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QUANTUM STATISTICS OF IDEAL PARA-FERMI GASES III: THE SLATER QUASI-DETERMINANT¹

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ABSTRACT

The wave function $\psi(1, 2, \dots, N)$ of a gas of N identical two-dimensional exotic particles of spin $1/M \pmod{1}$, $M = 2, 3, 4, \dots$ is constructed. It is shown that $\psi(1, 2, \dots, N)$ obeys a generalized Pauli exclusion principle according to which no more than $(M - 1)$ identical particles of spin $1/M \pmod{1}$ can live in the same quantum state. The Fermi nilpotency and the Bose condensation are recovered as special cases. Examples are given

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1 Introduction

Recently, fractional spin symmetries have been attracting a great interest of attention because of their natural generalization of the Bose-Fermi symmetry and due to the role they play in two-dimensional conformal minimal models and their integrable deformation [1,2]. Well common examples are given by the $c = 7/10$ tricritical Ising model, the $c = 6/7$ three states tricritical Potts model and their integrable $\phi_{(1,3)}$ deformations [3]. In this letter, we continue a series of works [3,4,5] on this subject by examining one of the remarkable predictions of the quantized fractional spin supersymmetric models namely the existence, in their physical spectrum, of exotic particles of fractional spin $s = L/M$ where $L \in \mathbb{Z}^+$ and M is a positive integer greater than one. Our main purposes in this study is to build the wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ of a system of N identical exotic particles of spin $s = 1/M \pmod{1}$ and prove the generalization of the Pauli exclusion principle conjectured in [4,6]. Here M is a generic integer which can take the values $2, 3, \dots, \infty$. The boundary values $M = 2$ and $M = \infty$ describe obviously the fermions and the bosons respectively. They will be used in this discussion as two limiting cases for testing our results. Besides this property, our interest in spin $1/M$ particles, sometimes referred, hereafter, to as parafermions, is dictated by no selection rule apart that we do not know how to treat the general situation.

Moreover, the wave function that we want to build here can be thought of as the wave function of a quantum gas of parafermions of spin $1/M \pmod{1}$. This function, which in principle should generalize the wave functions of the quantum Bose and Fermi gases of statistical mechanics [7], along with the parafermi statistics [4,5] are expected to clarify much more the very different behaviours of the Bose and Fermi gases at low temperatures [6]. As the quantum properties of the gases depend strongly on the values of the spin of the particles, we would like to emphasize once more that the values of the spin we will consider hereafter are limited to $1/M \pmod{1}$. The presentation of this letter is as follows: First, we recall the fractional supersymmetric algebra and describe its scalar representation, a matter of exhibiting the varieties of exotic particles one encounters in invariant fractional supersymmetric quantum field theory. Then we introduce one of our main results namely the quasi determinant Δ . This is a generalized Slater determinant carrying the quantum features of the spin $1/M$ parafermions considered in this study. After that we give the wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ and describe some of its peculiar properties such as the manifestation of the generalized Pauli exclusion principle

obeyed by this specific type of particles. Finally, we give our conclusion.

2 Two-dimensional $(1/M, 0)$ supersymmetry

Roughly speaking the two-dimensional $(1/M, 0)$ supersymmetry is generated, in addition to the spin one energy momentum vector operator P , by a fractional spin $1/M$ charge operator Q satisfying the following relations:

$$Q^M = P; \quad QP = PQ, \quad M = 2, 3, \dots \quad (1)$$

These equations reduce to the standard two-dimensional $(1/2, 0)$ supersymmetry for the leading value $M = 2$. They admit also a Z_M automorphism symmetry acting on Q and P as:

$$Q \mapsto qQ, \quad P \mapsto P, \quad (2)$$

where $q = \exp(2i\pi/M)$ is the Z_M group generator. Irreducible representations of the algebra Eq.(1) are obtained in the same way as in the usual supersymmetric case. They are constructed by repetitive actions of the Q operator on a given one particles state $|\psi_\alpha\rangle$ of spin α . The simplest multiplet is given by the scalar representation consisting of $(M - 1)$ exotic particles of spin L/M , with $L = 1, 2, \dots, M - 1$, along with the scalar state $|\psi_0\rangle$. Denoting by $|\psi_{L/M}\rangle$ the states of this representation, we have:

$$|\psi_{1/M}\rangle \sim Q|\psi_0\rangle, \quad (3)$$

and more generally

$$|\psi_{L/M}\rangle \sim Q^L|\psi_0\rangle, \quad L = 1, 2, \dots, M - 1. \quad (4)$$

The one particle states of this multiplet have different spins. They have then different quantum behaviours exactly as in the $M = 2$ leading case. In what follows, we shall focus our attention on the spin $1/M \pmod{1}$ particles only. These particles coincide with the fermions and the bosons for the limiting cases $M = 2$ and $M = \infty$ respectively

3 The quasi-determinant Δ

This is the object that we are going to propose for describing the wave function of a system of N identical spin $1/M$ particles. To better illustrate how this quantity is built and how things work in practice, let us first introduce the quasi-determinant $\Delta_3^{(1)}$ of 3×3

matrices (a_{ij}) $i, j = 1, 2, 3$. By definition it reads as:

$$\begin{aligned} \Delta_3^{(1)} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_q \\ &\equiv a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_q + q a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}_q + q^2 a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}_q, \end{aligned} \quad (5)$$

where the a_{ij} 's are $\mathbb{C}$ -numbers, $q = \exp(2i\pi/M)$ and where

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}_q \equiv ad + qbc. \quad (6)$$

Note that Eq.(5) is just a kind of a generalization of the usual Sarrus decomposition of the determinant of 3×3 matrices [8]. The upper index (1) carried by $\Delta_3^{(1)}$ means that the generalized Sarrus decomposition is made with respect to the first column of the (3×3) (a_{ij}) matrix. Later on we shall see that, up to an S_3 automorphism transformation, this detail may be ignored. Putting Eq.(6) back into Eq.(5), one gets

$$\begin{aligned} \Delta_3^{(1)} &= (a_{11}a_{22}a_{33} + q^3a_{31}a_{22}a_{13}) + q(a_{11}a_{32}a_{23} + a_{21}a_{12}a_{33}) \\ &\quad + q^2(a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}). \end{aligned} \quad (7)$$

Introducing the rank $(3!)$ permutation group S_3 and following Ref.[9] by expressing all the S_3 permutations in terms of its simple transpositions P_{12} and P_{23} namely

$$P_{13} = P_{12}P_{23}P_{12} \quad (8.a)$$

for the transposition of the first and the third rows and

$$\begin{aligned} P_{(123)} &= P_{23}P_{12} \\ P_{(132)} &= P_{12}P_{23}, \end{aligned} \quad (8.b)$$

for the two S_3 cyclic permutations, Eq.(7) can conveniently be rewritten as:

$$\Delta_3^{(1)} = (1 + qP_{12})(1 + qP_{23} + q^2P_{23}P_{12})a_{11}a_{22}a_{33}. \quad (9)$$

Already at this level, one may learn some remarkable features of the quasi-determinant $\Delta_N^{(1)}$. Choosing at least two columns of the (a_{ij}) 3×3 matrix to be equal by setting for instance $a_{i1} = a_{i2}$, $i = 1, 2, 3$, the quasi-determinant $\Delta_3^{(1)}$ reduces to:

$$\Delta_3^{(1)} = (1 + q) [1 + (q + q^2)P_{23}] a_{11}a_{22}a_{33}. \quad (10)$$

This equation vanishes identically for $q = -1$, i.e. for fermions, in agreement with the Pauli exclusion principle according to which two fermions in the same quantum state

cannot live together. On the other hand taking the three columns of the 3×3 matrix to be equal, i.e. $a_{11} = a_{12} = a_{13}$, one discovers another remarkable feature of the quasi-determinant $\Delta_3^{(1)}$. Indeed, we have in this case:

$$\Delta_3^{(1)} = (1+q)(1+q+q^2) a_{11}a_{22}a_{33} . \quad (11)$$

This quantity vanishes not only for the case $M = 2$ as required by the Pauli exclusion principle but also for $M = 3$. Actually this property is a manifestation of a generalized exclusion principle obeyed by the spin $1/3 \pmod{1}$ particles. To fix the ideas, let us anticipate on this point by saying that the generalized Pauli exclusion principle states that no more than $(M-1)$ exotic particles of spin $1/M \pmod{1}$ can live altogether on the same quantum state [6]. The cases $M = 2$ and $M = 3$ described by Eqs.(10,11) are just the leading examples. The general proof will be given later on. For the moment let us complete this discussion by giving the quasi-determinant Δ_N of an arbitrary $N \times N$ (A_{ij}) matrix. Expanding Δ_N with respect to the first column in terms of quasi-determinants of rank $(N-1)$ as follows:

$$\begin{aligned} \Delta_N^{(1)}(A) = & A_{11} \begin{vmatrix} A_{22} & A_{23} & \dots & A_{2N} \\ A_{32} & A_{33} & \dots & A_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N2} & A_{N3} & \dots & A_{NN} \end{vmatrix}_q + q A_{21} \begin{vmatrix} A_{11} & A_{13} & \dots & A_{1N} \\ A_{31} & A_{32} & \dots & A_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N3} & \dots & A_{NN} \end{vmatrix}_q \\ & + \dots + q^{(N-1)} A_{N1} \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1,N-1} \\ A_{21} & A_{22} & \dots & A_{2,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N-1,1} & A_{N-1,2} & \dots & A_{N-1,N-1} \end{vmatrix}_q . \end{aligned} \quad (12)$$

and repeating the same procedure $(N-1)$ times, one gets at the end the series:

$$\Delta_N^{(1)}(A) = \sum_{n=0}^{N(N-1)/2} q^n \mathcal{P}_n(A_{ij}) , \quad (13)$$

where the $\mathcal{P}_n(A_{ij})$'s are homogeneous polynomials in the A -matrix entries. Note that this equation can also be brought to a form similar to that given by Eq.(9). To do that, one introduces the rank $(N!)$ permutation group S_N and expresses all its elements in terms of the $(N-1)$ simple transpositions $P_{i,i+1}$, $i = 1, \dots, N-1$ exactly as we have done in Eqs.(8)

Using identities type

$$\mathcal{P}_{ij} = \mathcal{P}_{j,j-1} \mathcal{P}_{i,j-1} \mathcal{P}_{j,j-1} , \quad (14)$$

one finds after some straightforward algebra

$$\Delta_N^{(1)}(A) = \prod_{\ell=2}^N [1 + \chi_\ell] a_{11}a_{22} \dots a_{NN} , \quad (15)$$

where the operator χ_ℓ is given by

$$\chi_\ell = \sum_{k=1}^{\ell-1} q^{(\ell-k)} P_{\ell,\ell-1} P_{\ell-1,\ell-2} \dots P_{k+1,k} . \quad (16)$$

Eqs.(13) and (15) admit several remarkable properties one of them is given by the case where all the columns of $N \times N$ A -matrix are equal i.e., $A_{ij} = A_{ik}$ for any integers j and k . In this case, Eqs.(13) and (15) reduce to:

$$\Delta_N^{(1)}(A) = A_{11}A_{22} \dots A_{NN} \cdot \prod_{j=1}^{(N-1)} \left[\frac{(1-q^{j+1})}{(1-q)} \right] . \quad (17)$$

Eq.(17) vanishes not only for $M = 2$ ($q = -1$) as required by the Pauli nilpotency principle but also for all the integer values of M lying between two and N . This means that for particles of spin $1/M \pmod{1}$ with $M \leq N$, the quasi-determinant $\Delta_N^{(1)}(A)$ vanishes identically whenever M columns of A at least are equal. It means also that no more than $(M-1)$ identical particles of spin $1/M \pmod{1}$ can live together on the same quantum state. This states the generalized Pauli exclusion principle conjectured in [4,7]. We end this section by noting that quasi-determinants $\Delta_N^{(j)}$ whose generalized Sarrus decompositions is done with respect to the j -th columns can also be introduced. However, it can be shown that all these quantities $\Delta_N^{(j)}(A)$, $1 \leq j \leq N$, are identified to $\Delta_N^{(1)}(A)$ up to S_N automorphism transformations:

$$\Delta_N^{(1)}(A) \equiv \Delta_N^{(j)}/S_N . \quad (18)$$

This equation means that given a permutation σ of S_N and denoting by $(A^\sigma)_{ij} = A_{i\sigma(j)}$, we have

$$\Delta_N^{(1)}(A^\sigma) = \sigma \Delta_N^{(j)}(A) . \quad (19)$$

Having defined the quasi-determinant $\Delta_N^{(1)}$ and described some of its remarkable features, we turn now to the other purpose of this paper.

4 The wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$

We start by considering a gas of N identical two-dimensional spin $1/M \pmod{1}$ particles enclosed within a container of volume V . Let ξ_j denote collectively all the coordinates of the j -th particle namely its two position coordinates (τ, σ) and its spin coordinate s . Let k_j be an index labelling the possible quantum states $\psi_{k_j}(\xi_j) = \psi_{k_j}(j)$ of this single particle. Each possible value of k_j corresponds to a specification of the energy ϵ_{k_j} and

the momentum p_{k_j} of the particle as well as the direction of the spin orientation $\pm s$. The state of the whole gas is then described by the set of quantum numbers $\{k_j, 1 \leq j \leq N\}$ characterizing the wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ of the gas in this state. To write down explicitly this wave function in terms of the individual quantum states $\psi_{k_j}(j)$, one should take into account the indistinguishability of the particles of the gas since it imposes definite symmetry requirements on $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$. Indeed, the interchange of any two particles do not lead to a new state of the gas; i.e.:

$$\psi(1, 2, \dots, i, \dots, j, \dots, N) = \omega(i, j) \psi(1, 2, \dots, j, \dots, i, \dots, N) \quad (20)$$

where we have dropped out the subscripts $\{k_1, k_2, \dots, k_N\}$ for sake of simplicity and where the phase $\omega(i, j)$ is given by

$$\omega(i, j) = \exp \in (2\pi i s) \ell(P_{ij}) = q^{\ell(P_{ij})} \quad (21)$$

In this equation $\in = \text{sign}(j - i)$ and $\ell(\sigma)$ is the minimal length of the permutation σ expressed as a product of simple transpositions. Taking into account the results describing the limiting cases $M = 2$ and $M = \infty$, the solution of Eqs.(20,21) is given by the following quasi-determinant of the individual wave functions $\psi_{k_i}(j)$:

$$\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N) \simeq \begin{vmatrix} \psi_{k_1}(1) & \psi_{k_1}(2) & \dots & \psi_{k_1}(N) \\ \psi_{k_2}(1) & \psi_{k_2}(2) & \dots & \psi_{k_2}(N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{k_N}(1) & \psi_{k_N}(2) & \dots & \psi_{k_N}(N) \end{vmatrix}_q \quad (22)$$

5 More on the generalized Pauli exclusion principle

According to the usual Pauli exclusion principle two fermions in the same quantum state cannot live together. This principle extends to spin $1/M \pmod{1}$ particles as follows. No more than $(M - 1)$ spin $1/M \pmod{1}$ particles can live together on the same quantum state. The fermion nilpotency and the Bose condensation appear then as a natural consequence of this generalized principle. To illustrate the idea of this generalization, we study hereafter the wave functions of a gas of four identical particles of spin $1/M$ by taking two particles in the same quantum states; three particles in the same quantum states and finally all the particles in the same quantum state.

5.1 Two particles in the same quantum state

Taking, for instance, the third and the fourth particles in the same quantum state, i.e., $k_3 = k_4 = k$, the wave function Eq.(22) then reads:

$$\begin{aligned} \psi_{\{k_1, k_2, k, k\}}(1, 2, 3, 4) = & (1 + q) \left\{ \psi_{k_1}(1) [\psi_{k_2}(2) \psi_k(3) \psi_k(4) + q \psi_k(2) \psi_{k_2}(3) \psi_k(4) \right. \\ & \left. + q^2 \psi_k(2) \psi_k(3) \psi_{k_2}(4)] \right. \\ & + q \psi_{k_2}(1) [\psi_{k_1}(2) \psi_k(3) \psi_k(4) + q \psi_k(2) \psi_{k_1}(3) \psi_k(4) \\ & \left. + q^2 \psi_k(2) \psi_{k_1}(3) \psi_{k_1}(4)] \right. \\ & + q^2 \psi_k(1) [\psi_{k_1}(2) (\psi_{k_2}(3) \psi_k(4) + q \psi_k(3) \psi_{k_2}(4)) \\ & + q \psi_{k_2}(2) (\psi_{k_1}(3) \psi_k(4) + q \psi_k(3) \psi_{k_1}(4)) \\ & \left. + q^2 \psi_k(2) (\psi_{k_1}(3) \psi_{k_2}(4) + q \psi_{k_2}(3) \psi_{k_1}(4))] \right\} \quad (23) \end{aligned}$$

This function vanishes only for $q = -1 (M = 2)$ i.e. for the fermions. Thus two spin $1/M \pmod{1}$ with M greater than two can live on the same quantum state.

5.2 Three particles in the same quantum state

Choosing $k_2 = k_3 = k_4 = k$, the wave function $\psi_{\{k_1, k, k, k\}}(1, 2, 3, 4)$ reads in this case as:

$$\begin{aligned} \psi_{\{k_1, k, k, k\}}(1, 2, 3, 4) = & (1 + q)(1 + q + q^2) \left\{ \psi_{k_1}(1) \psi_k(2) \psi_k(3) \psi_k(4) \right. \\ & + q \psi_k(1) \psi_{k_1}(2) \psi_k(3) \psi_k(4) \\ & + q^2 \psi_k(1) \psi_k(2) \psi_{k_1}(3) \psi_k(4) \\ & \left. + q^3 \psi_k(1) \psi_k(2) \psi_k(3) \psi_{k_1}(4) \right\} \quad (24) \end{aligned}$$

As expected, this quantity vanishes for $M = 2$ but also for $M = 3$. Therefore three particles of spin $1/3 \pmod{1}$ in the same quantum state are not allowed to live altogether. However, three particles of spin $1/M$ with M greater than three are still authorized to live on the same quantum state.

5.3 Four particles in the same quantum state

In this case, $k_1 = k_2 = k_3 = k_4 = k$, the wave function reduces to a single term namely:

$$\psi_{\{k, k, k, k\}}(1, 2, 3, 4) = (1 + q)(1 + q + q^2)(1 + q + q^2 + q^3) \psi_k(1) \psi_k(2) \psi_k(3) \psi_k(4) \quad (25)$$

This equation vanishes for $M = 2$, $M = 3$ and $M = 4$ in agreement with the generalized Pauli exclusion principle stated above.

6 Conclusion

In this paper we have built the wave function $\psi_{\{c_1, \dots, c_N\}}(1, 2, \dots, N)$ of a gas of N identical particles of spin $1/M \pmod{1}$, $M = 2, 3, \dots$. We have shown that this function is proportional to the quasi-determinant $\Delta_N^{(1)}$ which reduces for $M = 2$ and $M = \infty$ to the well known Slater determinant and Slater permanent, respectively. We have also shown that this kind of exotic particles obey a generalized Pauli exclusion principle according to which no more than $(M - 1)$ identical particles of spin $1/M$ can live on the same quantum state. This principle can be thought of as a trait d'union between the Fermi nilpotency and the Bose condensation. Two interesting questions remain, however, without answer in this study. First, what is the wave function of particles of spin L/M with L greater than one? Second, is there any connection between our quasi-determinant $\Delta_N^{(1)}$ using commutative $\mathbb{C}$ -numbers and the deformed determinant based on non commutative algebras? Progress in these directions will be reported elsewhere.

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