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OSCILLATOR AND NONLINEAR ALGEBRAS

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ABSTRACT

Generalized deformed commutation relations for a single mode para-Bose oscillator and for a system of two para-Bose oscillators are constructed. It turns out that generalized deformed para-Bose oscillators are not, in general, exactly independent. Furthermore, we also discuss about the Fock space corresponding to generalized deformed para-Bose oscillators. Finally, we show how $SU(2)$ and $SU(1,1)$ generators can be constructed in terms of generalized deformed para-Bose creation and annihilation operators. The algebras $SU(2)$ and $SU(1,1)$ of generalized deformed oscillators [14, 18] are the special cases of generalized deformed para-Bose oscillators algebras but, interestingly, they have the same form.

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1. Introduction

In the last few years there has been a great deal of interest in particles obeying statistics different from Bose or Fermi. These generalized statistics are called para-Bose and para-Fermi statistics [1-4]. Since the advent of the theory of parastatistics there have been many attempts to generalize the canonical commutation relations (CRs). In particular, the real impetus in the recent study of deformed CRs has been the discovery of quantum groups and algebras and the role played by the q -deformed boson [5-7] in their representation theory. Chatuvedi and Srinivasan [8] have shown that a single para-Bose oscillator may be regarded as a deformed Bose oscillator. In ref. [9] the CR for a single mode of the harmonic oscillator which contains para-Bose and q -oscillator CR have been constructed. The canonical q -transformations with two q -para-Bose oscillators has been described in ref. [10]. Chakrabarti and Jagannathan [11] have considered two-parameter deformation of a parabosonic algebra underlying the two-particle Calogero model and have discussed corresponding coherent states. It is noteworthy that starting from an arbitrary deformation of the oscillator Daskaloyannis [12] has constructed all the general deformed oscillator algebra and has studied its properties. Next, by using this, the generalized deformed centerless Virasoro algebra has been derived [13] and the realization of the generalized deformed $SU(2)$ algebra has been constructed [14]. Furthermore, the analogues of the single-photon and multiphoton coherent states for the generalized deformed bosonic oscillator system have been constructed in a unified way [15]. Taking into consideration the expressions of the quantum fields in terms of the generalized deformed oscillators we have found some special deformations, in which corresponding deformed quantum fields are local and deformed oscillators are linear [16].

This paper deals with generalized deformed para-Bose oscillators (GDPO) and some closely related problems. In section 2 we construct generalized deformed CR for a single mode para-Bose oscillator and determine the connection of GDPO with para-Bose oscillators. Section 3 is devoted to the discussion of generalized Fock space. In section 4 we analyze a possible generalized deformation of the two-dimensional para-Bose oscillator. We present the generalizations of Holstein-Primakoff and Jordan-Schwinger realizations of $SU(2)$ and $SU(1,1)$ algebras, based on the use of the generalized paraboson operators in section 5. We conclude with a short summary of the highlights obtained as well as a list of unsolved problems and topics to be investigated in the future.

2. Generalized deformed para-Bose oscillator algebras and its para-Boson realization

As is well known, a single mode para-Bose system is characterized by the CRs [8,9]

$$[a, \mathcal{N}] = a, \quad [a^+, \mathcal{N}] = -a^+ \quad (1)$$

where

$$\mathcal{N} = \frac{1}{2}(aa^+ + a^+a) - \frac{p}{2} \quad (2)$$

and p is the order of the para-Bose system.

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Also

$$aa^+ = f(\mathcal{N} + 1), \quad a^+a = f(\mathcal{N}) \quad (3)$$

with

$$f(n) = n + \frac{1}{2} \{1 - (-1)^n\} (p - 1). \quad (4)$$

Hence

$$[a, a^+] = f(\mathcal{N} + 1) - f(\mathcal{N}) = 1 + (-1)^{\mathcal{N}} (p - 1). \quad (5)$$

From these relations, an operator A^+ was constructed so that [8,9]

$$A^+ = a^+ \frac{\mathcal{N} + 1}{f(\mathcal{N} + 1)} \quad (6)$$

$$[a, A^+] = 1, \quad [A^+, \mathcal{N}] = -A^+ \quad (7)$$

By the results just above mentioned the number operator $\mathcal{N}$ can be written as follows

$$\mathcal{N} = A^+a. \quad (8)$$

Let us now turn to the question of generalized deformed para-Bose oscillator algebra corresponding to the annihilation and creation operators $\tilde{a}$ and $\tilde{a}^+$ respectively. According to the method of Daskaloyannis [12, 13] we begin by establishing an operator $\tilde{A}^+$ and consider the generalized deformed CR

$$[\tilde{a}, \tilde{A}^+] = F(N + 1) - F(N) = [N + 1] - [N] \quad (9)$$

where $\tilde{A}^+$ is defined by [9, 10]

$$\tilde{A}^+ = \frac{\tilde{a}^+(N + 1)}{f(N + 1)}. \quad (10)$$

In the q-deformed oscillator algebra relating to works [8, 9] the function $F(x)$ takes the form

$$F(x) = \frac{q^x - 1}{q - 1}. \quad (11)$$

For the q-deformed oscillator algebra in the form first discovered by Macfarlane [5] and Biedenharn [6] we have

$$F(x) = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin(\tau x)}{\sin \tau}, \quad \tau = -i \ln q. \quad (12)$$

The number operator N satisfies, by definition, the CRs

$$[\tilde{a}, N] = \tilde{a}, \quad [\tilde{A}^+, N] = -\tilde{A}^+. \quad (13)$$

It can be shown that N is given by the relation [12, 13]

$$N = F^{-1}(\tilde{A}^+ \tilde{a})$$

or equivalently

$$N = \tilde{A}^+ \frac{N + 1}{F(N + 1)} \tilde{a} \quad (14)$$

As an important immediate consequence of (10) and (13) we obtain

$$[\tilde{a}^+, N] = -\tilde{a}^+. \quad (15)$$

By using (10) and (13) we also obtain

$$\tilde{a}^+ \tilde{a} = f(N) \frac{[N]}{N} \quad (16)$$

$$\tilde{a} \tilde{a}^+ = f(N + 1) \frac{[N + 1]}{N + 1} \quad (17)$$

With the help of (16) and (17) we get

$$[\tilde{a}, \tilde{a}^+] = f(N + 1) \frac{[N + 1]}{N + 1} - f(N) \frac{[N]}{N} \equiv G(N) \quad (18)$$

In so doing, we are led to the most general CR (18) for GDPO. It includes as special cases the various forms of the CRs defined in the literature [9, 11, 17]. For instance, the q-deformed para-Bose CR [9] can be reproduced with the choice (11) of the structure functions.

According to refs. [15, 18, 19] the boson realization of the operators $\tilde{a}$ and $\tilde{A}^+$ which satisfy CR (9) takes the form:

$$\tilde{a} = \sqrt{\frac{[N + 1]}{N + 1}} a \quad (19)$$

$$\tilde{A}^+ = A^+ \sqrt{\frac{[N + 1]}{N + 1}}, \quad (20)$$

where

$$N = A^+a = \mathcal{N}. \quad (21)$$

Combining (6), (10) and (20) yields

$$\tilde{a}^+ = a^+ \sqrt{\frac{[N + 1]}{N + 1}}. \quad (22)$$

So, owing to the relations (19) and (22) we have just constructed a nonlinear realization of the single-mode generalized deformed para-Bose algebra in terms of a single para-boson. It is interesting to note that by using the method of Jannussis [18] we may

rewrite the relations (19) and (22) as

$$\tilde{a} = \sqrt{\frac{\left(\sum_{l=0}^n G(l)\right)_{n=N}}{f(N+1)}} a \quad (23)$$

$$\tilde{a}^+ = a^+ \sqrt{\frac{\left(\sum_{l=0}^n G(l)\right)_{n=N}}{f(N+1)}}. \quad (24)$$

The above results are general and can be applied for any deformed para-Bose oscillator, including the q-deformed ones [17].

3. Generalized Fock space for the operators $\tilde{a}$ and $\tilde{a}^+$

The generalized deformed para-Bose oscillator algebra possess a Fock space of eigenvectors of the number operator N ,

$$N|n\rangle = n|n\rangle \quad (25)$$

$$\langle n|m\rangle = \delta_{nm} \quad (26)$$

if the vacuum state $|0\rangle$ satisfies the following relations,

$$\tilde{a}|0\rangle = 0 \quad (27)$$

$$\tilde{a}\tilde{a}^+|0\rangle = p[1]|0\rangle. \quad (28)$$

It is easily seen that relation (28) generalizes corresponding one in the case of para-Bose oscillator [8]. The orthonormalized eigenstates $|n\rangle$ of the operator N may be obtained by repeated applications of $\tilde{a}^+$ on $|0\rangle$:

$$|n\rangle = c_n (\tilde{a}^+)^n |0\rangle. \quad (29)$$

The numerical norm c_n can then be determined from (26) and (17) to obtain

$$|n\rangle = \frac{1}{\sqrt{f(n)! \frac{[n]!}{n!}}} (\tilde{a}^+)^n |0\rangle \quad (30)$$

where $[n]! = [n][n-1]\dots[1]$, $[0]! = 1$ and $f(n)! = f(n)f(n-1)\dots f(1)$.

With the help of (30) and (17) we find

$$\begin{aligned} \tilde{a}|n\rangle &= \sqrt{f(n)\frac{[n]}{n}} |n-1\rangle \\ \tilde{a}^+|n\rangle &= \sqrt{f(n+1)\frac{[n+1]}{n+1}} |n+1\rangle. \end{aligned} \quad (31)$$

As $[n] \rightarrow n$ this becomes

$$\begin{aligned} a|n\rangle &= \sqrt{f(n)} |n-1\rangle \\ a^+|n\rangle &= \sqrt{f(n+1)} |n+1\rangle. \end{aligned} \quad (32)$$

Therefore the relation (31) reduces to the corresponding one for the para-Bose oscillator [8].

4. Two-dimensional generalized deformed para-Bose oscillators

In this section we will analyse a possible generalized deformation of the two-dimensional para-Bose oscillator. In accordance with eqs. (18), (19), (22) we consider two generalized deformed para-Bose oscillators $(\tilde{a}_1, \tilde{a}_1^+)$ and $(\tilde{a}_2, \tilde{a}_2^+)$ which satisfy the relations

$$[\tilde{a}_i, \tilde{a}_i^+] = f(N_i + 1) \frac{[N+1]}{N+1} - f(N_i) \frac{[N]}{N} \quad (33)$$

$$\tilde{a}_i = \sqrt{\frac{[N+1]}{N+1}} a_i, \tilde{a}_i^+ = a_i^+ \sqrt{\frac{[N+1]}{N+1}}. \quad (34)$$

Here the operators a_i, a_i^+ ($i = 1, 2$) obey the usual para-Bose CR:

$$[a_i, a_i^+] = f(N_i + 1) - f(N_i) \quad (35)$$

and the number operator N is defined as follows : $N = N_1 + N_2$.

From (34) and (35) we get

$$[\tilde{a}_1, \tilde{a}_2] = [\tilde{a}_1^+, \tilde{a}_2^+] = 0 \quad (36)$$

$$[\tilde{a}_i, \tilde{a}_i^+] = f(N_i + 1) \frac{[N+1]}{N+1} - f(N_i) \frac{[N]}{N} \quad (37)$$

$$[\tilde{a}_1, \tilde{a}_2^+] = a_1 a_2^+ \left\{ \frac{[N+1]}{N+1} - \frac{[N]}{N} \right\} \quad (38a)$$

$$[\tilde{a}_2, \tilde{a}_1^+] = a_2 a_1^+ \left\{ \frac{[N+1]}{N+1} - \frac{[N]}{N} \right\} \quad (38b)$$

$$\begin{aligned} &(\tilde{a}_1 \tilde{a}_1^+ + \tilde{a}_2 \tilde{a}_2^+) - (\tilde{a}_1^+ \tilde{a}_1 + \tilde{a}_2^+ \tilde{a}_2) = \\ &= \left\{ f(N_1 + 1) + f(N_2 + 1) \right\} \frac{[N+1]}{N+1} - \left\{ f(N_1) + f(N_2) \right\} \frac{[N]}{N}. \end{aligned} \quad (39)$$

Equation (38b) is the Hermitian conjugate of (38a). Thus we see that while $(\tilde{a}_1, \tilde{a}_2), (\tilde{a}_1^+, \tilde{a}_2^+)$ commute ordinary and $(\tilde{a}_i, \tilde{a}_i^+)$ satisfy eq. (37) which is similar to eq. (18); $(\tilde{a}_i, \tilde{a}_j^+)$ ($i \neq j$) obey non-trivial CRs. This means that the generalized deformed para-Bose oscillators, generally speaking, are not exactly independent (this is a different instance of the nonlinearity of the deformed oscillators). It is worth pointing out that in particular deformations corresponding to $\frac{[x+1]}{x+1} = \frac{[x]}{x}$ (for example in the case of $[x] = x \cos(2\pi x)$)

we have usual CR $[\tilde{a}_i, \tilde{a}_j^\dagger] = 0 (i \neq j)$. In addition, the right hand side of eq. (39) can be replaced by the term

$$\sum_{i=1}^2 \left\{ f(N_i + 1) \frac{[N_i + 1]}{N_i + 1} - f(N_i) \frac{[N_i]}{N_i} \right\}. \quad (40)$$

Therefore, in these cases we have independent deformed para-Bose oscillators. Several comments are called for in connection with the above-mentioned result. Starting from $[x] = (q^x - q^{-x})/(q - q^{-1})$ Gangopadhyay [20] concluded that his paper seems to indicate the existence of some no-go theorem whereby the realization of canonical q transformations with two or more independent q-oscillators is not possible. Clearly this conclusion is correct and can be easily seen from (38), (39) when $p = 1$, $x = (q^x - q^{-x})/(q - q^{-1})$. Moreover, by considering general deformations we can find out some systems of independent q-deformed para-Bose oscillators with choices of corresponding special deformations. Furthermore, at this point we remark that these particular types of deformation also concerned with local deformed quantum fields [16].

5. Realizations of $SU(2)$ and $SU(1, 1)$ algebras in terms of one or two generalized deformed para-Bose oscillators

We shall begin with the Holstein-Primakoff realizations of generalized deformed $SU(2)$ and $SU(1, 1)$ algebras. For $SU(2)$ we get

$$\tilde{J}_+ = \tilde{a}^\dagger \sqrt{[2\sigma - N]} \quad (41)$$

$$\tilde{J}_- = \sqrt{[2\sigma - N]} \tilde{a} \quad (42)$$

and define the functional form of $\tilde{J}_3$ by applying the CR

$$[\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_3]. \quad (43)$$

Using equations (16) and (17) we find that

$$[2\tilde{J}_3] = [2\sigma + 1 - N] \frac{[N]}{N} f(N) - [2\sigma - N] \frac{[N + 1]}{N + 1} f(N + 1). \quad (44)$$

This gives

$$[\tilde{J}_3, \tilde{J}_+] = \left\{ \tilde{J}_3(N) - \tilde{J}_3(N - 1) \right\} \tilde{J}_+ \quad (45)$$

$$[\tilde{J}_3, \tilde{J}_-] = -\tilde{J}_- \left\{ \tilde{J}_3(N) - \tilde{J}_3(N - 1) \right\}. \quad (46)$$

Similarly, the Holstein-Primakoff realization of $SU(1, 1)$ can be formulated as

$$\tilde{K}_+ = \tilde{a}^\dagger \sqrt{[2\sigma + N]} \quad (47)$$

$$\tilde{K}_- = \sqrt{[2\sigma + N]} \tilde{a} \quad (48)$$

$$-[2\tilde{K}_3] = [\tilde{K}_+, \tilde{K}_-] = [2\sigma - 1 + N] \frac{[N]}{N} f(N) - [2\sigma + N] \frac{[N + 1]}{N + 1} f(N + 1). \quad (49)$$

for which

$$[\tilde{K}_3, \tilde{K}_+] = \left\{ \tilde{K}_3(N) - \tilde{K}_3(N - 1) \right\} \tilde{K}_+ \quad (50)$$

$$[\tilde{K}_3, \tilde{K}_-] = -\tilde{K}_- \left\{ \tilde{K}_3(N) - \tilde{K}_3(N - 1) \right\} \quad (51)$$

For the case $p = 1$, with $F(x)$ satisfied the following realization [14]

$$F(x)F(y + 1) - F(x + 1)F(y) = F(x - y), \quad (52)$$

the form (44) can be written as

$$J_3 = N - \sigma. \quad (53)$$

From (53), (45) and (46) it follows that

$$[J_3, J_\pm] = \pm J_\pm. \quad (54)$$

Thus, the CRs (43), (45), and (46) reduce to the generalized deformed bosonic $SU(2)$ algebra was previously considered in ref. [14], which can also be reduced to the results in refs. [21, 22].

We now turn our attention to the following Jordan-Schwinger realization of generalized deformed para-bosonic $SU(2)$ algebra :

$$\tilde{J}_+ = \tilde{a}_1^\dagger \tilde{a}_2 \quad (55)$$

$$\tilde{J}_- = \tilde{a}_2^\dagger \tilde{a}_1 \quad (56)$$

$$[2\tilde{J}_3] = [\tilde{J}_+, \tilde{J}_-]. \quad (57)$$

The CR (57) can be written :

$$[2\tilde{J}_3] = \frac{[N_1]}{N_1} \frac{[N_2 + 1]}{N_2 + 1} f(N_1) f(N_2 + 1) - \frac{[N_1 + 1]}{N_1 + 1} \frac{[N_2]}{N_2} f(N_1 + 1) f(N_2). \quad (58)$$

Using (55), (56) and (58) it follows that

$$[\tilde{J}_3, \tilde{J}_+] = \left\{ \tilde{J}_3(N_1, N_2) - \tilde{J}_3(N_1 - 1, N_2 + 1) \right\} \tilde{J}_+. \quad (59)$$

$$[\tilde{J}_3, \tilde{J}_-] = -\tilde{J}_- \left\{ \tilde{J}_3(N_1, N_2) - \tilde{J}_3(N_1 - 1, N_2 + 1) \right\}. \quad (60)$$

Interestingly, the CRs (57), (59) and (60) coincide exactly with the corresponding CRs of the generalized deformed bosonic $SU(2)$ algebras in term of the generalized deformed usual oscillators [18].

Following essentially the same analysis as above, we obtain

$$[\tilde{K}_+, \tilde{K}_-] = -[2\tilde{K}_3] = \frac{[N_1]}{N_1} \frac{[N_2]}{N_2} f(N_1) f(N_2) - \frac{[N_1 + 1]}{N_1 + 1} \frac{[N_2 + 1]}{N_2 + 1} f(N_1 + 1) f(N_2 + 1) \quad (61)$$

$$[\tilde{K}_3, \tilde{K}_+] = \left\{ \tilde{K}_3(N_1, N_2) - \tilde{K}_3(N_1 - 1, N_2 - 1) \right\} \tilde{K}_+, \quad (62)$$

$$[\tilde{K}_3, \tilde{K}_-] = -\tilde{K}_- \left\{ \tilde{K}_3(N_1, N_2) - \tilde{K}_3(N_1 - 1, N_2 - 1) \right\} \quad (63)$$

where

$$\tilde{K}_+ = \tilde{a}_1^+ \tilde{a}_2^+ \quad (64)$$

$$\tilde{K}_- = \tilde{a}_1 \tilde{a}_2. \quad (65)$$

For the special case $p = 1$ and $F(x)$ defined by (12), i.e. the case of the deformed q-harmonic oscillator, the above commutations yield [23]

$$[K_+, K_-] = -[2K_3] \quad (66)$$

$$[K_3, K_\pm] = \pm K_\pm \quad (67)$$

with

$$K_3 = \frac{1}{2}(N_1 + N_2 + 1). \quad (68)$$

6. Conclusion

Our results are general because we did not choose the concrete form of the function $F(x)$.

In the same way as mentioned above we can extend our results to the case of generalized deformed parafermions. We think that our method of realizations of generalized deformed $SU(2)$ and $SU(1, 1)$ algebras will be also applied to other kinds of algebras, such as superalgebras and $SU(N)$ algebra.

An interesting open question is whether some special deformations corresponding to local and linear quantum fields [16] can be generalized to the case of quantum parafields. Works along these lines are in progress.

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