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ABSTRACT

Bose-Einstein condensation is considered within the Tsallis generalized thermostatics as an illustrative case of a situation where interactions are inexistent in the Hamiltonian and the system evolves in a fractal space. The $(1 - q)/kT \rightarrow 0$ asymptotic behavior is studied.

Key-words: Generalized Thermostatistics; Non-extensive Systems; Bose-Einstein Condensation; Ideal Gas.

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The Boltzmann-Gibbs statistics and its connection with thermodynamics are powerful tools in theoretical physics to study situations where the following conditions are satisfied (i) the effective microscopic interaction is short-ranged or inexistent, (ii) the microscopic memory is short-ranged or inexistent and (iii) the system evolves in a nonfractal space-time. This is to say, whenever the extensive (additive) properties of thermodynamics hold.

Whenever an Euclidean-like (nonfractal) space is involved, the ideal Bose-Einstein gas (no interaction exists in the Hamiltonian) is the simplest exactly solvable continuous system that has a phase transition. Therefore, it has its place in any course on statistical mechanics including quantum statistics [1]. An elementary, but rigorous, calculation is made in ref. [2] by controlling the difference between sums and integrals in the thermodynamic limit. This phase transition is the well known Bose-Einstein condensation (most of particles go to the zero momentum state). The Bose-Einstein condensation is a physical situation in vogue nowadays because of important experimental advancements have that occurred recently [3].

Let us start by considering an ideal gas of particles that obey to the Bose-Einstein statistics at temperature T and chemical potential μ , for a large (hyper)volume V and a large number of particles N_1 . The average number of particles is given by

$$N_1 = n_1^{(0)} + V \left(\frac{mkT}{2\pi\hbar^2} \right)^{D/2} g_{D/2}(e^{\mu/k_B T}). \quad (1)$$

Here D is the dimension of the system and we assume $D > 2$ in order to have a non-vanishing critical temperature; k_B is the Boltzmann constant and $\hbar = h/2\pi$ (h is the Planck constant). The function $g_{D/2}$ is defined as $g_{D/2}(y) = \sum_{n=1}^{\infty} y^n / n^{D/2}$. The quantity $n_1^{(0)} \equiv 1/(\exp(-\mu/k_B T) - 1)$ is called the zero momentum state and can be macroscopically occupied. The phase transition happens as $\mu \rightarrow 0$.

The occupation of the zero momentum state is of course the Bose-Einstein condensation. It occurs at temperature $T \leq T_{c1}$. The critical temperature T_{c1} is given by

$$T_{c1} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N_1}{\zeta(D/2)V} \right)^{2/D}. \quad (2)$$

where $\zeta(x)$ is the Riemann function. It is worthy to recall that, for $T \geq T_{c1}$, $n_1^{(0)} = 0$; for $T \leq T_{c1}$, $\mu = 0$; and that, for $T = T_{c1} + \epsilon$, where $\epsilon \rightarrow 0$, $n_1^{(0)} \propto \epsilon$, which could be compared with the mean field approximation, namely $\epsilon^{1/2}$.

The fraction of particles in the zero momentum state can be written

$$\frac{n_1^{(0)}}{N_1} = 1 - \left(\frac{T}{T_{c1}} \right)^{D/2}, \quad \text{for } T \leq T_{c1}. \quad (3)$$

Eqs.(1)-(3) is the more interesting set of results on the problem within the Boltzmann-Gibbs thermostatics. Some thermodynamic properties of an ideal gas in D dimensions have been discussed in ref. [4]. Nevertheless, this formalism fails whenever the physical system includes (i) long-range force and/or (ii) long-memory effect and/or (iii) a (multi)fractal space-time. In any of these cases, the system is expected to violate the standard extensive properties.

More precisely [5], the difficulties and their consequences are classified as follows:

- (i) For a relevant Euclidean-like space-time and if either the forces or the memory (or both) are long-ranged, but we are interested in an equilibrium state, the Boltzmann-Gibbs statistics is weakly violated, the formalism can be used to obtain an approximate description. However, if we are interested in a meta-equilibrium state [6], the Boltzmann-Gibbs description is strongly violated. Other formalism must be used.
- (ii) For a relevant (multi)fractal space, the Boltzmann-Gibbs formalism is strongly violated again and other formalism is needed.

The explicit need for a nonextensive thermodynamics has been well known in cosmology, gravitation and astrophysics [7], magnetic systems [8], Lévy-like anomalous diffusion [9], etc.

As a possible solution, Tsallis proposed a nonextensive thermostatics in his paper [10]. This formalism has already received some applications. Among them, let us mention: Self-gravitating systems, Stellar polytropes, Vlasov equation [11, 6]; Lévy-like anomalous diffusion [9, 12]; Simulated annealing [13]. Furthermore, its connection within quantum statistics [14], quantum groups [15], quantum uncertainty [16], fractals [17, 18], etc., has been established.

The generalized statistics relies on the so called Tsallis entropy, namely

$$S_q \equiv -k \frac{1 - \sum_R p_R^q}{1 - q}, \quad (4)$$

where $q \in \mathbb{R}$; k is a positive constant and S_q recovers the standard form $-k_B \sum_R p_R \ln p_R$, in the limit $q \rightarrow 1$.

Expression (4) has enabled various (nontrivial, though mathematically simple and natural) generalizations of important properties such as ref. [14]

- (i) The grand-canonical equilibrium distribution now becomes

$$p_R = \frac{[1 - \beta(1 - q)(\varepsilon_R - \mu N)]^{\frac{1}{1-q}}}{\Xi_q}, \quad (5)$$

with the generalized grand-partition function consistently given by

$$\Xi_q(\beta, \mu) = \sum_R [1 - \beta(1 - q)(\varepsilon_R - \mu N)]^{\frac{1}{1-q}}, \quad (6)$$

where $\beta \equiv 1/kT > 0$ and ε_R is the spectrum (R represents a set of given real numbers).

- (ii) The thermodynamics associated with Eq.(4) is invariant under Legendre transformations and preserves thermodynamic stability [19]; in particular, the fundamental equation for open system is

$$\Omega_q = -kT \frac{\Xi_q^{1-q} - 1}{1 - q}. \quad (7)$$

The q -expectation value of the particle number is given by,

$$N_q = \sum_R p_R^q N = \frac{\partial \Omega_q}{\partial \mu} = -kT \frac{1}{(\Xi_q)^q} \frac{\partial \Xi_q}{\partial \mu}. \quad (8)$$

- (iii) The corresponding Hilhorst integral transformation of the grand-partition function was obtained. In the same manner, the generalized distribution function as well as the q -expectation values of some thermodynamic quantities were established. The Hilhorst transformation (as discussed by Prato [20]) for the q -expectation value of the particle number can be obtained in a closed form as

$$N_q = \frac{\Gamma(\frac{1}{1-q})}{[\Xi_q(\beta)]^q} \frac{i}{2\pi} \oint_C dz (-z)^{\frac{-1}{1-q}} e^{-z} \Xi_1(-\beta(1-q)z, \mu) N_1(-\beta(1-q)z, \mu). \quad (9)$$

Now, within this formalism, we study the Bose-Einstein system (no interactions in the Hamiltonian) in a (non Euclidean) fractal space. The last fact makes necessary to use a nonstandard Boltzmann-Gibbs statistics. The Tsallis formalism will be used, where the information about the fractal dimension is kept in the parameter q [17].

The generalization of Eq.(1) is

$$N_q = n_q^{(0)} + V \left(\frac{mkT}{2\pi\hbar^2} \right)^{D/2} G_q(D/2, \mu), \quad (10)$$

defining

$$K_q(m, x, \mu) = \frac{(1-q)^{m-x} \Gamma(1/(1-q))}{\Gamma(1/(1-q) + x - m)} < (kT - (1-q)(\mathcal{H} - \mu\mathcal{N}))^{x-m} >_q,$$

where, $< \mathcal{O} >_q$ is the q -expectation values of the operator $\mathcal{O}$. As before, $n_q^{(0)}$ is the generalized occupation number of the zero momentum state and it is given by

$$n_q^{(0)} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(n\mu)^m}{m!} K_q(m, 0, \mu), \quad (11)$$

and

$$G_q(D/2, \mu) = \sum_{n=1}^{\infty} \left(\frac{kT}{n} \right)^{D/2} \sum_{m=0}^{\infty} \frac{(n\mu)^m}{m!} K_q(m, D/2, \mu). \quad (12)$$

According to Eq.(10) and Eq.(12), the critical temperature is obtained by requiring the following expression to be satisfied:

$$< (kT_{cq} - (1-q)\mathcal{H})^{D/2} >_q = \left(\frac{2\pi\hbar^2}{m} \right)^{D/2} \frac{N_q}{\zeta(D/2)V}, \quad (13)$$

which is the generalization of Eq.(2). We can also obtain the generalization of Eq.(3), this is:

$$\frac{n_q^{(0)}}{N_q} = 1 - \frac{< (kT - (1-q)\mathcal{H})^{D/2} >_q}{< (kT_{cq} - (1-q)\mathcal{H})^{D/2} >_q}. \quad (14)$$

Furthermore, it is convenient to remark that the Eq.(13) for the critical temperature is satisfied whenever the generalized fraction of particles in the zero momentum state vanishes ($n_q^{(0)}/N_q = 0$).

Due to the mathematical difficulties associated with a generic values of q , let us from now focus the $q \approx 1$ case. By using Eq.(7) from the ref. [21], Eq.(13) asymptotically becomes

$$kT_{cq}\Xi_1(T_{cq})^{2(1-q)/D} \left(1 + (1-q)(D/2 - 1) \frac{U_1(T_{cq})}{kT_{cq}} \right)^{2/D} = k_B T_{c1}. \quad (15)$$

When $T \leq T_{c1}$, the internal energy U_1 for a Bose-Einstein system is written as a function of the temperature, namely

$$U_1(T) = \frac{D}{2} V \left(\frac{m}{2\pi\hbar^2} \right)^{D/2} (kT)^{D/2+1} \zeta(D/2 + 1); \quad (16)$$

for the same case, the thermodynamic potential is given by

$$\Omega_1 = -V \left(\frac{m}{2\pi\hbar^2} \right)^{D/2} (kT)^{D/2+1} \zeta(D/2 + 1).$$

So, as $\Xi_1 = \exp(-\beta\Omega_1)$, the grand partition function is written as

$$\Xi_1 = \exp \left(V \left(\frac{mkT}{2\pi\hbar^2} \right)^{D/2} \zeta(D/2 + 1) \right). \quad (17)$$

Evaluating Eq.(15), we obtain an equation for T_{c1} , and by inverting for T_{cq} , we have

$$T_{cq} = \left[1 - (1-q) \left(\frac{D}{2} + \frac{2}{D} - 1 \right) \frac{\zeta(D/2 + 1)}{\zeta(D/2)} N_1 \right] T_{c1}, \quad (18)$$

whenever $k = k_B$ by the first order correction in $(1-q)$. This approximation shows that, if $q < 1$ the critical temperature decreases. According to Eq.(2) there exists an apparent modification on the density (*i.e.*, if the average number of particles N_1 is constant the volume increases as $q < 1$) in the $q \rightarrow 1^-$ limit. Now, we calculate

$$\frac{1}{N_1} \frac{\partial}{\partial q} \left(\frac{T_{cq}}{T_{c1}} \right) = \frac{(D-2)^2 \zeta(D/2 + 1)}{2D \zeta(D/2)}; \quad (19)$$

so, we remark that, the first derivative of T_{cq} with respect to q is positive for all values of $D > 2$. In Fig.1 is depicted (solid line) the profile of the Eq.(19) as a function of the D parameter; (if $V = \ell^D$, where ℓ is the side of a box in D dimensions) the quantity $N_1^{2/3-2/D} T_{c1}(D)/T_{c1}(3)$ versus D (dashed line).

Summarizing, the standard results are recovered from the generalized results when $q = 1$, as it should be. It is shown that, in the present approximation, the critical temperature for the phase transition of the Bose-Einstein condensation is modified whenever the nonextensive thermostatistics is considered. Furthermore, the critical temperature in other problems change as well if $q \neq 1$; for instance, in the canonical ensemble, the self-dual planar lattice Ising ferromagnet within renormalization group calculation [22] and a z-coordinated spin- $\frac{1}{2}$ Ising ferromagnet within molecular field approximation [23] suffer modifications perfectly consistent to our result. In particular, all these theoretical approaches provide a critical temperature which increases with q .

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Fig.1: Profile of the Eq.(19) (solid line) and the quantity $N_1^{2/3-2/D} T_{c1}(D)/T_{c1}(3)$ (dashed line) as a function of D .

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