertson-Walker (FRW) cosmological model coupled to a free massive scalar field. We show further that in the coherent state of a nonadiabatic eigenstate of a particular invariant the quantum back reaction of the scalar field has exactly the same form as the time-time component of the energy-momentum tensor in the classical Einstein equation by matching appropriately quantum numbers to the amplitudes of classical field.

The FRW cosmology has a homogeneous and isotropic spacetime manifold with the metric

\[ ds^2 = -N^2(t)dt^2 + R^2(t)d\Omega_3^2, \]

where \( N \) is the lapse function and \( R(t) \) is the scale factor depending only on \( t \). The time will be scaled in unit of \( c = 1 \), and the Planck mass squared will thus denote \( m^2_{\text{P}} = \frac{\hbar}{8\pi\kappa} \).

The action for the FRW cosmology coupled to a free massive scalar field, which is also homogeneous and isotropic, i.e., depends only on time \( t \), takes the form

\[ S = \int dt \left[ -\frac{m^2_{\text{P}}}{\hbar} R^3 \left( \frac{1}{2N} \left( \frac{\dot{R}}{R} \right)^2 + N \left( -\frac{k}{2R^2} + \frac{\Lambda}{6} \right) \right) + R^3 \left( \frac{1}{2N} \phi^2 - \frac{Nm^2}{2} \phi^2 \right) \right]. \]

The classical equations of motion are obtained by taking variation with respect to \( N \):
\[- \frac{m_p^2}{\hbar} \left( \frac{\dot{R}^2}{2} + \frac{kR}{2} - \frac{\Lambda R^3}{6} \right) + R^3 \left( \frac{\dot{\phi}^2}{2} + \frac{m^2 \phi^2}{2} \right) = 0, \quad (3)\]

with respect to $R$:

\[
\frac{m_p^2}{\hbar} \left( \frac{d}{dt}(\dot{R}) - \frac{\dot{R}^2}{2} + \frac{k}{2} - \frac{\Lambda R^2}{2} \right) + 3R^2 \left( \frac{\dot{\phi}^2}{2} - \frac{m^2 \phi^2}{2} \right) = 0, \quad (4)
\]

and with respect to $\phi$:

\[
\frac{d}{dt} \left( R^2 \dot{\phi} \right) + m^2 R^3 \phi = 0. \quad (5)
\]

We rewrite the classical equation (3) in the form

\[
\left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = 16\pi \kappa \frac{1}{R^3} T_{00} \quad (6)
\]

where $T_{00}$ is the time-time component of the energy-momentum stress tensor

\[
T_{00} = \frac{1}{2} R^3 \dot{\phi}^2 + \frac{1}{2} m^2 R^3 \phi^2. \quad (7)
\]

Up to some operator ordering ambiguity, we quantize the Hamiltonian a la the Dirac quantization method to obtain the WD equation

\[
\left[ \frac{\hbar^2}{2m_p^2 R} \frac{\partial^2}{\partial R^2} + \frac{m_p^2}{\hbar} \left( -\frac{kR}{2} + \frac{\Lambda R^3}{6} \right) - \frac{\hbar^2}{2R^3} \frac{\partial^2}{\partial \phi^2} + \frac{m^2 R^3}{2} \phi^2 \right] \Psi(R, \phi) = 0. \quad (8)
\]

The gist of the new asymptotic expansion method is to search for a wave function of the form

\[
\Psi(R, \phi) = \psi(R) \Phi(\phi, R). \quad (9)
\]

Here $\psi$ and $\Phi$ are still quantum states of the gravity and the scalar field, respectively. The classical spacetime will emerge later only after finding the wave function for the gravity in a WKB form with a quantum back reaction of the scalar field. The wave function (9) has the same form obtained from the adiabatic expansion or superadiabatic expansion of the WD equation [4]. Any quantum state of the scalar field can be expanded by the basis of some Hermitian operator relevant to the matter field Hamiltonian.
The main result of the new asymptotic expansion method is the separation of the WD equation into the gravitational field equation

\[
\left[ \frac{h^3}{2m_P^2 R} \frac{\partial^2}{\partial R^2} + \frac{m_P^2}{\hbar} \left( -\frac{kR}{2} + \frac{\Lambda R^3}{6} \right) + :H:_{nn}(R) \right] \psi(R) = 0,
\]

and the asymptotic matter field equation

\[
i\hbar \frac{\partial}{\partial t} c_n + \left( \Omega^{(1)}_{nn} - H_{nn} + :H:_{nn} \right) c_n + \sum_{k \neq n} \left( \Omega^{(1)}_{nk} - H_{nk} \right) c_k = 0,
\]

valid in the limit \( \frac{\hbar^2}{m_P^2} \to 0 \). Here

\[
\Omega^{(1)}_{nk} = i\hbar \left\langle \Phi_n(\phi, R) \left| \frac{\partial}{\partial t} \right| \Phi_k(\phi, R) \right\rangle,
\]

is an induced gauge potential and

\[
H_{nk} = \left\langle \Phi_n(\phi, R) | \hat{H}_m | \Phi_k(\phi, R) \right\rangle,
\]

is the expectation value of the Hamiltonian operator and \( :H:_{nn} \) is that of the normal ordered Hamiltonian operator. We have introduced a cosmological time

\[
\frac{\partial}{\partial t} = \text{Im} \left( \frac{h^2}{m_P^2 R} \frac{1}{\psi} \frac{\partial \psi}{\partial R} \frac{\partial}{\partial R} \right),
\]

which for a gravitational wave function in the WKB form

\[
\psi(R) = f(R) \exp \left[ i \frac{m_P^2}{\hbar^2} S(R) \right]
\]

becomes

\[
\frac{\partial}{\partial t} = \frac{1}{R} \frac{\partial S(R)}{\partial R} \frac{\partial}{\partial R}.
\]

From the gravitational wave equation (11) we obtain the Einstein-Hamilton-Jacobi equation

\[
\frac{1}{2R} \left( \frac{\partial S}{\partial R} \right)^2 + \frac{kR}{2} - \frac{\Lambda R^3}{6} = 8\pi\kappa :H:_{nn}(R),
\]

in the asymptotic limit \( m_P^2 \to \infty \). We rewrite Eq. (18) as
\[
\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = 16\pi k \frac{1}{R^3} : H_{nn} (R). \tag{19}
\]

Now we consider the time-dependent matter field Hamiltonian of a harmonic oscillator form
\[
\hat{H}_m = T_{00} = \frac{1}{2R^3} \pi^2 \phi^2 + \frac{1}{2} m^2 R^3 \phi^2. \tag{20}
\]

For a generalized invariant \( \hat{I} \) that obeys the equation
\[
\frac{\partial}{\partial t} \hat{I} - \frac{i}{\hbar} [\hat{I}, \hat{H}_m] = 0, \tag{21}
\]
the nonadiabatic basis of its eigenstates
\[
\hat{I}(\pi_\phi, \phi, R) \ket{\Phi_n(\phi, R)} = \lambda_n \ket{\Phi_n(\phi, R)}. \tag{22}
\]
has equal contributions from the induced gauge potential and Hamiltonian operator
\[
H_{nk}(R) = \Omega^{(1)}_{nk} \tag{23}
\]
for \( n \neq k \). It is the decoupling theorem [5] that compels one to choose specially the nonadiabatic basis of some generalized invariant for a time-dependent quantum system.

When one uses the nonadiabatic basis for the quantum states in Eq. (12), the asymptotic matter field equation becomes a diagonal equation, whose solution is given by
\[
c_n(t) = c_n(t_0) \exp \left[ \frac{i}{\hbar} \int \left( \Omega^{(1)}_{nn} - H_{nn} + i :H_{nn} : \right) dt \right]. \tag{24}
\]
It should be noted that the asymptotic quantum state is an exact quantum state of the Schrödinger equation
\[
i\hbar \frac{\partial}{\partial t} \Phi(\phi, R) = \hat{H}_m(\pi_\phi, \phi, R) \Phi(\phi, R) \tag{25}
\]
except for \( \exp \left[ \frac{i}{\hbar} \int :H_{nn} : dt \right] \), a time-dependent phase factor.

First, we find a particular second order generalized invariant of the form [6]
\[
\hat{I} = \frac{1}{2} (\hat{I}_+ \hat{I}_- + \hat{I}_- \hat{I}_+), \tag{26}
\]
6
\[ \hat{I}_+ = \phi^*(t)\hat{\pi}_\phi - R^3(t)\dot{\phi}^*(t)\hat{\phi}, \quad \hat{I}_- = \phi(t)\hat{\pi}_\phi - R^3(t)\dot{\phi}(t)\hat{\phi}, \] (27)

in terms of complex classical solutions of Eq. (5) that satisfy the boundary condition

\[ R^3(t) \left( \phi^*(t)\dot{\phi}(t) - \phi(t)\dot{\phi}^*(t) \right) = i, \quad \text{Im} \left( \frac{\dot{\phi}(t)}{\phi(t)} \right) < 0. \] (28)

Then the \( \hat{I}_+ \) acts as the creation operator \( \hat{A}^\dagger(t) \) and the \( \hat{I}_- \) as the annihilation operator \( \hat{A}(t) \).

The ground quantum state is given by

\[ \langle \phi | \Phi_0(\phi, R) \rangle = \frac{1}{(2\pi\hbar|\phi(t)|^2)^{1/4}} \exp \left[ i \frac{R^3\dot{\phi}(t)}{2\hbar\phi(t)} \phi^2 \right], \] (29)

and the \( n \)th quantum state by

\[ \langle \phi | \Phi_n(\phi, R) \rangle = \frac{1}{(2\pi\hbar|\phi(t)|^2)^{1/4}} \frac{1}{\sqrt{2^n n!}} \left( \frac{i}{|\phi(t)|} \right)^n H_n \left( \frac{\phi}{\sqrt{2\hbar|\phi(t)|}} \right) \exp \left[ i \frac{R^3\dot{\phi}(t)}{2\hbar\phi(t)} \phi^2 \right], \] (30)

where \( H_n \) is the \( n \)th Hermite polynomial. Each eigenstate is a coherent state following the classical trajectory. From the quantum back reaction of the scalar field

\[ :H :_{nn}(t) = n\hbar R^3(t) \left( \phi^*(t)\dot{\phi}(t) + m^2\phi^*(t)\phi(t) \right) \] (31)

we obtain the Einstein-Hamilton-Jacobi equation with the quantum back reaction

\[ \left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} - \Lambda = 16\pi n\hbar\kappa \left( \phi^*(t)\dot{\phi}(t) + m^2\dot{\phi}^*(t)\phi(t) \right). \] (32)

The ground state \( (n = 0) \) of the scalar field leads to the vacuous Einstein equation. Furthermore, one can show that the semiclassical gravity reduces the classical Einstein gravity by identifying the amplitude of the classical field \( \phi_c \) with \( \phi_0 \phi_q \) where \( \phi_0 = \sqrt{2n\hbar} \) and \( \phi_q \) is subject to the boundary condition (28). In particular, the classical field energy is proportional to the field squared, whereas the quantum energy to \( n\hbar \); therefore for a large quantum number \( n \) one may expect the correspondence \( \phi_0 = \sqrt{2n\hbar} \). A factor of 2 is a special feature of the coherent state of the nonadiabatic eigenstate subject to the boundary condition (28).

To summarize, we showed how the classical spacetime obeying the Einstein-Hamilton-Jacobi equation with the back reaction of a classical matter field, which is equivalent to the
classical Einstein equation equated with the matter field, can emerge through the investigation of a quantum FRW cosmological model minimally coupled to a free massive scalar field. The result of this letter differs somewhat conceptually and technically from the others on the semiclassical gravity. The quantum status of the matter field has still been kept in the gravitational field equation throughout the asymptotic limiting procedure, a Born-Oppenheimer expansion. There needs no restriction on the form of the wave function of the matter field as far as the cosmological time is appropriately defined. The remaining open question is to see how the method used in this letter can be applied successfully to the quantum FRW cosmological model minimally coupled to a generic scalar field with an arbitrary potential and whether classical spacetime emerges really from quantum gravity. Even though to find the generalized invariants is not so simple for a potential other than a quadratic one, it is likely that the quantum back reaction of a minimal scalar field will be proportional to the classical one in the leading order of the Planck constant.

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