Extreme dyonic black holes 
in string theory

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Abstract

Supersymmetric extreme dyonic black holes of toroidally compactified heterotic or type II string theory can be viewed as lower-dimensional images of solitonic strings wound around a compact dimension. We consider conformal sigma models which describe string configurations corresponding to various extreme dyonic black holes in four and five dimensions. These conformal models have regular short-distance region equivalent to a WZW theory with level proportional to magnetic charges. Arguments are presented suggesting a universal relation between the black hole entropy (area) and the statistical entropy of BPS-saturated oscillation states of solitonic string.

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1 Introduction

String theory generalises the Einstein theory in many ways. In addition to the Einstein term, the low-energy effective action of string theory contains also a combination of extra massless ‘matter’ fields with special couplings; there are classical higher derivative $\alpha'$-corrections; exact classical string solutions are described by conformal field theories which often correspond to higher dimensional backgrounds and their properties should be determined by studying propagation of test strings instead of test particles, etc. As a result, string theory introduces several new aspects in the black-hole physics. At the same time, it is also likely to provide answers to some old questions.

In general, properties of leading-order string solutions are modified by both classical ($\alpha'$) and quantum ($e^\Phi$) string corrections. There exists a special class of extreme (supersymmetric, BPS saturated) black hole solutions for which these corrections are expected (see e.g. [1]) to be more under control and which, hopefully, may be used as ‘solvable models’ to study some aspects of black hole physics.

The extreme field configurations contain vector fields of Kaluza-Klein origin and thus are actually 4-dimensional ‘images’ (or dimensional reductions) of higher dimensional backgrounds. It turns out that electrically charged extreme $D = 4$ black holes are, in fact, approximate descriptions of higher (e.g. $D = 5$) dimensional fundamental string backgrounds [2]. The latter are field configurations produced by classical fundamental strings wound around compact spatial directions, i.e. they are supported by $\delta$-function sources [3]. This provides an explanation [4, 5] of why the electrically charged BPS saturated extreme black holes are in one-to-one correspondence with BPS states in elementary string spectrum in flat space [6, 7].

The condition of BPS saturation is still satisfied if, in addition to some winding and momentum numbers (related to electric charges of $D = 4$ black hole), the fundamental string source is also oscillating in compact directions (e.g., in the left-moving sector in the heterotic string case) [3, 4, 5]. Higher-dimensional backgrounds produced by fundamental strings with different oscillation patterns reduce to a family of extreme black holes with the same electric charges but different non-vanishing massive ‘Kaluza-Klein’ fields (‘hair’) which are invisible at scales larger than the compactification scale. This suggests a natural way of understanding the thermodynamic black-hole entropy in terms of degeneracy of string configurations [8, 10] which give rise to black holes with the same values of asymptotic charges [4, 5]. To reproduce the black hole entropy as a statistical entropy $\ln d(N)$ it is important that the oscillating object is a string-like, i.e. has exponentially growing number of states at a given oscillator level $N$ (it is not enough to consider just a Kaluza-Klein field theory).\(^2\)

\(^2\)Though singular at the classical level the fundamental string solutions should become regularised (get $\sqrt{\alpha'}$-'thickness') once quantum oscillations of the source (i.e. $\alpha'$-corrections) are taken into account [9]. This expectation is essentially equivalent in the present context to the suggestion [10] that the thermodynamic black-hole entropy, which vanishes when evaluated at the singular horizon
The magnetically charged extreme black holes are dimensional reductions of certain $D \geq 5$ solitonic configurations \cite{11,12}. They are regular, i.e. have a finite size of the ‘core’.\(^3\) While for the extreme electric black hole or fundamental string solution the effective string coupling (exponential of the dilaton, $e^{2\Phi} = e^{2\Phi_{\infty}}(1 + \frac{Q}{r})^{-1}$) goes to zero at small scales, it blows up near the origin for the magnetic solution ($e^{2\Phi} = e^{2\Phi_{\infty}}(1 + \frac{P}{r})$). As a result, string loop corrections may not be ignored.

Remarkably, this problem can be avoided if one considers certain extreme dyonic black holes. The condition of preservation of supersymmetry implies \cite{1,14} that the electric ($Q$) and magnetic ($P$) charges should correspond to different vector fields. In addition, since we are interested in string-theory solutions which have exact conformal $\sigma$-model interpretation, these vector fields (at least part of them) should originate from different internal dimensions.\(^4\) These black holes are reductions of higher ($D \geq 6$) dimensional string solitons \cite{18,19} with all of the background fields being regular. As in the purely electric case there is actually a large class of $D = 4$ dyonic BPS-saturated black hole backgrounds which differ only by their short-distance structure and correspond to BPS-saturated excitations of the solitonic string \cite{20,19}.

The dilaton field is generically given by a ratio of the magnetic and electric charge factors, e.g., $e^{2\Phi} = e^{2\Phi_{\infty}}(r + P)/(r + Q)$. The approximate constancy of the dilaton both at large and small distances ensures that these four-dimensional dyonic black holes are more ‘realistic’ than purely electric or purely magnetic ones. Indeed, these solutions have global space-time structure of extreme Reissner-Nordström black holes. Also, in contrast to the purely electric or magnetic extreme black holes which have zero area of the horizon, here the area is proportional to the product of electric and magnetic charges \cite{14}, opening a possibility to compare the statistical entropy to Bekenstein-Hawking one \cite{20}. By analogy with the corresponding counting for purely electric black holes (oscillating fundamental string counting) \cite{10,4,5} the entropy should now have an interpretation in terms of counting of different BPS-saturated solitonic string states with given values of asymptotic charges \cite{20,19}.

Below we shall consider an example of such background described by magnetically charged (via coupling to sixth internal dimension) solitonic string wound around fifth compact dimension (with winding and momentum numbers giving rise to electric

\(^3\)The magnetic charge plays the role of a short-distance regulator, providing an effective shift $r \to r + P$ (analogous to $r \to r + \sqrt{\alpha'}$ that should happen in the exact fundamental string solution).

\(^4\)Supersymmetric dyonic solutions with $Q$ and $P$ corresponding to the two vector fields ($G_{5\mu}$ and $B_{5\mu}$) associated with the same Kaluza-Klein dimension \cite{15,16} can be obtained from purely-electric or purely-magnetic solutions by $SL(2,R)$ duality rotations (which change $\sigma$-model metric and in general may not preserve manifest conformal invariance of $\sigma$-model since $S$-duality is not a symmetry at string tree level). Dyonic backgrounds with both $Q$ and $P$ associated with the same vector field (see, e.g., \cite{17}) completely break supersymmetry \cite{14} and do not correspond to a manifestly conformal 2d theory, i.e. are non-trivially deformed by $\alpha'$ corrections. I am grateful to T. Ortín for clarifying discussions of related issues.
charges of the $D = 4$ black hole) [18]. The important point is that here one is dealing with explicitly known well-defined conformal $\sigma$-model. For large values of charges the level of the WZW-type conformal field theory which describes the short-distance ('throat') region is large and thus the counting of excited states should proceed in more or less the same way as in flat space case up to a rescaling of the string tension by a product of magnetic charges [19], as suggested in [20]. As a result, one is able to reproduce the Bekenstein-Hawking expression for the black hole entropy using semiclassical string-theory considerations [20, 19]. The universality of this approach is confirmed by a similar analysis of the case of (both non-rotating and rotating) $D = 5$ extreme dyonic black holes (the special case of $D = 5$ dyonic black holes with all charges being equal was considered using different methods in [21, 22]).

We shall start with listing the conformal $\sigma$-models describing string configurations which represent, upon dimensional reduction, to various $D = 4$ and $D = 5$ extreme black hole backgrounds (Section 2). We shall then concentrate on the simplest models (corresponding to 4-charge dyonic black hole in four dimensions [14, 18] and 3-charge dyonic black hole in five dimensions) and explain, both for $D = 4$ and $D = 5$ cases, how one may relate their thermodynamic entropy to the statistical entropy of excited BPS-saturated solitonic string states (Section 3). In Section 4 we shall consider a conformal $\sigma$-model describing a rotating $D = 5$ supersymmetric extreme dyonic black hole which generalises the solution in [22]. We shall find that conditions of conformal invariance and regularity at short distances demand the presence of two equal components of rotation in two orthogonal planes. The consistency of 2d conformal model implies a quantization condition for the angular momentum and a bound on its maximal value.

Most of the results about $D = 4$ dyonic black holes have already appeared in [18, 19] while the construction of $D = 5$ dyonic black holes and statistical understanding of their entropy using the conformal $\sigma$-model approach are new.

2 Conformal models for fundamental and solitonic strings, and extreme black holes

Classical string solutions are described by conformal 2d $\sigma$-models. In the case when the background fields ($\sigma$-model couplings) do not depend on some compact isometric coordinates one may re-interpret the $\sigma$-model action as representing a lower-dimensional background with extra Kaluza-Klein fields (for a review see, e.g., [23]).

Below we shall discuss the conformal models for certain fundamental and solitonic string backgrounds, which, upon dimensional reduction, may be identified as $N \geq 1$ supersymmetric $D = 4$ or $D = 5$ extreme black holes. We shall give only the bosonic parts of the corresponding (2, 1) (in type II theory case) or (2, 0) (in heterotic string theory case) world-sheet supersymmetric Lagrangians.
2.1 Fundamental strings and electric black holes

The fundamental string solutions are described by the $D = 10$ conformal chiral null models [24] with flat transverse part. An example is provided by ($u = t + y, \, v = t - y$)

$$L = F(x) \partial u \left[ \partial v + K(x) \partial u \right] + \partial x^i \partial x^i + R \Phi(x). \quad (1)$$

This model is conformal to all orders if

$$\partial^2 F^{-1} = 0, \quad \partial^2 K^{-1} = 0, \quad \Phi = \Phi_{\infty} + \frac{1}{2} \ln F. \quad (2)$$

If the functions depend on $D - 2$ transverse dimensions, then

$$F^{-1} = 1 + \frac{Q_2}{r^{D-4}}, \quad K = K_0 + \frac{Q_1}{r^{D-4}}, \quad r^2 = x^i x^i, \quad (3)$$

describes (for $K_0 = 0$) a background produced by a fundamental string [3, 5] wound around the compact dimension $y \equiv y + 2\pi R$ with the winding number $w$ and the momentum number $m$ along $y$ being proportional to $Q_2$ and $Q_1$,

$$Q_1 = \frac{16\pi G_{(N)}^{(D-1)}}{(D-4)\omega_{D-3}} \cdot \frac{m}{R}, \quad Q_2 = \frac{16\pi G_{(N)}^{(D-1)}}{(D-4)\omega_{D-3}} \cdot \frac{wR}{\alpha'}. \quad (4)$$

Here $\omega_{D-3} = 2\pi^{\frac{D-2}{2}}/\Gamma(\frac{D-2}{2})$ is the area of unit sphere in $D - 3$ dimensions and $G_{(D)}_{(D-1)} = G_{(D)}/2\pi R$ is the Newton’s constant in $D - 1$ dimensions (which is fixed under $T$-duality, $m \leftrightarrow w, \, R \leftrightarrow \alpha'/R, \, Q_1 \leftrightarrow Q_2$). In the case of $D = 5$ fundamental string $x^i = (x^s, y^n)$ where $x^s \ (s = 1, 2, 3)$ are three non-compact spatial coordinates and $y^n \ (n = 1, ... , 5)$ are toroidally compactified coordinates (with $y \equiv y^6$). The dimensional reduction along $y$ gives extreme electrically charged $D = 4$ black hole. Equivalent $D = 4$ background is found by choosing $K_0 = 1, \, u = y', \, v = 2t$ and dimensionally reducing along the ‘boosted’ coordinate $y'$. Starting with the $D = 5$ model (1), (4) with

$$F^{-1} = 1 + \frac{Q_2}{r}, \quad K = 1 + \frac{Q_1}{r}, \quad r^2 = x^s x^s, \quad (5)$$

$$Q_1 = 4G_N \frac{m}{R} \equiv 4G_N \bar{Q}_1, \quad Q_2 = 4G_N \frac{Rw}{\alpha'} \equiv 4G_N \bar{Q}_2, \quad G_N \equiv G_{(N)}^{(4)}, \quad (6)$$

5We shall use the following notation. The free string action is $I = (\pi \alpha')^{-\frac{1}{2}} \int d^2 \sigma d\xi d\bar{\xi} = (4\pi \alpha')^{-\frac{1}{2}} \int d^2 \sigma \partial_\sigma x \partial_\bar{\sigma} x$. The string effective action is $S_D = (16\pi G_N^{(D)})^{-\frac{1}{2}} \int d^D x \sqrt{-G} e^{-2\Phi}(R + ...)$ = $(16\pi G_{(D)}^{(D)})^{-\frac{1}{2}} \int d^D x \sqrt{-g}(R_g + ...)$ Here $G_{\mu\nu}$ and $g_{\mu\nu}$ are the string-frame and Einstein-frame metrics (both equal to $\eta_{\mu\nu}$ at $r \rightarrow \infty$), $\Phi'$ is the non-constant part of the dilaton (i.e. $\Phi_{\infty} = 0$). In general, the Newton’s constant is $G_{(D)} \sim e^{2\Phi_{\infty}} \alpha'^4/V_{10-D}$ where $V_{10-D}$ stands for the volume of internal $(10 - D)$-dimensional space.
one finds the following $D = 4$ string-frame metric [24]
\[ ds^2 = -\lambda^2(x)dt^2 + dx^i dx^i , \quad \lambda^2 = FK^{-1} . \] (7)
The Einstein-frame metric and the (non-constant part of) $D = 4$ dilaton are
\[ ds^2_E = -\lambda(r)dt^2 + \lambda^{-1}(r)(dr^2 + r^2 d\Omega^2_2) , \] (8)
\[ \lambda(r) = e^{2\Phi(r)} = \frac{r}{\sqrt{(r + Q_1)(r + Q_2)}} . \] (9)
In addition, there are two vector fields and a scalar modulus (radius of $y$-direction).
The $T$-duality in the $y$-direction interchanges the ‘winding’ and ‘momentum’ charges $Q_2$ and $Q_1$ (i.e. interchanges the vector fields associated with $B_{\mu y}$ and $G_{\mu y}$).
The special cases of $Q_2 = 0$ and $Q_1 = 0$ describe the Kaluza-Klein ($a = \sqrt{3}$) extreme electric black hole and its $T$-dual, while the case of $Q_1 = Q_2$ corresponds to the dilatonic ($a = 1$) extreme black hole [25].

Other supersymmetric BPS-saturated black hole solutions with the same values of electric charges $Q_1, Q_2$ but different short-distance structure are found by starting with a more general conformal model (which can be viewed as an integrated marginal deformation of the model (1)) describing fields produced by BPS-saturated oscillating string states of the free string spectrum with fixed values of the winding and momentum numbers in the $y$-direction. This model is itself a special case of the chiral null model [24]
\[ L = F(x)\partial u \left[ \bar{\partial}v + K(u, x)\bar{\partial}u + 2\mathcal{A}_i(u, x)\bar{\partial}x^i \right] + \partial x^i \bar{\partial}x^i + R\Phi(x) , \] (10)
where, for conformal invariance, $\mathcal{A}_i$ should satisfy $\partial_i F^{ij} = 0$, $F_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$. For example, one may choose $K = f_n(u)y^n + Q_1/r$, $\mathcal{A}_5 = q/r$ where $f_n$ will then be related to the profile of ‘left’ oscillations of the string source in ‘transverse’ directions $y^n$ ($\partial^2 y^n = -2f^n$) and $q$ will be the (‘left’) electric charge of the string. The level matching condition for the string source will then imply $Q_1 Q_2 - q^2 \sim (\partial u y^n)^2 > [5]$. Viewed from 4 dimensions (i.e. after averaging in $u$), this background will describe a family of extreme electrically charged black holes with the same electric charges ($Q_1, Q_2; q, -q$) (corresponding to 6-th and 5-th internal dimensions) but different short-distance structure (massive ‘hair’) depending on a choice of the oscillation profile function [4, 5].

### 2.2 Superconformal $\sigma$-models and magnetic black holes

The magnetic counterparts of the extreme $D = 4$ electric black holes can be found by $D = 4$ $S$-duality transformation, or by considering the above fundamental string model as a 6-dimensional background and applying the $S$-duality transformation in
6 dimensions \((G \rightarrow e^{-2\Phi} G, \ dB \rightarrow e^{-2\Phi} dB, \ \Phi \rightarrow -\Phi)\).  At the string-theory level the corresponding solitonic configurations are described by \(N = 4\) superconformal models with the bosonic part of the Lagrangian given by [18]

\[
L = -\partial t \partial t + \partial y^n \partial y^n + f(x)k(x)[\partial y + a_s(x)\partial x^s][\bar{\partial} y + a_s(x)\bar{\partial} x^s]
+ f(x)k^{-1}(x)\partial x^s \bar{\partial} x^s + b_s(x)(\partial y \bar{\partial} x^s - \bar{\partial} y \partial x^s) + R \phi(x) .
\]  

(11)

As above, \(x^s\) are three non-compact spatial dimensions, while \(y\) and \(y^n\) are compact coordinates. To have \(N = 4\) ((4,0) in the heterotic or (4,1) in the type II case) world-sheet supersymmetry and conformal invariance we shall assume that all the fields depend only on \(x^s\) and \(f, k, a_s, b_s, \phi\) are subject to

\[
\partial_s \partial^s f = 0 , \quad \partial_s \partial^s k^{-1} = 0 , \quad (12)
\]

\[
\partial_p b_q - \partial_q b_p = -\epsilon_{pq} \partial^s f , \quad \partial_p a_q - \partial_q a_p = -\epsilon_{pq} \partial^s k^{-1} , \quad \phi = \frac{1}{2} \ln f . \quad (13)
\]

Under \(T\)-duality in \(y\)-direction \(f \rightarrow k^{-1}, \ k \rightarrow f^{-1}, \ a_s \rightarrow b_s, \ b_s \rightarrow a_s, \) (provided \(a[p,bq] = 0)\). The simplest 1-center choice for the harmonic functions

\[
f = 1 + \frac{P_2}{r}, \quad k^{-1} = 1 + \frac{P_1}{r}, \quad a_s dx^s = P_1(1 - \cos \theta) d\varphi, \quad b_s dx^s = P_2(1 - \cos \theta) d\varphi , \quad (14)
\]

leads, upon dimensional reduction along \(y\) and \(y^n\) directions, to the \(D = 4\) extreme magnetic black hole backgrounds parametrised by the two charges \(P_1\) and \(P_2\). The special cases of this model include: the Kaluza-Klein monopole or \(a = \sqrt{3}, \ D = 4\) magnetic black hole \((P_2 = 0)\), its \(T\)-dual – the \(H\)-monopole background [11] \((P_1 = 0)\), and the \(T\)-self-dual model corresponding to the \(a = 1\) dilatonic magnetic black hole [12] \((P_1 = P_2)\). A generalization of (11) to the case of several magnetic charges (obtained by adding extra free coordinates and applying \(T\)-duality transformation) was discussed in [26].

The \(D = 4\) metric corresponding to (11) has the form (7),(8) with

\[
\lambda(r) = \sqrt{k f^{-1}} = e^{-2\Phi(r)} = \frac{r}{\sqrt{(r + P_1)(r + P_2)}} . \quad (15)
\]

As in the case of the ‘5-brane’ solitonic model [27], the short-distance \(r \rightarrow 0\) region of the model (11) is a regular ‘throat’ and is described by (a factor of) \(SU(2)\) WZW theory with level \(\kappa = 4P_1P_2/\alpha'\).

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\(^6\)This is a formal symmetry of the part of \(D = 6\) heterotic or type II effective action which does not include extra vector fields.
2.3 Solitonic strings and dyonic black holes

It may look rather surprising that the ‘electric’ and ‘magnetic’ models (1) and (11) can be directly superposed while preserving the exact conformal invariance and \( N = 2 \) world-sheet \((N = 1, D = 4\) space-time [14]) supersymmetry. The resulting conformal model (with non-trivial 6-dimensional part) is defined by [18]

\[
L = F(x)\partial u \left[ \bar{\partial}v + K(x)\partial u \right] + f(x)k(x)[\partial y_1 + a_s(x)\partial x^s][\bar{\partial}y_1 + a_s(x)\bar{\partial}x^s] + f(x)k^{-1}(x)\partial x^s\bar{\partial}x^s + b_s(x)(\partial y_1\bar{\partial}x^s - \bar{\partial}y_1\partial x^s) + \partial y^k\bar{\partial}y^k + R\Phi(x),
\]

\[
\Phi = \Phi_\infty + \frac{1}{2}\ln(Ff),
\]

where \( u = y_2, v = 2t, k = 3, 4, 5, 6 \), all functions depend only on \( x^s \) \((s = 1, 2, 3)\) and \( F^{-1}, K, f, k^{-1} \) are harmonic as in (2),(12),(13). With the functions chosen as in (5),(14) this model describes a magnetically charged solitonic string wound around a compact ‘fifth’ dimension.\(^7\)

The corresponding \( D = 4 \) background obtained by dimensional reduction along six compact \( y \)-coordinates is the supersymmetric BPS-saturated extreme dyonic black hole (first found as a leading order \( D = 4 \) solution of \( T^6 \)-compactified heterotic string in [14]).\(^8\) In addition to the 2 electric and 2 magnetic vector fields and two scalar moduli (radii of \( y_1 \) and \( y_2 \) circles) the dyonic background includes the Einstein-frame metric (8) with

\[
\lambda(r) = \sqrt{FK^{-1}kf^{-1}} = \frac{r^2}{\sqrt{(r + Q_1)(r + Q_2)(r + P_1)(r + P_2)}},
\]

and the dilaton \( \Phi \) or the effective \( D = 4 \) dilaton \( \Phi'_(4) \)

\[
e^{2\Phi} = e^{2\Phi_\infty}\sqrt{\frac{r + P_2}{r + Q_2}}, \quad e^{2\Phi'_(4)} = \sqrt{\frac{(r + P_1)(r + P_2)}{(r + Q_1)(r + Q_2)}}.
\]

Assuming all charges are positive, \( r = 0 \) is a regular horizon (for \( P_i = 0 \) the horizon coincides with singularity). There is also a time-like singularity at \( r_{sing} = -\min\{P_1, P_2, Q_1, Q_2\} \), i.e. the global structure of \( D = 4 \) space-time is that of extreme Reissner-Nordström black holes [14].\(^9\) It should be noted that these conclusions apply if one considers this background as a usual 4-dimensional one and applies the standard

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\(^7\)Some special cases and related models were discussed also in [28, 29].

\(^8\)It is possible, of course, to consider more general (e.g. multi-center) choices for the harmonic functions \( F^{-1}, K, f, k^{-1} \). In particular, taking all 4 functions to be \( O(3) \)-symmetric but having centers at different points one finds the solution discussed in [30].

\(^9\)If all charges are equal the dilaton is constant and the solution coincides with the extreme dyonic Reissner-Nordström black hole of the standard Einstein-Maxwell \((a = 0)\) theory. The case of \( Q_1 = Q_2 = P_1, P_2 = 0 \) corresponds to a solution in the case of dilatonic coupling constant \( a = 1/\sqrt{3} \) [14, 31].
rules of geodesic extension (what corresponds to analytic continuation of $r$ to negative values). At the same time, if such backgrounds are considered as string-theory solutions, they are effectively higher-dimensional when probed at small scales and thus their short-distance structure should actually be analyzed from a higher dimensional point of view (more precisely, from the point of view of underlying conformal field theory). The extension of $r$ to negative values seems not to make sense in that case.

The ADM mass of black hole and the area of the event horizon ($A \equiv 4\pi(\lambda^{-1}r^2)_{r=0}$) are found to be

$$M = \frac{1}{4G_N}(Q_1 + Q_2 + P_1 + P_2), \quad A = 4\pi\sqrt{Q_1Q_2P_1P_2}.$$  \hspace{1cm} (19)

The expression for the mass can be written also as

$$M = \bar{Q}_1 + \bar{Q}_2 + g^{-2}(\bar{P}_1 + \bar{P}_2),$$  \hspace{1cm} (20)

where we have chosen $G_N \equiv G_N^{(4)} = \frac{1}{8}\alpha'\alpha^{2\infty} \equiv \frac{1}{8}\alpha'\alpha^2$ and $Q_i \equiv 4G_N\bar{Q}_i = \frac{1}{2}\alpha'\alpha^2\bar{Q}_i$, $P_i \equiv \frac{1}{2}\alpha'\bar{P}_i$ (cf.(6)). One needs all four charges to be non-vanishing to get a non-zero area of the horizon and thus a non-zero value of the analogue of the Bekenstein-Hawking entropy$^{10}$

$$S_{BH} \equiv \frac{A}{4G_N}.$$  \hspace{1cm} (21)

As in the purely electric case, there is a whole family of black hole solutions with the same values of electric and magnetic charges but different short-distance structure. They are 4-dimensional ‘images’ of certain (supersymmetric BPS-saturated) excited states of the solitonic string (16) which are described by a marginal ‘deformation’ of (16) which includes extra ‘chiral’ couplings (see (10) and the next subsection). The knowledge of underlying 2d conformal field theory makes possible to count the number $d(N)$ of such states for a given set of (large) charges $(Q_1, Q_2, P_1, P_2)$ and to identify [19] the Bekenstein-Hawking entropy (21) with the statistical entropy $\ln d(N)$, supporting the proposal of [20]. This will be discussed in Section 3.

### 2.4 Generalisations and related models

The models (11),(16) are special cases of a chiral null model with curved transverse part$^{11}$

$$L = F(x)\partial u \left[ \partial v + K(u, x)\partial u + 2A_i(u, x)\partial x^i \right] + \frac{1}{2}\mathcal{R}\ln F(x) + L_\perp,$$  \hspace{1cm} (22)

$^{10}$Extremal black holes have zero temperature and simplest examples of them were shown to have zero thermodynamic entropy [32]. This zero entropy conclusion does not, however, apply in the present 4-charge case as one can show by first defining the entropy in the non-extremal case (the corresponding solution was given in [14]) and then taking the extremal limit. It is important that in contrast to the simple Reissner-Nordström case here the limit involves several different parameters. Thus the Bekenstein-Hawking formula is still expected to apply to the case of these extreme dyonic black holes (cf.[33]). I am grateful to M. Cvetič for explanations on this point.

$^{11}$When both $K$ and $A_i$ depend on $u$ one can redefine $v$ to absorb $K$ into $A_i$ [24].
\[ L_\perp = (G_{ij} + B_{ij}) (x) \partial x^i \bar{\partial} x^j + \mathcal{R} \phi (x) \, . \]  

There exists a renormalisation scheme in which (22) is conformal to all orders in \( \alpha' \) provided the ‘transverse’ \( \sigma \)-model (23) is conformal and the functions \( F^{-1}, K, A_i, \Phi \) satisfy certain conformal invariance conditions. The simplest tractable case is when the transverse theory has at least \((4,0)\) extended world-sheet supersymmetry so that the conformal invariance conditions essentially preserve their 1-loop form [34], i.e. are the ‘Laplace’ equations in the ‘transverse’ background (for simplicity here we assume that \( K \) and \( A_i \) are \( u \)-independent) [24, 19]

\[ \partial_i (e^{-2\phi} \sqrt{G} G^{ij} \partial_j F^{-1}) = 0 \; , \; \partial_i (e^{-2\phi} \sqrt{G} G^{ij} \partial_j K) = 0 \, , \]  

\[ \hat{\nabla}_i (e^{-2\phi} \mathcal{F}^{ij}) = \frac{1}{\sqrt{G}} \partial_i (e^{-2\phi} \sqrt{G} \mathcal{F}^{ij}) - \frac{1}{2} e^{-2\phi} H^{kij} \mathcal{F}_{ki} = 0 \, , \]  

where \( \hat{\Gamma}_{jk} = \Gamma_{jk} + \frac{1}{2} H_{jk} \), \( \mathcal{F}_{ij} \equiv \partial_i A_j - \partial_j A_i \).

For example, the 8-dimensional transverse space may be chosen as a direct product of some 4-space \( \mathcal{M}^4 \) and a 4-torus. If the functions defining \( \mathcal{M}^4 \)-model have \( SO(3) \) symmetry we are led back to the case of (16) with 1-center harmonic functions. Choosing \((s = 1, 2, 3)\)

\[ A_i (x) \bar{\partial} x^i = A (x) [\bar{\partial} y_1 + a_s (x) \bar{\partial} x^s ] \, , \; \; \; A = \frac{q}{r} \; \frac{r + \frac{1}{2} (P_1 + P_2)}{r + P_1} \, , \]  

we find a generalisation [19] of the ‘dyonic’ model (16) which describes a spherically symmetric \( D = 4 \) background with two extra electric charges \((q, -q)\) and the metric function and the area of event horizon being (the mass remains the same as in (19))

\[ \lambda = \frac{r^2}{\sqrt{(r + Q_1)(r + Q_2)(r + P_1)(r + P_2) - q^2 [r + \frac{1}{2} (P_1 + P_2)]^2}} \, ; \]  

\[ A = 4\pi \sqrt{Q_1 Q_2 P_1 P_2 - \frac{1}{4} q^2 (P_1 + P_2)^2} \, . \]  

Further generalisations of the dyonic model (16) (responsible for the ‘hair’ of 4-dimensional dyonic backgrounds) are obtained, e.g., by introducing \( \mathcal{A}_i \) which describes rotation and/or Taub-NUT charge (see also [24] and Section 4) or by switching on the \( u \)-dependence in \( K \) (or, equivalently, in \( A_i \)) as in the fundamental string case (10) discussed above.

Another possibility is to consider 6-dimensional models with \( M^4 \)-part having (locally) \( SO(4) \)-invariant structure. In this case the transverse theory is the same as in the 5-brane model [27]

\[ L_\perp = f (x) \partial x^m \bar{\partial} x^m + B_{mn} (x) \partial x^m \bar{\partial} x^n + \mathcal{R} \phi (x) \, , \; \; \; \phi = \frac{1}{2} \ln f \, , \; \partial^2 f = 0 \, , \]  

\[ G_{mn} = f (x) \delta_{mn} \, , \; \; \; H_{mnk} = -2 \sqrt{G} G_{pl} \epsilon_{mnpk} \partial_l \phi = -\epsilon_{mnlk} \partial_l f \, , \]
where \( m, n, ... = 1, 2, 3, 4 \). One finds that the Laplace-type equations for \( F^{-1}, K \) preserve their flat-space form (2) and thus (we assume that \( \mathcal{A}_i = 0; \text{cf.}(5),(14))

\[
    f = 1 + \frac{P}{r^2}, \quad F^{-1} = 1 + \frac{Q_2}{r^2}, \quad K = 1 + \frac{Q_1}{r^2},
\]

\( e^{2\Phi} = Ff = \frac{r^2 + P}{r^2 + Q_2}, \quad r^2 \equiv x^m x^m. \)

The special cases of this model are: (1) six-dimensional fundamental string \((P = 0)\) with momentum and winding numbers \( \sim Q_{1,2} \) [3]; (2) its \( D = 6 \) \( S \)-dual – solitonic string solution [37] \((Q_1 = Q_2 = 0, P \neq 0)\) which is a \( D = 6 \) reduction of \( D = 10 \) 5-brane solution [27, 35]; (3) ‘dyonic’ \( D = 6 \) string of [38] \((Q_1 = 0, Q_2 \neq 0, P \neq 0)\). A generalisation of this background containing one extra (angular momentum) parameter will be constructed in Section 4.

Dimensionally reducing the corresponding \( D = 6 \) model to 5 dimensions along \( u \) (or, from the point of view of the corresponding \( D = 10 \) solution, wrapping the 5-brane around \( S^1 \times T^4 \)) one finds the 3-charge \((Q_1, Q_2, P)\) extreme dyonic \( D = 5 \) black hole background.\(^{13}\) The 5-dimensional string-frame metric and the Einstein-frame metric are related by \( g_{\mu\nu} = e^{-\frac{4}{3} \Phi(\nu)} G_{\mu\nu}, e^{4\Phi(\nu)} = f^2 F K^{-1}, e^{2\Phi} = e^{2\Phi} F f \), so that

\[
    ds_5^2 = -\lambda^2(r) dt^2 + \lambda^{-1}(r)(dr^2 + r^2 d\Omega_3^2),
\]

\[
    \lambda = (F^{-1} K f)^{-1/3} = \frac{r^2}{[(r^2 + Q_1)(r^2 + Q_2)(r^2 + P)]^{1/3}}.
\]

The mass of this \( D = 5 \) black hole, the 3-area of the regular \( r = 0 \) horizon and the Bekenstein-Hawking entropy are (cf.(19),(21))\(^{14}\)

\[
    M = \frac{\pi}{4G_N} (Q_1 + Q_2 + P), \quad G_N \equiv G^{(5)}_N, \quad (32)
\]

\[
    A = 2\pi^2 \sqrt{Q_1 Q_2 P}, \quad S_{BH} = \frac{A}{4G_N}. \quad (33)
\]

\(^{12}\)The \( D = 10 \) model (22),(29) may thus be considered as an anisotropic generalisation of the 5-brane model of [35, 27]: different isometric 5-brane coordinates are multiplied by different functions of the 4 orthogonal coordinates, cf.[36]. In a sense, it describes a fundamental string ‘lying’ on a 5-brane: the 5-brane is wrapped around 4-torus times \( S^1 \) (\( u \)-circle) with the fundamental string wound around the latter. The non-singular solitonic nature of the 5-brane is responsible for regularity of the ‘combined’ background.

\(^{13}\)In five dimensions an antisymmetric tensor is dual to a vector so that \( H_{mnp} \) is a magnetic dual of an electric field with charge \( P \). The duality maps a \( D = 5 \) solution with 2 electric and one ‘magnetic’ charge of one theory into a solution with 3 electric charges of a dual theory.

\(^{14}\)In general, for a black hole in \( D \) dimensions with \( g_{tt} = -1 + \mu/r^{D-3} + ... \) the ADM mass is (see, e.g., [39]) \( M = \mu(D - 2) \omega_{D-2}/16\pi G^{(D)}_N \) \((\omega_2 = 4\pi, \omega_3 = 2\pi^2, \text{etc}.\)). The Bekenstein-Hawking entropy is expressed in terms of the volume \( A \) of \((D - 2)\)-dimensional horizon surface by \( S_{BH} = A/4G^{(D)}_N \).
If $Q_1 = Q_2 = P$ we find $\lambda^{-1} = f = 1 + P/r^2$, $\Phi = \Phi_\infty$. Introducing $\rho^2 = r^2 + \rho_0^2$, $\rho_0 \equiv \sqrt{P}$ we then finish with the $D = 5$ extreme Reissner-Nordström metric

$$ds_E^2 = -\lambda^2 dt^2 + \lambda^{-2}d\rho^2 + \rho^2 d\Omega_3^2, \quad \lambda = 1 - \rho_0^2/\rho^2.$$ \hspace{1cm} (34)

This special case ($Q_1 = Q_2 = P$) was discussed in [21] in connection with reproducing the Bekenstein-Hawking entropy as a statistical entropy using the D-brane approach to count the number of corresponding microscopic BPS states. As in the case of the $D = 4$ dyonic black hole model, this counting can be done also in a direct way (for generic large values of $(Q_1, Q_2, P)$) using the fact that the small-scale (‘throat’) region of the corresponding $D = 6$ conformal model is described by a WZW model with level $\kappa = P/\alpha'$ (see Section 3.3).

There are also many other exact $D = 10$ superstring solutions described by (22),(23). For example, we may take the 8-dimensional transverse model to be a product of the two ‘5-brane’ models (29)

$$L = F(x, y) \partial u \left[ \partial v + K(x, y) \partial u \right] + f_1(x) \partial x^m \partial x^m + f_2(y) \partial y^m \partial y^m$$

$$+ B_{1mn}(x) \partial x^m \partial x^n + B_{2mn}(y) \partial y^m \partial y^n + \frac{1}{2} R \ln[F(x, y)f_1(x)f_2(x)],$$

where $f_1$ and $f_2$ are harmonic. The conformal invariance conditions $\nabla_i(e^{-2\phi}(F^{-1})) = 0$, $\nabla_i(e^{-2\phi}(K)) = 0$ reduce to\hspace{1cm} (35)

$$[f_2(y)\partial_x^2 + f_1(x)\partial_y^2]F^{-1}(x, y) = 0, \quad [f_2(y)\partial_x^2 + f_1(x)\partial_y^2]K(x, y) = 0,$$

and thus are solved, e.g., by

$$F(x, y) = F_1(x)F_2(y), \quad K(x, y) = K_1(x)K_2(y),$$ \hspace{1cm} (36)

where $F_{1,2}$ and $K_{1,2}$ are independent harmonic functions. For the simplest 1-center choice of the functions this $D = 10$ ‘dyonic’ background is parametrised by 6 charges. The special case ($K = 1$) of this solution was found in [38].

### 3 Dyonic black hole entropy from string theory

The $D = 4$ dyonic black holes (with four (17) or five (27) parameters) discussed above belong to the set of $N = 1$ supersymmetric BPS saturated extreme dyonic solutions of the leading-order effective equations of $T^6$-compactified heterotic string which are parametrised by 28 electric and 28 magnetic charges [14, 19, 40].\hspace{1cm} (36)

15I am grateful to J. Maldacena for pointing out a mistake in the equation below in the original version of this paper.

16These more general $D = 4$ solutions can be constructed [19, 40] by applying special $T$- and $S$-duality transformations to the 5-parameter solution but not all of them directly correspond to exact string solutions, i.e. to manifestly conformal $D = 10 \sigma$-models.
of charges \((Q_i, P_i)\) one expects to find a subfamily of \(N = 1\) supersymmetric BPS-saturated black hole backgrounds which all look the same at large distances but differ in their short-distance structure (i.e. at scales of order of compactification scale where their higher dimensional solitonic string origin becomes apparent). This classical ‘fine structure’ or ‘degeneracy’ should be responsible for the black hole entropy \([4, 5, 20]\).

To try to check the proposal \([20]\) that one can indeed reproduce the Bekenstein-Hawking entropy as a statistical entropy of degenerate black hole configurations it is sufficient to consider the simplest non-trivial choice of the charges \((Q_1, Q_2, P_1, P_2)\). The aim is to explain the expression for the corresponding entropy \((21)\) in terms of degeneracy of BPS states originating from possible small-scale oscillations of underlying six-dimensional string soliton \((16),(5),(14)\). The relevant ‘oscillating’ \(D = 6\) backgrounds are described by the general model \((22)\) and can be thought of as special marginal deformations of \((16)\). Since \((16)\) defines a non-trivial CFT, the counting is not as simple as in the fundamental string (‘pure electric’) case, where, because of the matching onto string sources, one expects that BPS-saturated oscillating fundamental string configurations are in one-to-one correspondence with analogous BPS states in the free string spectrum \([4, 5]\). Still, it can be done at least approximately (for large charges) in a rather straightforward way.

The key observation is that since the degeneracy is present only at small scales (all corresponding \(D = 4\) black hole backgrounds look the same at large \(r\)) to find the number of different states one may first replace \((16)\) by its short-distance \((r \to 0)\) limit and then do the counting. In this ‘throat’ limit \((16)\) reduces to a WZW-type model with level \(\kappa = 4P_1P_2/\alpha'\). Then for large values of the level the counting of BPS states should proceed essentially as in the free string case, with only two important differences. One is that now the string tension of the ‘transverse’ part of the action is proportional to \(\kappa\). Another is that in contrast to what happens in the free string case, here the number of BPS oscillation states we should count is the same in the heterotic and type II theories. Indeed, the relevant marginal perturbations of the model \((16)\) are only ‘left’, not ‘right’ (the functions in \((22)\) can depend only on \(u\), not on \(v\) to preserve the conformal invariance when both \(F\) and \(K\) are non-trivial). This is also related to the fact that the background in \((16)\) is ‘chiral’ and has the same amount of space-time supersymmetry \((N = 1, D = 4)\) in both theories (only ‘left’ perturbations will be supersymmetric). As a result, the entropy should be the same in heterotic and type II cases, in agreement with the fact that the corresponding black hole solutions (and thus the Bekenstein-Hawking entropy) are the same.

This approach is legitimate if all four charges \((Q_1, Q_2, P_1, P_2)\) are large compared to the compactification scales (radii \(R_n\) of compact coordinates \(y_1, y_2\) which we shall assume to be of order of \(\sqrt{\alpha'}\)). Indeed, the scale of the soliton is determined by \(\sqrt{P_1P_2}\) and thus is large (i.e. the curvature at the throat is small) provided the magnetic charges are large. At the same time, if the large-distance value of string coupling is taken to be small \((e^{\Phi_{\infty}} < 1)\), to ensure that its short distance (‘throat’) value \((e^{\Phi_0})\) is also small (see \((18)\)), one is to assume that the electric charges are of the same order.
as the magnetic ones. Similar remarks apply to the case of the $D = 5$ dyonic black hole model discussed in Section 2.4.

3.1 Throat region model and magnetic ‘renormalisation’ of string tension

The model (16),(5),(14) has a regular $r \to 0$ limit described by [18, 19]

$$I = \frac{1}{\pi\alpha'} \int d^2 \sigma L_{r \to 0} = \frac{\kappa}{4\pi} \int d^2 \sigma \left( \partial z \partial \bar{z} + \partial \bar{u} \partial u + e^{-z} \partial \bar{u} \partial \bar{v} \right)$$

$$+ \partial \bar{y}_1 \partial y_1 + \partial \bar{\varphi} \partial \varphi + \partial \theta \partial \bar{\theta} - 2 \cos \theta \partial \bar{y}_1 \partial \varphi)$$

$$= \frac{1}{\pi\alpha'} \int d^2 \sigma \left( e^{-z} \partial u \partial \bar{v} + Q_2 Q_1^{-1} \partial u \partial \bar{u} \right)$$

$$+ \frac{\kappa}{4\pi} \int d^2 \sigma \left( \partial z \partial \bar{z} + \partial \bar{y}_1 \partial y_1 + \partial \varphi \partial \bar{\varphi} + \partial \theta \partial \bar{\theta} - 2 \cos \theta \partial \bar{y}_1 \partial \varphi) \right).$$

Here

$$\kappa = \frac{4}{\alpha'} P_1 P_2, \quad z \equiv \ln \frac{Q_2}{r} \to \infty,$$

$$\tilde{u} = (Q_1^{-1} Q_2 P_1 P_2)^{-1/2} u, \quad \tilde{v} = (Q_1 Q_2^{-1} P_1 P_2)^{-1/2} v, \quad \tilde{y}_1 = P_1^{-1} y_1 + \varphi.$$

As already mentioned above, an important property of this model is that here (in contrast to, e.g., the 5-brane model [27]) the dilaton is constant in the wormhole region.\(^{17}\)

The throat region model (37) is equivalent to a direct product of the $SL(2,\mathbb{R})$ and $SU(2)$ WZW theories (divided by discrete subgroups) with both levels equal to $\kappa$.\(^{18}\) Since the level of $SU(2)$ must be integer, we get the quantisation condition for the product of magnetic charges, $P_1 P_2 = \frac{1}{4} \alpha' \kappa$.\(^{19}\)

For large values of charges, i.e. large $\kappa$, one may expect that the counting of states in this model may proceed essentially as if this was a flat space case. The only (but crucial) difference is that the transverse ($z, \tilde{y}_1, \varphi, \theta$) part of the action has now ‘renormalised’ string tension

$$\frac{1}{\alpha'} \to \frac{1}{\alpha'_{\perp}} = \frac{P_1 P_2}{\alpha'^2}.$$

\(^{17}\)Assuming that all 4 charges are of the same order we may ignore the difference between the values of the dilaton (string coupling) at $r = \infty$ and at $r = 0$ (see (18)).\(^{18}\)Similar model (corresponding to the special case of $P_1 = P_2$, $Q_1 = Q_2$) was previously discussed in [41].\(^{19}\)Moreover, each of $P_1$ and $P_2$ should also be quantized. For example, the requirement of regularity of the 6-metric $\sim [dy_1 + P_1 (1 - \cos \theta) d\varphi]^2 + \ldots$ (the absence of Taub-NUT singularity) implies that one should be able to identify $y_1$ with period $4\pi P_1$, which is possible if $2P_1 / R_1 = k$. By $T$-duality in $y_1$-direction the same constraint should apply also to $P_2$, i.e. $2P_2 = l R_1 = l \alpha' / R_1$.\(^{13}\)
Though this rescaling seems to be the correct final result, to think that counting can be done exactly as in flat space is an oversimplification: the presence of non-vanishing electric charges is actually as important as the presence of the magnetic ones. This is crucial to take into account in order to demonstrate that all the transverse coordinates get renormalised tension. While the coefficients in front of ‘longitudinal’ terms $\partial u \partial v$ and $\partial u \partial u$ can be rescaled, this does not apply to the coefficient of $\partial z \partial z$ term (which determines the level of $SL(2, R)$). Since $(u, v, z)$ are coupled, the information about the scale of $\partial z \partial z$ ‘propagates’ into the rest of the transverse part of the action (omitted in (37)) through the dependence of perturbations on $u$. The relevant (supersymmetric, left-moving) perturbations in all other free transverse compact coordinates are represented by the $A_i$-term in (22):

$$L' = L + V, \quad V = 2F(x)A_n(u, x)\partial u \partial y^n. \quad (40)$$

They are marginal if $A_n = q_n(u)/r$ where $q_n$ are related to profiles of oscillations in $y_n$ directions. These perturbations vanish at large $r$ while near $r = 0$ we get $V \approx 2q_n(u)Q_2^{-1}\partial u \partial y^n$. The dependence on magnetic charges enters indirectly – due to the fact that $u$ is coupled to $z = \ln(Q_2/r)$. For $r \to 0$ we get (cf.(37))

$$L' = e^{-2}\partial u \partial v + Q_1Q_2^{-1}\partial u \partial u + P_1P_2\partial z \partial z + 2q_n(u)Q_2^{-1}\partial u \partial y^n + \partial y_n \partial y_n + \ldots, \quad (41)$$

where dots stand for the 3-sphere and constant dilaton terms. To see the effect of non-vanishing $q_n$ let us integrate out $y_n$. Then

$$L' = e^{-z}\partial u \partial v + c(u)\partial u \partial u + P_1P_2\partial z \partial z + \ldots, \quad c(u) \equiv Q_2^{-2}[Q_1Q_2 - q_n(u)q_n(u)]. \quad (42)$$

Note that all transverse ‘charges’ $q_n(u)$ enter on an equal footing. Since $(u, v, z)$-theory is $SL(2, R)$ WZW model with level $\kappa = 4\alpha'P_1P_2 \gg 1$ the coefficient in front of $<q_nq_n>$ term in the stress tensor or Virasoro operators will thus be rescaled by a factor of $1/\kappa$. At the same time, $<q_nq_n>$ should have an interpretation of the oscillator level. Solving the classical equations for $v, z$ using $1/\kappa$ expansion (assuming that $z$ is finite, i.e. that $\kappa$ goes to infinity before $r$ goes to zero) and computing the Virasoro operators one finds that for a winding (in $u$) string state the level matching condition becomes $mw \sim \frac{1}{\kappa} <q_nq_n>$ or $N_L \sim <q_nq_n> \sim \kappa mw$.

More explicitly, using the flat-space counting picture one finds that if $w$ and $m$ are the winding and momentum numbers of a free heterotic string compactified on a circle of radius $R$, then the mass and oscillator level of BPS-saturated left-moving oscillation modes ($\hat{N}_R = 0, \hat{N}_L = N_L - 1$) are given by the standard expressions (cf.(19),(20))

$$M = \frac{m}{R} + \frac{wR}{\alpha'} = \frac{1}{4G_N}(Q_1 + Q_2), \quad \hat{N}_L = mw = \frac{\alpha'}{16G_N^2}Q_1Q_2, \quad (43)$$

\(^{20}\text{I am grateful to F. Larsen for raising this issue.}\)
where we have used (6) to express $m, w$ in terms of the ‘space-time’ charges $Q_1, Q_2$. The relations (43) are true in the case when both $S^1$ and non-compact terms in the string action have the same overall coefficient (tension). The magnetic renormalisation (39) of the tension of the transverse part of the action implies that $N_L$ is to be rescaled by the ratio of the ‘longitudinal’ $(1/2\pi\alpha')$ and ‘transverse’ $(1/2\pi\alpha'_\perp)$ tensions (the oscillator level is proportional to the inverse of string tension)

$$N_L \rightarrow N_L = \frac{\alpha'}{\alpha'_\perp} mw = \frac{P_1 P_2 Q_1 Q_2}{16 G_N^2}.$$  \hspace{1cm} (44)

Since the charges are large we ignore the quantum ‘$-1$’ shift.

### 3.2 Statistical entropy

The number of BPS states in the free string spectrum with a given left-moving oscillator number $N_L \gg 1$ is (see, e.g., [8]) $d(N_L)_{N_L \gg 1} \approx c_0 N_L^\nu \exp(4\pi \sqrt{N_L})$. Then the statistical entropy of an ensemble of states with the same charges but different left-moving oscillation modes is given by

$$S_{stat} = \ln d(N_L)_{N_L \gg 1} \approx 4\pi \sqrt{N_L}.$$  \hspace{1cm} (45)

Using the expression (44) for $N_L$ we then find that, for large charges, $S_{stat}$ coincides with the Bekenstein-Hawking entropy (21),

$$S_{stat} \approx 4\pi \sqrt{N_L} = 2\pi \sqrt{\kappa mw} = \frac{\pi \sqrt{P_1 P_2 Q_1 Q_2}}{G_N} = \frac{A}{4G_N} = S_{BH}.$$  \hspace{1cm} (46)

It should be emphasized that there is no ambiguity in the overall coefficients in these expressions for the entropy. First, there is no ambiguity in the coefficient in front of the thermodynamic entropy (in contrast to the stretched horizon approach to the purely electric case where the numerical coefficient in front of $S_{BH}$ depends on a choice of a position of the stretched horizon [10, 13]). Also, the coefficient in front of $\sqrt{N_L}$ in $d(N_L)$ is related to the fact that underlying degrees of freedom correspond to a (‘half’ of) 1-dimensional extended object, i.e. are described by a 2d field theory (see also [20]). Note also that the dependence on the Newton’s constant effectively drops out.\textsuperscript{21}

\textsuperscript{21}The statistical entropy is given by a classical string-theory expression and does not depend on the string coupling. The constant $G_N$ needs to be introduced to express (cf.(6)) the string ‘microscopic’ quantum numbers in terms of ‘space-time’ charges (which appear in the effective field theory description). This is necessary in order to be able to compare with the Bekenstein-Hawking entropy. Since $S_{BH}$ itself originates from the on-shell value of the (euclidean) space-time effective action it also contains a factor of $1/G_N$.  

15
3.3 $D = 5$ extreme dyonic black holes

The universality of the relation between the statistical and Bekenstein-Hawking entropies is confirmed by analogous consideration of the case of $D = 5$ extreme dyonic black holes described by (22),(29)–(33). Here the throat limit $r \to 0$ is described by a similar $SL(2, R) \times SU(2)$ WZW model (cf.(37)–(39))$^{22}$

$$I_{r \to 0} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left( e^{-z} \partial u \partial \bar{v} + Q_1 Q_2^{-1} \partial u \partial \bar{u} \right)$$

$$+ \frac{\kappa}{4\pi} \int d^2 \sigma \left( \partial z \partial z + \partial \psi \partial \bar{\psi} + \partial \varphi \partial \bar{\varphi} + \partial \theta \partial \bar{\theta} - 2 \cos \theta \partial \bar{\psi} \partial \bar{\varphi} \right),$$

$$\kappa = \frac{1}{\alpha'} P, \quad z \equiv \ln \frac{Q_2}{\tau^2} \to \infty, \quad \frac{1}{\alpha'_\perp} = \frac{P}{4\alpha'^2}. \quad (48)$$

Its integer level $\kappa = P/\alpha'$ again rescales the tension of the ‘transverse’ part of the action by $\alpha'/\alpha'_\perp = P/4\alpha'$. The important factor of 4 difference compared to the expressions in (38),(39) is related to the increased dimensionality of the spatial sphere (here the harmonic functions depend on $r^2$; note also that the standard metric on $S^3$ with unit radius is $ds^2 = \frac{1}{4} [d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi - \cos \theta d\varphi)^2]$, where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, $0 \leq \psi \leq 4\pi$).

The analogues of the relations (43),(44) are found by using the expressions for the charges (we apply (6) for $D = 6$ fundamental string; as in (32) here $G_N \equiv G_N^{(5)}$)

$$Q_1 = \frac{4}{\pi} G_N \cdot \frac{m}{R}, \quad Q_2 = \frac{4}{\pi} G_N \cdot \frac{wR}{\alpha'}, \quad P = \kappa \alpha', \quad (49)$$

$$M = \frac{m}{R} + \frac{wR}{\alpha'} = \frac{\pi}{4G_N} (Q_1 + Q_2), \quad \hat{N}_L = mw = \frac{\alpha' \pi^2}{16G_N^2} Q_1 Q_2, \quad \hat{N}_L \to N_L = \frac{\alpha'}{\alpha'_\perp} mw = \frac{\pi^2 P}{64G_N^2} Q_1 Q_2. \quad (50)$$

The expression for the statistical entropy is then (cf.(33),(46))

$$S_{stat} \approx 4\pi \sqrt{N_L} = 2\pi \sqrt{\kappa mw} = \frac{\pi^2 \sqrt{P Q_1 Q_2}}{2G_N} = \frac{A}{4G_N} = S_{BH}. \quad (52)$$

Notice again a remarkable consistency of numerical factors.

In the special case of $Q_1 = Q_2 = P$ the same conclusion was reached in [21] by first transforming the heterotic string solution into a type II string one using the $D = 6$ $S$-duality and then counting the number of corresponding BPS states as $D$-brane bound states. As the approach (described in [19] and above) based directly on

$^{22}$The ‘transverse’ $(z, \psi, \varphi, \theta)$ part of the throat region is exactly the same as in the 5-brane model [27] (where $Q_1 = Q_2 = 0$) except for the fact that here the dilaton $\Phi$ is constant in the $r \to 0$ region, i.e. the string coupling is not blowing up.
the underlying conformal field theory, this ‘$D$-brane counting’ approach also uses the assumption of large charges and also reduces to evaluation of a number of states in some 2d conformal (hyperkähler) $\sigma$-model. A simpler $D$-brane counting derivation of the entropy of $D = 5$ extreme dyonic black holes which applies to the general case of arbitrary large charges $Q_1, Q_2, P$ was recently given in [42].

4 Rotating $D = 5$ extreme dyonic black holes

The conformal chiral null model (22) with non-vanishing $A_i$-term describes also rotating purely electric extreme supersymmetric black holes in various dimensions. Indeed, (22) with flat transverse part represents [24] IWP backgrounds [16] which include, in particular, extreme Taub-NUT and rotating solutions (for axi-symmetric choices of harmonic functions). The corresponding extreme solutions (with rotation in only one plane) have naked ring singularity in $D = 4,5$ but get regular horizon and saturate Bogomol’nyi bound in $D \geq 6$ [46]. More complicated rotating solutions in $D = 4$ (which have the same large $r$ asymptotics as the extreme supersymmetric low-energy solutions obtained by applying $T$-duality transformations to Kerr solution [7] but do not have naked singularities) can be constructed by taking a periodic array of $D > 5$ black holes [46] or by starting with the chiral null model (22) with $u$-dependent coupling $K$ or $A_i$ (which describes a background produced by oscillating or rotating $D = 5$ fundamental string) and averaging over the compact direction [5] (in this latter case there is a natural Regge bound on the maximal value of the angular momentum but the short-distance form of the solution is not explicit).

One may wonder how the properties of $D = 4,5$ rotating solutions change once one adds magnetic charges, i.e. considers extreme supersymmetric dyonic black holes described by (22) with curved transverse part.\(^{23}\) It turns out that the rotating version of the $D = 4$ background corresponding to the model (16) still has a ring singularity as in the pure electric case.\(^{24}\) At the same time, there exists a natural rotating generalisation of the 3-charge $D = 5$ dyonic model (30),(31) which describes a non-singular rotating dyonic black hole solution of $D = 10$ heterotic or type II theory compactified on $S^1 \times T^4$. In the special case when all 3 charges are equal this model is a conformal $\sigma$-model behind the solution found in [22] (which itself is a rotating version of the solution of [21]).

As we shall see below, the requirement of conformal invariance of underlying $\sigma$-model (which reduces to a condition of self-duality of the strength of 4-potential $A_i$ in (22)) implies that (as in [39, 22]) one needs two equal components of rotation in the two orthogonal planes. A remarkable feature of this model is that like in the absence of rotation (Section 3.3) it has a regular throat region described by a similar

\(^{23}\)The importance of this problem was emphasized to me by M. Cvetić. I would like to thank her for helpful discussions of several aspects of rotating solutions.

\(^{24}\)The presence of the singularity may, in principle, be harmless at the string-theory level provided the corresponding 5-dimensional conformal $\sigma$-model is regular.
$SL(2,R) \times SU(2)$ WZW model. The bound on the maximal value of the angular momentum and its quantisation then follows directly from the conformal field theory considerations.\textsuperscript{25} The expression for the entropy of the resulting rotating black hole can again be understood in terms of counting of possible BPS deformations of the conformal model (a $D$-brane counting derivation of the entropy in the special equal-charge case was given in [22]).

4.1 $D=6$ conformal $\sigma$-model and $D=5$ black hole

Let us consider a 6-dimensional model (22) with the ‘transverse’ $M^4$-part $(G_{mn}, B_{mn}, \phi)$ having torsion related to dilaton in the following specific way ($m,n,...=1,2,3,4$)

$$H^{mnk} = -\frac{2}{\sqrt{G}}\epsilon^{mnkl}\partial_l\phi.$$ (53)

This relation is satisfied both when the $M^4$-part is represented by $SO(4)$-invariant ‘5-brane’ $\sigma$-model (29) and in the case of $SO(3)$-invariant model in (16) (there one needs to assume that $y_1 = x_4, x_s = (x_1, x_2, x_3)$ and that $a_s\partial_s\phi = 0$ which is satisfied for the 1-center background (14)). Then the conformal invariance condition for 4-vector $A_m$ (25) takes remarkably simple form

$$\partial_m(e^{-2\phi}\sqrt{G} F^m_{+}) = 0,$$ (54)

$$F^m_{+} \equiv F^{mn} + F^{*mn}, \quad F^{*mn} = \frac{1}{2\sqrt{G}}\epsilon^{mnkl}F_{kl}.$$ (55)

The simplest solutions existing for arbitrary $\phi$ are thus given by abelian 4-dimensional instantons

$$(\sqrt{G} G^{mn} G^{kl} + \frac{1}{2}\epsilon^{mknl})F_{nl} = 0.$$ (55)

Since this (anti)selfduality equation does not depend on a scale of $G_{mn}$, in the $SO(4)$-invariant case of (29) (corresponding to the $D=5$ solution discussed below) it does not depend on the ‘magnetic’ function $f$, i.e. takes the flat space form. Choosing the 4 spatial coordinates as $x^1 + ix^2 = r\sin\theta e^{i\varphi}, \quad x^3 + ix^4 = r\cos\theta e^{i\psi},$

$$dx^m dx_m = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2),$$ (56)

the equation (55) can be solved by a simple ansatz with rotational symmetry in the two orthogonal planes

$$A_\varphi = A_\varphi(r, \theta), \quad A_\psi = A_\psi(r, \theta), \quad A_r = A_\theta = 0,$$ (57)

\textsuperscript{25}This solution has a non-singular horizon only if all 3 charges are non-vanishing. A generalisation of extreme purely electric $D=5$ solution in [46] to the case of two components of rotation remains singular.
\[ F_{r\varphi} = \frac{r \cot \theta \partial_r A_\varphi - \partial_\varphi A_r}{r^3 \sin \theta \cos \theta} , \quad F_{\theta r} = \frac{r^{-1} \cot \theta \partial_\theta A_r + \partial_r A_\theta}{r^3 \sin \theta \cos \theta} , \quad (58) \]

\[ F_{\theta \psi} = -r^{-1} \tan \theta F_{r \varphi} , \quad F_{r \psi} = r \tan \theta F_{\theta \varphi} . \]

Imposing \( F^m_m = 0 \) one finds\(^{26}\)

\[ A_\varphi = \frac{\gamma}{r^2} \sin^2 \theta , \quad A_\psi = \frac{\gamma}{r^2} \cos^2 \theta , \quad \gamma = \text{const.} \quad (59) \]

[ In the \( SO(3) \)-invariant case (16) (related to the \( D = 4 \) dyonic solution) one finds that the self-duality condition becomes (we assume that the fields do not depend on the internal coordinate \( y_1 \))\(^{27}\)

\[ F_{pq} + a_p \partial_q A - a_q \partial_p A + k^{-1} \epsilon_{pqrs} \partial_s A = 0 , \quad A(x) \equiv A_{y_1} . \]

Since \( \partial_p a_q - \partial_q a_p = -\epsilon_{pqrs} \partial_s k^{-1} \) this becomes \( \partial_p \tilde{A}_q - \partial_q \tilde{A}_p = -k^{-2} \epsilon_{pqrs} \partial_s (kA) \), where \( \tilde{A}_p \equiv A_p - A a_q \). Then \( \partial^s [k^{-2} \partial_s (kA)] = 0 \), i.e. \( A(r) = q/r \) and \( \tilde{A}_\varphi = q (1 - \cos \theta) \) in the simplest 1-center case, so that we get an extra Taub-NUT term in the resulting \( D = 4 \) metric (i.e. \( (8) \) with \( dt^2 \to (dt + A_\varphi d\varphi)^2 \) and \( \lambda^{-2} \to F^{-1} k f k^{-1} - k^{-2} A^2 \), cf.(17)). It is interesting that all the charges can be superposed in a way consistent with conformal invariance with the Taub-NUT charge being related to the charge of \( A \). The solution (26) found in [19] corresponds to \( \tilde{A}_p = 0 \) but with \( A \) subject not to (55) but the full second order equation (54). The \( n = y_1 \) component of it reduces to \( \partial^s [f^{-1} k^{-3} \partial_s (kA)] = 0 \) (which is solved by (26)) while three other components are satisfied identically. ]

Since the presence of \( A_m \)-coupling does not change the equations (24) for the functions \( F, K \) they can still be chosen as in (30). Dimensionally reducing this \( D = 6 \) model (22),(29) to 5 dimensions along \( u \) as in Section 2.3 one finds the rotating generalisation of the 3-charge \( (Q_1, Q_2, P) \) extreme dyonic \( D = 5 \) black hole background (31). The resulting 5-dimensional Einstein-frame metric is

\[ ds^2_E = -\lambda^2(r)(dt + A_m dx^m)^2 + \lambda^{-1}(r) dx^m dx_m \quad (60) \]

\[ = -\lambda^2(r)(dt + \frac{\gamma}{r^2}(\sin^2 \theta d\varphi + \cos^2 \theta d\psi))^2 + \lambda^{-1}(r)(dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2)) , \]

\[ \lambda = \frac{r^2}{[(r^2 + Q_1)(r^2 + Q_2)(r^2 + P)]^{1/3}} , \]

where \( \lambda(r) \) is the same as in the static case (31). As for \( \gamma = 0 \) the surface \( r = 0 \) is a regular horizon. The mass of this \( D = 5 \) rotating dyonic black hole is still given by

\(^{26}\)In addition, there is a growing solution \( A_\varphi = \gamma r^2 \sin^2 \theta , \quad A_\psi = -\gamma r^2 \cos^2 \theta \), which describes a rotating magnetic universe (a similar solution with one rotational plane was discussed in [24]).

\(^{27}\)Here \( H_{y_1 y_2} = \partial_q b_p - \partial_p b_q \), \( G^{y_1 y_2} = f^{-1}(k^{-1} + k a_s a_s) \), \( G^{pp} = f^{-1}k \delta^{pp} \), \( G^{p1} = -f^{-1} k a_p \), \( \sqrt{G} = f^2 k^{-1} \). \( p, q, .. = 1, 2, 3 \). All repeated indices are contracted with flat metric.
while the angular momenta in the two planes are
\[ J_\varphi = J_\psi = J = \frac{\pi}{4G_N}\gamma. \] (61)

In the \( r \to 0 \) limit the metric becomes
\[ (ds^2)_{r \to 0} = (Q_1Q_2P)^{1/3}[ (d\ln r)^2 + d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2 \]
\[ - \frac{\gamma^2}{Q_1Q_2P}(\sin^2 \theta d\varphi + \cos^2 \theta d\psi)^2 ] , \] (62)
leading to the following expression for the area of the \( r = 0 \) horizon (cf.(33))
\[ A = 2\pi^2 \sqrt{Q_1Q_2P - \gamma^2}. \] (63)

The combination \( Q_1Q_2P - \gamma^2 \) should be positive in order for the signature of the metric (62) to remain euclidean. Thus there is no regular rotating solution in the electric limit \( P \to 0 \).

Expressed in terms of quantised charges (49) and the angular momentum (61) the entropy becomes
\[ S_{BH} = \frac{A}{4G_N} = \frac{\pi^2 \sqrt{PQ_1Q_2 - \gamma^2}}{2G_N} = 2\pi \sqrt{\kappa m w - J^2}. \] (64)

The special case \( Q_1 = Q_2 = P \) of this solution (written in terms of \( \rho \) in (34)) was found in [22] where the corresponding thermodynamic entropy was also obtained using D-brane counting approach. Interpreting the above general \( D = 6 \) background as a solution of type IIB theory in 10 dimensions and applying \( SL(2,\mathbb{Z}) \)-duality to transform it into a solution supported by RR-charges it should be straightforward to reproduce the entropy (64) by generalising the counting done in the non-rotating case in [42].

### 4.2 Throat limit, bound on angular momentum and entropy

Having identified a conformal \( \sigma \)-model behind this rotating solution one is able to derive some of its properties directly from the corresponding conformal theory. As in the non-rotating case (47) the throat limit \( r \to 0 \) of this model is regular and is described by \( SL(2,\mathbb{R}) \times SU(2) \) WZW theory. The angles \( (\theta, \varphi, \psi) \) in (56) are related to the standard \( S^3 \) Euler angles used in (47) by (these Euler angles in (47) here will be denoted by primes)
\[ \theta = \frac{1}{2}\theta', \quad \varphi = \frac{1}{2}(\varphi' + \psi'), \quad \psi = \frac{1}{2}(\psi' - \varphi'), \quad 0 \leq \theta' \leq \pi, \quad 0 \leq \varphi' \leq 2\pi, \quad 0 \leq \psi' \leq 4\pi. \]

\(^{28}\)In general, for a black hole in \( D \) dimensions with the metric \( g_{\mu\nu} = \gamma \epsilon_{ik} x^k / r^{D-2} + \ldots \) the angular momentum in \( (i, k) \) plane is \([39]\) \( J = \gamma \omega_{D-2} / 8\pi G_N^{(D)} \).
Defining \( z \equiv \ln \frac{Q_2}{r} \) we get (cf. (47))

\[
I_{r \to 0} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ e^{-z} \partial u \bar{u} v + Q_1 Q_2^{-1} \partial u \bar{u} u + \gamma Q_2^{-1} \partial u (\bar{\psi}' - \cos \theta' \bar{\phi}') \right] \\
+ \frac{P}{4 \pi \alpha'} \int d^2 \sigma \left( \partial z \bar{z} + \partial \psi' \bar{\psi}' + \partial \phi' \bar{\phi}' + \partial \theta' \bar{\theta}' - 2 \cos \theta' \partial \psi' \bar{\phi}' \right).
\]

(65)

The \( \gamma \)-term \( (\sim \partial u \bar{J}_3) \) can be interpreted as an integrable marginal deformation of the \( SL(2, R) \times SU(2) \) WZW model (similar models were discussed in [43, 44]). Introducing

\[
\psi'' = \psi' + 2\gamma P^{-1} Q_2^{-1} u
\]

we can represent this action in the \( SL(2, R) \times SU(2) \) WZW form as in (47)

\[
I_{r \to 0} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ e^{-z} \partial u \bar{u} v + (PQ_2^2)^{-1} (Q_1 Q_2 P - \gamma^2) \partial u \bar{u} u \right] \\
+ \frac{P}{4 \pi \alpha'} \int d^2 \sigma \left( \partial z \bar{z} + \partial \psi'' \bar{\psi}'' + \partial \phi'' \bar{\phi}'' + \partial \theta'' \bar{\theta}'' - 2 \cos \theta'' \partial \psi'' \bar{\phi}'' \right).
\]

(66)

The two WZW models (described by \( (z, u, v) \) and \( (\theta', \phi', \psi'') \)) have the same level \( \kappa = P/\alpha' \). The consequences of this representation are:

1. Quantization of parameters: \( P = \alpha' \kappa \) is quantized since the level of \( SU(2) \) is. Demanding that \( \psi'' \) should have the same \( (4\pi) \) periodicity as \( \psi' \) we conclude that the value of \( \gamma \) should be quantized \( \gamma R P^{-1} Q_2^{-1} = l = \text{integer} \) (\( R \) is the radius of the compact coordinate \( u \)).

29 Relating \( Q_2 \) to the 'winding number' \( w \) (49) and \( \gamma \) to \( J \) (61) we conclude that

\[
P = \alpha' \kappa, \quad \gamma = \frac{PQ_2}{R} l, \quad \text{i.e.} \quad J = \kappa mw,
\]

so that \( J \) should take only integer values.

2. Bound on angular momentum: In order for \( u \) to have positive norm in (66) we should have (we assume that all the charges are positive)

\[
Q_1 Q_2 P - \gamma^2 = \left( \frac{4G_N}{\pi} \right)^2 (\kappa mw - J^2) > 0, \quad \text{i.e.} \quad J^2 < \kappa mw,
\]

(68)

29From the low-dimensional classical effective action point of view \( \gamma \) can, of course, take continuous set of values. In principle, it is not necessary to insist on representing (65) in the 'factorised' \( SL(2, R) \times SU(2) \) form (66) with \( 4\pi \)-periodic \( \psi'' \). However, if \( \gamma \) is not quantised it breaks space-time supersymmetry at the string-theory level (as in the 'magnetic' models in [45, 44, 43]). Note that like (world-sheet supersymmetric) \( SL(2, R) \times SU(2) \) WZW model the theory (65) has zero central charge deficit. Continuous supersymmetry breaking does not contradict standard lore since this model is non-compact. Supersymmetry is important in the present context in order to be able to reproduce the entropy (64) as a statistical one (it is only if \( J \) is quantised that \( \kappa mw - J^2 \) may be related to a number of microscopic states).
getting a constraint on maximal value of \( \gamma \) or \( J \). This ‘Regge bound’ thus follows directly from the regularity of underlying conformal \( \sigma \)-model.\(^{30}\)

The structure of the throat region theory and the fact that the angular momentum terms \( (A_i \partial u \bar{\partial} x^i) \) appear in the conformal \( \sigma \)-model action (22) in the same way as (and indeed are particular examples of) marginal perturbations corresponding to left-moving oscillations of the solitonic string (‘fundamental string wound around 5-brane’) strongly suggests a possibility to understand the expression for the entropy (64) along the same lines as in the non-rotating case discussed in Section 3. Introducing a \( u \)-dependence into \( \gamma \) or \( J \), \( J(u) = J + \tilde{J}(u) \), and treating \( \tilde{J}(u) \) as a perturbation of the original model\(^{31}\) one should get, using \( 1/\kappa \) expansion, the following level matching condition

\[
mw = \frac{1}{\kappa} < J^2(u) > = \frac{1}{\kappa} J^2 + \frac{1}{\kappa} \tilde{J}^2(u) >, \quad N_L \sim < \tilde{J}^2(u) > = \kappa mw - J^2. \tag{69}
\]

This argument is exactly parallel to the one given in Section 3 with \( J(u) \) playing the role of a ‘charge’ \( q(u) \) (cf.(40)–(42)). As in (52) the corresponding statistical entropy then reproduces the expression for the Bekenstein-Hawking one (64).

Let us finish with remarks on other related solutions. The \( D \)-brane counting arguments in \cite{22} imply that there should exist a solution with independent values of angular momenta \( J_\varphi \) and \( J_\psi \) in the two planes. Such a solution is not described by the chiral null model (22) (which, at the same time, represents almost all known exact extreme solutions).\(^{32}\) One may also consider a generalisation to the case when the harmonic functions \( F^{-1}, K, f \) are not spherically but only axially symmetric \((x^1 + ix^2 = \sqrt{r^2 + a^2} \sin \theta e^{i\varphi}, \quad F^{-1} = 1 + \frac{Q}{r^2 + a^2 \cos^2 \theta}, \text{etc.})\). It may be of interest to construct rotating \( D = 4 \) solutions by reducing these \( D = 5 \) solutions (their multicenter generalisations) down to 4 dimensions as in \cite{46, 47}.

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\(^{30}\)The same condition is necessary to avoid closed time-like geodesics in the resulting dimensionally reduced \((D = 5)\) metric \cite{22}.

\(^{31}\)The perturbation \( \sim \gamma(u) \partial u (\bar{\partial} \psi' - \cos \theta' \bar{\partial} \varphi') \) of \( SL(2, R) \times SU(2) \) WZW theory (cf.(65)) remains marginal for arbitrary function \( \gamma(u) \) (as follows from the presence of the mixed \( \partial u \bar{\partial} v \)-term in \( SL(2, R) \) action).

\(^{32}\)In particular, a solution with an angular momentum in only one of the two planes (e.g. \( A_\varphi \neq 0, A_\psi = 0 \)) must satisfy, according to (54),(58) \( \partial_r f^{-1} \partial_\theta A_\varphi = 0 \) so that \( A_\varphi = A_\varphi(r) \) if \( P \neq 0 \). Then \( A_\varphi = -c \ln r + \frac{P}{2\pi}, \ a = \text{const} \), which does not decay at large \( r \).
References


