SINGULARITY THEORY for RIEMANNIAN DISTANCE FUNCTIONS
on NON-POSITIVELY CURVED SURFACES

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0. ABSTRACT

We construct Morse theory and singularity theory for Riemannian distance functions d_p on a compact non-positively curved surface M_2 . Stable critical points and their indexes are described due to connection between Min-type functions and Riemannian distance functions on non-positively curved manifolds; Min-type functions are minima of a finite number of smooth functions. We prove that the set of non-positively curved metrics, such that (for a given $p \in M_2$) the distance function d_p is Morse, is open and everywhere dense in the set of non-positively curved Riemannian metrics.

For a generic metric on M_2 distance functions d_q are not Morse for some points $q \in M_2$; the description of singularities of d_q is based on Morse theory for non-positively curved Riemannian metrics $\rho(x, y)$ which are Min-type functions on $M_2 \times M_2$. We prove that a generic negatively curved metrics are Morse.

I. DISTANCE FUNCTIONS

0. Introduction

Several generalizations of the Morse theory to the different classes of non-smooth functions are due to the fact that one does not need Hessian, but only the index of Hessian to construct a Morse theory. This index can be interpreted as the dimension of the maximal submanifold where the function has a strict maximum.

In this way Morse theory can be extended to several classes of non-smooth functions. Stable singularities of Min-Max functions (which can be obtained by a superposition of several operations minimum and maximum from continuous families of smooth functions) were studied in Arnold's school; results, obtained in [B1], [B2], [M1], [M2], show that Morse theory can be constructed for this class of functions (though it has not been done yet).

One of the purposes of such generalizations is an approach to optimization, (including non-smooth optimization) via singularity theory, see [A], [CGR1], [CGR2], [CGR3].

Another application is in geometry, where non-smooth functions arise in a natural way in "completely smooth" problems of Riemannian geometry.

In this paper we develop a Morse and singularity theory for two closely connected classes of non-smooth functions: Min-type functions, which are minima of a finite number of smooth functions on a manifold, and Riemannian distance functions on non-positively curved manifolds.

To make this paper independent we present principal facts related to Min-type functions on manifolds and to their connection with Riemannian distance functions on non-positively curved manifolds. In the first section we recall necessary facts related to Morse theory of Min-type functions obtained in [GR]; second section contains results on approx-

imation of Morse Min-type functions by Morse smooth functions, from [Ge]. In section three we reproduce results related to relations between Min-type functions and Riemannian distance functions on negatively curved manifolds, from [GR]. These results are presented without proofs.

The rest of the first Chapter contains new results related to series of distance functions, interpretation of metrics on a non-positively curved manifold M_n as a distance function on $M_n \times M_n$, and initial results related to singularity theory for Riemannian distance functions.

The second chapter is devoted to Morse theory for Riemannian distance functions on negatively curved surfaces; we prove that, for a given point p the subset of non-positively curved metrics on M_2 , such that d_p is a Morse distance function, is open and everywhere dense.

The study of singularity theory (in Chapter III) is based on the fact that any non-positively curved Riemannian metric is a Min-type function. This allows us to describe degenerations of distance functions d_q for all $q \in M_2$ for a generic metric on M_2 .

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We use the following notations in this paper.

NOTATIONS.

$r_{inj}(x)$ is the injectivity radius at x and $r_{inj} = \min_{x \in M_n} r_{inj}(x)$;

$diam(M_n)$ is the diameter of M_n ;

$Sect_{x,\alpha}$ is the sectional curvature on a 2-plane $\alpha \in T_x M_n$, $x \in M_n$, and $Sect(M_n) = \min_{x \in M_n, \alpha \subset T_x M_n} Sect_{x,\alpha}$.

μ is the Riemannian measure on M_n ;

$D_r(x)$ is the ball of radius r centered at x ;

$S_r(x)$ is the sphere of radius r centered at x ;

$\pi : (M_n, \tilde{\rho}) \rightarrow (M_n, \rho)$

is the universal covering; we denote all objects (points, balls, geodesics, etc.) related to the universal covering $\tilde{M}_n$ by the same letters as the corresponding objects for M_n but with tilde.

We use the letter γ with different indexes to denote geodesics; we use the notation $\dot{\gamma}(p)$ to denote the $\frac{d\gamma}{dt}(t)$ for t such that $\gamma(t) = p$; of course, we use such notation only when p is not a point of self-intersection for γ .

We use the symbol $\odot$ to indicate the end of the proof.

1. Morse Min-type functions

DEFINITION 1.1 (germs of Min-type functions). Let M_n be a C^∞ -smooth n -manifold. A germ of a function $f : M_n \rightarrow \mathbb{R}^1$ at $x \in M_n$ is said to be a germ of a Min-type function if there exist germs of smooth functions $\alpha_1, \dots, \alpha_l$ at x such that

$$f = \min \{ \alpha_1, \dots, \alpha_l \} \quad (1)$$

in a neighbourhood $U(x)$.

DEFINITION 1.2 (minimal representations). A representation (1) of a germ of a Min-type function f is said to be minimal if it includes the minimal possible number of smooth functions.

DEFINITION 1.3 (global Min-type functions). A function $f : M_n \rightarrow \mathbb{R}^1$ is said to be a Min-type function if each germ of f is a germ of a Min-type function.

PROPOSITION 1.1. Let M_n be a compact manifold and $f : M_n \rightarrow \mathbb{R}^1$ be a function such that each germ of f is a germ of a Min-type function. Then there exist smooth functions $\alpha_1, \dots, \alpha_N : M_n \rightarrow \mathbb{R}^1$ such that

$$f = \min \{ \alpha_1, \dots, \alpha_N \} \quad (2)$$

on M_n .

DEFINITION 1.4 (C^k -Min-type functions). A Min-type function f is said to be C^k -smooth if there exists a minimal representation (1) with C^k -smooth functions α_i .

DEFINITION 1.5 (C^k -topology). We define C^k -topology on the space of C^k -Min-type functions in such a way that a small perturbation of f is a result of independent C^k -small perturbations of α_i , $i = 1, \dots, N$ in (2).

REMARK. The C^k -topology does not depend on a choice of a representation (2).

DEFINITION 1.6 (non-degenerate points). Let $f : M_n \rightarrow \mathbb{R}^1$ be a Min-type function, $x \in M_n$ and $f = \min \{\alpha_1, \dots, \alpha_m\}$ be a minimal representation of f at x ; x is said to be a **non-degenerate point** if the germ of f at x satisfies the following two conditions:

- 1) Any $m - 1$ gradients among $\text{grad } \alpha_1(x), \dots, \text{grad } \alpha_m(x)$ are linearly independent;
- 2) $\text{grad } (\alpha_1 - \alpha_2), \dots, \text{grad } (\alpha_{m-1} - \alpha_m)$ are linearly independent at x ;

A non-degenerate point x is said to be **regular** if the convex hull $\text{Conv} \{\text{grad } \alpha_i(x)\}$ does not include the origin, and **critical** otherwise.

REMARK. The three previous definitions do not depend on a choice of a minimal representation (1).

DEFINITION 1.7 (Morse Min-type functions). A Min-type function $f : M_n \rightarrow \mathbb{R}^1$ is said to be a **Morse Min-type function** if f is non-degenerate at all points.

LEMMA 1.1. Let $f : M_n \rightarrow \mathbb{R}^1$ be a Morse Min-type function, x be a critical point of f and $f = \min \{\alpha_1, \dots, \alpha_m\}$ be a minimal representation of a germ of f at x . Then the equations

$$\alpha_1(y) = \dots = \alpha_m(y)$$

define a germ for a smooth submanifold of dimension $(n - m + 1)$ at x .

Denote this submanifold $\mathcal{N}_x^f$. The restriction of f on $\mathcal{N}_x^f$ is smooth (f coincides with α_1 on this submanifold, and then coincides with any of α_i); in fact $\mathcal{N}_x^f$ is the germ of the maximal submanifold where f is smooth.

LEMMA 1.2. The restriction of f to $\mathcal{N}_x^f$ is smooth and x is a non-degenerate critical point for this restriction.

Let us define the index for Min-type functions at non-degenerate critical points; we denote the index of f at x as $\text{ind}_x f$.

DEFINITION 1.8 (index). Let $f : M_n \rightarrow \mathbb{R}^1$ be a Min-type function and x be a non-degenerate critical point for f . We define

$$\text{ind}_x f = (m - 1) + \text{ind}_x f|_{\mathcal{N}_x^f}.$$

THEOREM 1.1. The set of Morse Min-type functions is open and everywhere dense in the space of C^2 -Min-type functions.

As well known, see [AGV], smooth functions have simple normal forms at regular and at non-degenerate critical points. All functions on $\mathbb{R}^n$ which are regular at the origin O , and vanishing at O are on the same orbit of the action of the group of local diffeomorphisms Diff_{loc}^n of $\mathbb{R}^n$. Therefore the normal form can be chosen as $f \sim \text{Const} + x_1$ at a regular point.

All smooth functions vanishing at O that have a non-degenerate critical point at O are on one of $n + 1$ orbits of the action of Diff_{loc}^n ; these orbits are numbered by the index of the Hessian of f . The normal form of a germ of f at x is

$$f = \text{Const} - \sum_{i=1}^k x_i^2 + \sum_{i=k+1}^n x_i^2,$$

where $\{x_i\}_{i=1}^n$ is a local coordinate system around x , k is the index of f . To obtain a sensible orbit structure for Min-type functions one has to increase the transformation group, compare [Mal].

DEFINITION 1.9 (group of almost smooth homeomorphisms). A homeomorphism $\phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is said to be **almost smooth** if ϕ and ϕ^{-1} are smooth outside a submanifold of a positive codimension.

Denote the group of almost smooth local homeomorphisms of $\mathbb{R}^n$ as Hom_{as}^n .

THEOREM 1.2 (non-degenerate regular points). All germs of C^2 -Min-type functions on $\mathbb{R}^n$, vanishing at the origin and such that the origin is a non-degenerate regular point, are on the same orbit of action of Hom_{as}^n . The normal form of a Min-type function at a non-degenerate regular point is $f \sim Const + x_1$.

THEOREM 1.3 (non-degenerate critical points). All germs of Min-type functions, such that the origin is a non-degenerate critical point and vanishing at the origin, are on one of $n + 1$ orbits of the action of Hom_{as}^n . The normal form of a Min-type function at such point is

$$f \sim Const - \sum_{i=1}^k x_i^2 + \sum_{i=k+1}^n x_i^2.$$

REMARKS.

1) The number k coincides with the index of Min-type function defined above, see Definition 1.9; k is the dimension of the maximal submanifold where f has a strict local maximum.

2) Certain germs of Min-type functions at degenerate critical points are equivalent to germs of Min-type functions at non-degenerate critical points under the action of Hom_{as}^n . The simplest example is birth-death singularities, defined below.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ has a birth-death singularity at the origin; then the normal form of f is

$$f = Const + x_n^3 - \sum_{i=1}^k x_i^2 + \sum_{k+1}^{n-1} x_i^2,$$

see [AGV]. This function is equivalent to

$$f_1 = Const + x_n - \sum_{i=1}^k x_i^2 + \sum_{k+1}^{n-1} x_i^2$$

under the action of Hom_{as}^n ; (the transformation is $x_n \rightarrow x_n^{1/3}$); in its turn, f_1 is smoothly equivalent to $Const + x_n$.

A critical point, where f has a birth-death singularity, can be considered as a “degenerate regular” point in the following sense: the topological type of $f^{-1}(-\infty, c)$ does not change when c goes through zero. (The “classical” explanation of this effect is that the decomposition of the birth-death singularity at the origin gives two non-degenerate critical points with indexes differing by one. Such points can be eliminated, see [Mi2], and contribute (when taken together) to the change of homotopy type of $f^{-1}(-\infty, c)$).

PROPOSITION 1.2. Let a degenerate critical point y of f be equivalent to a non-degenerate critical point of index k . Then the decomposition of y (by a small perturbation of f) gives a non-degenerate critical point $\tilde{y}$ of index k and a number of pairs of non-degenerate critical points with indexes differing by one, and a number of regular points.

This statement follows from the theorem on approximation of a Morse Min-type function by a smooth Morse function.

2. Approximation by smooth Morse functions

DEFINITION 2.1 (non-smooth set). Let $f : M_n \rightarrow \mathbb{R}^1$ be a Min-type function, denote $U_f \subset M_n$ the set of points where f is not smooth.

LEMMA 2.1. Let $f : M_n \rightarrow \mathbb{R}^1$ be a Morse Min-type function. Then U_f has a positive codimension in M_n .

Denote U_f^ϵ the ϵ -neighbourhood of U_f .

THEOREM 2.1 (on an approximation). Let M_n be a compact manifold, $f : M_n \rightarrow \mathbb{R}^1$ be a Morse Min-type function. Let $x_1, \dots, x_l$ be critical points of f and c_i be the index of f at x_i , $i = 1, \dots, l$. Then for any positive ϵ there exists a smooth Morse function h with the same set of critical points (satisfying $\text{index}_{x_i} h = c_i$, $i = 1, \dots, l$), which are ϵ -close to f in C^∞ -topology on $M_n \setminus U_f^\epsilon$ and ϵ -close to f on U_f^ϵ in C^0 -topology.

The approximation by a smooth Morse function can be constructed easily under the assumption that the Riemannian metric on M_n is generic.

Define $h(x, t)$, $x \in M_n, t \in \mathbb{R}_+^1$ by the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta \right) h(x, t) = 0$$

$$h(x, 0) = f_0(x),$$

and denote $f_t(x) = h(x, t)$.

THEOREM 2.2 (approximation using heat flow). Let $f_0 : M_n \rightarrow \mathbb{R}^1$ be a Morse Min-type function. Then for small enough positive ϵ f_ϵ is a C^∞ -smooth Morse function.

For small enough ϵ the function f_ϵ satisfies the conditions of Theorem 1.4 except one. Namely f_ϵ has critical points $\{x_1^\epsilon, \dots, x_l^\epsilon\}$ such that x_i^ϵ is ϵ -close to x_i , $i = 1, \dots, l$ (and index of f_ϵ at x_i^ϵ is c_i), but x_i and x_i^ϵ do not coincide. One can easily correct this using an isotopy of M_n close to the identity.

We present one more result related to heat flows on manifolds with a generic Riemannian metric. K. Uhlenbeck proved, see [U1], [U2], that a residual set of Riemannian metrics on a compact Riemannian manifold satisfies the following properties:

- a) The spectrum of the corresponding Laplace operator Δ is simple;
- b) All eigenfunctions of Δ are Morse.

DEFINITION 2.2 (Uhlenbeck metrics). Let (M_n, ρ) be a compact Riemannian manifold. We call ρ Uhlenbeck metric if the spectrum and the eigenfunctions of the Laplacian Δ on (M_n, ρ) satisfy conditions a and b above.

THEOREM 2.3 (evolution of Morse functions). Let M_n be a compact manifold. Let ρ be an Uhlenbeck metric on M_n . Then almost all (open and everywhere dense subset of) Morse functions $f : M_n \rightarrow \mathbb{R}^1$ satisfy the following conditions:

- 1) There exists a finite set $T = \{t_1, \dots, t_l\} \in \mathbb{R}^1$ such that f_t is Morse for $t \in \mathbb{R}^1 \setminus T$.
- 2) For every $i = 1, \dots, l$ there exists a point $x_i \in M_n$ such that the restriction $f_{t_i}|_{M_n \setminus \{x_i\}}$ is Morse.
- 3) f_{t_i} has a birth-death singularity at x_i .

3. Distance functions on M_n

One can extend Morse inequalities, handlebody decomposition of the manifold corresponding to a Morse function, etc, to the class of Morse Min-type functions. A connection with Riemannian distance functions on non-positively curved manifolds is clarified by the following theorem.

THEOREM 3.1. *Let (M_n, ρ) be a non-positively curved Riemannian manifold. Then every distance function d_p , for $p \in M_n$ is a Min-type function.*

To present a germ of a distance function d_p as a germ of a Min-type function, consider the universal covering

$$\pi : (\tilde{M}_n, \tilde{\rho}) \rightarrow (M_n, \rho),$$

choose $\tilde{q} \in \pi^{-1}(q)$ and denote $\pi^{-1}p = \{\tilde{p}_1, \dots, \tilde{p}_m, \dots\}$. Hereafter we order the points $\tilde{p}_i$ in such a way that

$$\tilde{\rho}(\tilde{p}_i, \tilde{q}) \leq \tilde{\rho}(\tilde{p}_{i+1}, \tilde{q}).$$

LEMMA 3.1. *There exists only a finite number of points $\tilde{p}_i$ such that $\tilde{\rho}(\tilde{p}_i, \tilde{q}) = \rho(p, q)$.*

Denote by m the number of such points. We obtain the following (minimal) representation for a germ of distance function d_p at q

$$d_p(y) = \min \{\tilde{d}_{\tilde{p}_1}(\tilde{y}), \dots, \tilde{d}_{\tilde{p}_m}(\tilde{y})\}$$

for all $\tilde{y}$ in a small neighbourhood of $\tilde{q}$ and $y = \pi(\tilde{y})$, (where $\tilde{d}_{\tilde{p}_i}$ is a distance function for $\tilde{p}_i \in \tilde{M}_n$).

Let us reformulate the definition of non-degenerate points in terms of shortest geodesics $\gamma_1, \dots, \gamma_m$ connecting p and q . (Hereafter we assume that the geodesics are parametrized by arc length.)

PROPOSITION 3.1. *A point q is a non-degenerate point for d_p iff it satisfies the following two conditions:*

- 1) *For any $1 \leq i_1 < \dots < i_{m-1} \leq m$ the vectors $\dot{\gamma}_{i_1}(q), \dots, \dot{\gamma}_{i_{m-1}}(q)$ are linearly independent.*
- 2) *The vectors $\dot{\gamma}_1(q), \dots, \dot{\gamma}_m(q)$ are not in an $(m-2)$ -dimensional affine plane (that is $\{\dot{\gamma}_i(q) - \dot{\gamma}_{i+1}(q)\}$ are linearly independent).*

This allows us to use the concept of index, defined for non-degenerate critical points of Min-type functions, for Riemannian distance functions on non-positively curved manifolds.

DEFINITION 3.1. (Morse distance functions). *Let M_n be a non-positively curved manifold, $p \in M_n$. The distance function d_p , $p \in M_n$ is said to be a Morse distance function if d_p is Morse Min-type function.*

Let us compare Gromov's and the "classical" definition of regular and critical points. The definition in the "classical style" was given above. We recall Gromov's definition now.

DEFINITION 3.2 (Gromov's definition). *Let (M_n, ρ) be a Riemannian manifold, $p, q \in M_n$. The point q is said to be regular for d_p if all shortest geodesics $\gamma_1^{p,q}, \dots, \gamma_m^{p,q}$ connecting p and q satisfy the following property. There exists an open halfspace $T_q^+ M_n \subset T_q M_n$ such that $\dot{\gamma}_1^{p,q}(q), \dots, \dot{\gamma}_m^{p,q}(q) \in T_q^+ M_n$. A point q is said to be critical for d_p if q is not regular.*

DEFINITION 3.3 (multiplicity). *The number $m_p(q)$ of shortest geodesics connecting p and q is said to be p -multiplicity of q .*

THEOREM 3.2. *Let (M_n, ρ) be a Riemannian manifold, d_p is a Morse distance function. Then Gromov's definition of critical points is equivalent to the classical one. The index of a critical point q is $m_p(q) - 1$.*

4. Distance functions on $M_n \times M_n$

Consider a Riemannian metric $\rho : M_n \times M_n \rightarrow \mathbb{R}^1$.

PROPOSITION 4.1. Let M_n be a non-positively curved manifold. Then ρ is a Min-type function on $M_n \times M_n$.

DEFINITION 4.1 (Morse metric). A non-positively curved metric ρ is said to be Morse if ρ is a Morse Min-type function on $M_n \times M_n$.

Let us discuss properties of Morse metric $\rho : M_n \times M_n \rightarrow \mathbb{R}^1$. Let $T_{p,q}(M_n \times M_n)$ denote the tangent space to $M_n \times M_n$ at $(p, q) \in M_n \times M_n$.

PROPOSITION 4.2. A Riemannian metric $\rho : M_n \times M_n \rightarrow \mathbb{R}^1$ is Morse iff for any points $p, q \in M_n$ the shortest geodesics $\gamma_1^{p,q}, \dots, \gamma_m^{p,q}$, connecting p and q satisfy the following conditions.

1) For any $(m-1)$ shortest geodesics $\gamma_{i_1}, \dots, \gamma_{i_{m-1}}$ tangent vectors $\{\dot{\gamma}_{i_j}(p), \dot{\gamma}_{i_j}(q)\}_{j=1}^{m-1} \subset T_{p,q}(M_n \times M_n)$ are linearly independent.

2) Tangent vectors $\{\dot{\gamma}_{i_1}(p), \dot{\gamma}_{i_1}(q)\}_{i_1=1}^m$ do not lie in an $(m-2)$ -dimensional affine plane in $T_{p,q}(M_n \times M_n)$.

Let us present several examples related to Morse metrics on surfaces.

1) Let κ be a periodic geodesic on M_2 such that

$$\kappa(0) = \kappa(2d) = p, \kappa(d) = q, |\dot{\kappa}| \equiv 1, \rho(p, q) = d. \quad (3)$$

Then (p, q) is a critical point of index one for ρ .

2) Let κ be a geodesic loop with a corner at p , satisfying (3). Then (p, q) is a regular point of ρ .

3) Suppose that $p, q \in M_2$ are connected by three shortest geodesics such that p is a non-degenerate critical point for d_q (that is p is a local maximum), and q is a regular point for d_p . Then (p, q) is a regular point for ρ .

4) Let $(p, q) \in M_2 \times M_2$ be a non-degenerate critical point of index three for ρ . Then p and q are connected by four shortest geodesics. Denote these geodesics as $\gamma_1, \dots, \gamma_4$.

PROPOSITION 4.3. Let ρ be a negatively curved Morse metric on M_2 . Let $(p, q) \in M_2 \times M_2$ be a non-degenerate critical point of index three. Then

- a) d_p has a non-degenerate local maximum at q ;
- b) d_q has a non-degenerate local maximum at p ;
- c) (p, q) is not a local maximum for ρ .

This proposition admits the following “more geometric” formulation.

PROPOSITION 4.4. Let (M_2, ρ) be a negatively curved surface. Suppose points $p, q \in M_2$ are connected by four shortest geodesics which are in generic position. Then

a) There exists tangent vectors $v_p \in T_p M_2$ and $v_q \in T_q M_2$ such that

$$\rho \left(\exp(\epsilon v_p), \exp(\epsilon v_q) \right) > \rho(p, q)$$

for small positive ϵ .

b) There is no a tangent vector $w_p \in T_p M_2$ such that

$$\rho \left(\exp(\epsilon w_p), q \right) > \rho(p, q)$$

for a small positive ϵ .

5) Let (p, q) be a non-degenerate critical point of index four for ρ . Then p and q are connected by five shortest geodesics; denote these geodesics as $\gamma_1, \dots, \gamma_5$. Denote $\alpha_i = \dot{\gamma}_i(p)$, $\beta_i = \dot{\gamma}_i(q)$, $\xi_i = \alpha_i \oplus \beta_i$. Then

- a) Any four vectors among $\{\xi_i\}_{i=1}^5$ are linearly independent;
- b) $\{\xi_{i+1} - \xi_i\}_{i=1}^4$ are linearly independent.
- c) *Convex* $\{\gamma_i(p), \gamma_i(q)\}_{i=1}^5 = T_{p,q}(M_2 \times M_2)$.

The condition c) means that for any $v_p \in T_p M_2$, $v_q \in T_q M_2$ there exists a shortest geodesic γ connecting p and q such that

$$\langle \dot{\gamma}(p), v_p \rangle + \langle \dot{\gamma}(q), v_q \rangle > 0. \quad (4)$$

5. l -th distance function

Let us define a sequence of functions $d_p^1, \dots, d_p^j, \dots$ on a negatively curved manifold, such that $d_p^1 = d_p$.

DEFINITION 5.1 (geodesic spectrum). Let M_n be a compact negatively curved manifold, $p, q \in M_n$. Let $\gamma_1^{p,q}, \gamma_2^{p,q}, \dots$, be the sequence of geodesics connecting p, q and ordered with accordance with their lengths $l_i^{p,q} = \text{length } \gamma_i^{p,q}$. We call

$$l_1^{p,q} \leq l_2^{p,q} \leq \dots \leq l_k^{p,q} \leq \dots$$

the geodesic spectrum of M_n at p, q .

REMARK 5.1. The traditional notion of the geodesic spectrum relates to the case $p = q$; the difference between definitions in this case is that our definition takes into account all periodic geodesics and all geodesics self-intersecting at p , in opposite to the classical definition which considers only periodic geodesics.

DEFINITION 5.2 (d_p^j). We define the j -th distance function $d_p^j(q) = l_j^{p,q}$ for $j = 1, \dots, \infty$.

The distance functions d_p^j satisfy the following properties.

PROPOSITION 5.1. Let M_n be a negatively curved manifold, $p \in M_n$. Then

- 1) $d_p^j(q)$ is a continuous function of q for any j ;
- 2) d_p^j depends continuously on p and on the Riemannian metric ρ on M_n .

COROLLARY 5.1 (semi-continuity). Let (M_n, ρ) be a negatively curved Riemannian manifold, $p, q \in M_n$. Then $m_{\bar{\rho}}^j(q) \leq m_{\rho}^j(q)$ for all metrics $\bar{\rho}$ close enough to ρ , and all $(\bar{p}, \bar{q}) \in M_2 \times M_2$ close enough to (p, q) .

PROOF

We have $d_p^{m_j(q)+1}(q) - d_p^j(q) > 0$, then the same inequality is correct for all metrics $\bar{\rho}$ close enough to ρ and for all pairs of points $(\bar{p}, \bar{q})$ close enough to (p, q) . $\odot$

A distance functions d_p^i is not a Min-type functions for $i > 1$. However they also belong to a nice class of non-smooth functions.

DEFINITION 5.3 (Min-Max functions). Let $\{f_{i,j} : M_n \rightarrow \mathbb{R}^1\}$, $i = 1, \dots, l$, $j = 1, \dots, m$ be a two-parametric family of smooth functions. Then

$$f = \max_{1 \leq j \leq m} \min_{1 \leq i \leq l} f_{i,j}$$

is said to be a Min-Max function.

We begin with a local result.

LEMMA 5.1. Let (M_n, ρ) be a non-positively curved manifold. Then for any $p, q \in M_n$ and any integer $i > 0$ the germ of d_p^i at q is a germ of a Min-Max function.

PROOF

- 1) Let $\pi : (\tilde{M}_n, \tilde{\rho}) \rightarrow (M_n, \rho)$ be the universal covering, $\tilde{q} \in \pi^{-1}(q)$, $\{\tilde{p}_1, \dots, \tilde{p}_m, \dots\} = \pi^{-1}(p)$.
- 2) Then for any $y \in M_n$ there exists positive constants K, ϵ such that d_p^i satisfies the following property:

$$d_p^i(y) = \max_{P \in T_{i-1}} \min_{j \leq K, i \notin T_{i-1}} \tilde{d}_j(\tilde{y}),$$

where $y \in D_\epsilon(q)$, T_{i-1} is the set of all subsets of $\{1, \dots, K\}$, consisting of $i-1$ elements. $\odot$

Such representation (for any germ of d_p^i) gives also the global result.

PROPOSITION 5.2. Let (M_n, ρ) be a non-positively curved compact manifold. Then d_p^i is a Max-Min function on M_n .

One can define also i -th distance function, $\rho^i : M_n \times M_n \rightarrow \mathbb{R}^1$.

DEFINITION 5.4 (i -th metric).

$$\rho^i(p, q) = d_p^i(q).$$

PROPOSITION 5.3. Let M_n be a non-positively curved compact manifold. Then ρ^i is a Min-Max function on $M_n \times M_n$.

DEFINITION 5.5 (p_i -multiplicity). Let (M_n, ρ) be a negatively curved manifold, $p, q \in M_n$, $l_1^{p,q} \leq \dots \leq l_m^{p,q} \leq \dots$ be the length spectrum at $(p, q) \in M_n \times M_n$. We denote $m_p^i(q)$ the number of geodesics connecting p and q with length equal to $l_i^{p,q}$.

6. Degenerations (estimates from above)

Let M_n be an n -manifold; degenerations of a distance function d_p at q are connected with a large number (more than $n+1$) of shortest geodesics connecting pairs $p, q \in M_n$; codimension of degeneration grows when the number of “additional” shortest geodesics increases. The upper bound for $m_p(q)$ is determined by the geometric characteristics of M_n . Hence only a finite number of codimensions of degeneration are admitted for Riemannian distance functions, in opposite to the classical Morse theory, where there exists an infinite series of degenerations ordered by codimension of degeneration.

PROPOSITION 6.1. Let (M_n, ρ) be a compact negatively curved manifold, such that $\text{Sect}(M_n) \geq -C$, $r_{inj} \geq r$, and $\text{diam}(M_n) \leq d$. Then

$$m_p(q) \leq \frac{\mu D_{2d}^C}{\mu D_{\frac{r}{2}}^0},$$

where D_{2d}^C is the ball of radius $2d$ in the (simply connected) hyperbolic manifold of the constant sectional curvature equal to $-C$, and $D_{\frac{r}{2}}^0$ is the Euclidean ball of the radius $\frac{r}{2}$.

PROOF

- 1) Let $p, q \in M_n$.

$$\pi : (\tilde{M}_n, \tilde{\rho}) \rightarrow (M_n, \rho)$$

be the universal covering, $\tilde{q} \in \pi^{-1}(q)$, $\tilde{p}_1, \dots, \tilde{p}_m \in \pi^{-1}(p)$ be the points such that $\tilde{\rho}(\tilde{p}_i, \tilde{q}) = \rho(p, q)$, $i = 1, \dots, m$.

- 2) Note that $\rho(p, q) \leq d = \text{diam}(M_n)$ and then $\tilde{p}_1, \dots, \tilde{p}_m \in D_d(\tilde{q})$.

- 3) $\mu D_d(p) \leq \mu D_d^C$, where D_d^C is the d -ball in the (simply connected) hyperbolic manifold with curvature $-C$; $\mu D_r^0 \leq \mu D_r$ ($\tilde{p}_i$), where D_r^0 is the Euclidean ball of radius r .
- 4) The balls $D_{\frac{r}{2}}(\tilde{p}_i)$, $i = 1, \dots, m$ are disjoint and

$$\bigcup_{i=1}^{m_p(q)} D_{\frac{r}{2}}(\tilde{p}_i) \subset D_{d+\frac{r}{2}}(p) \subset D_{2d}(p). \quad (5)$$

5) Comparison of volumes in both hand sides of (5) gives

$$m_p(q) \leq \frac{\mu D_{2d}^C}{\mu D_{\frac{r}{2}}^0},$$

where $D_{\frac{r}{2}}^0 \subset \mathbb{R}^n$ is the Euclidean ball of radius $r/2$. $\odot$

7. Degenerations (estimates from below)

Estimates from below for the number of shortest geodesics were obtained in [H].

THEOREM 7.1 ([H]). *Let (M_n, ρ) be a non-positively curved manifold, $p \in M_n$. Then there exists $q \in M_n$ such that $m_p(q) \geq n + 1$.*

THEOREM 7.2 ([H]). *Let (M_n, ρ) be a negatively curved manifold. Then there exist $p, q \in M_n$ such that $m_p^{\rho}(q) \geq 2n + 1$.*

These theorems are very natural in the context of the theory which we develop in this paper; the first theorem just states that there exist $n + 1$ shortest geodesics, connecting p, q , for a point q where the distance function d_p has a local maximum; this estimate is precise when d_p is Morse, then the index d_p at q is n and there exist precisely $n + 1$ shortest geodesics connecting p and q . When q is a degenerate critical point (a degenerate local maximum) the number of shortest geodesics is greater than $(n + 1)$.

The second theorem can be discussed from two different (but connected) points of view.

REMARK 7.1. *The first point of view is that one has to consider distance functions d_p for all $p \in M_n$. The number of “additional” shortest geodesics, connecting p and q is the the codimension of degeneration at a critical point q . One can disturb slightly the Riemannian metric to obtain the Morse distance function d_p for a given p . For some Riemannian manifolds (such as the torus T^2 with the flat metric) one can disturb the metric $\rho \rightarrow \tilde{\rho}$ in such a way that the distance functions $d_p^{\tilde{\rho}}$ are Morse for all $p \in M_2$.*

For some Riemannian manifolds this is impossible, one can obtain a Morse distance function d_p , by a small perturbation of a Riemannian metric, for a given p but not for all $p \in M_2$ simultaneously. When one considers the distance functions d_x for all $x \in M_n$ one can expect degeneration of codimension n at a finite number of points (under corresponding condition of jet transversality). However, some manifolds could admit smaller degenerations. Theorem 2 states that for any strictly negatively curved manifold there exists at least one point with degeneration of codimension n . Horja, see [H], proved that for any negatively curved n -manifold there exists a pair of points connected by $2n + 1$ shortest geodesics. Such points are local maxima of the distance function $\rho(x, y)$ on $M_n \times M_n$; in particular, this is always the case for the global maximum (p, q) for the metric $\rho : M_n \times M_n$, (where $\rho(p, q) = \text{diam } M_n$).

REMARK 7.2. *The second point of view is that we have to consider a Morse metric $\rho : M_n \times M_n \rightarrow \mathbb{R}^1$. Then ρ has index $2n$ at a local maximum (p, q) and then $2n + 1$ shortest geodesics connecting p and q .*

From this point of view, the explanation of the phenomenon for the torus is that the Morse metric ρ , such that the distance functions d_p are Morse for all $p \in M_2$, is not a Morse metric. The maximum for

$\rho : M_2 \times M_2 \rightarrow \mathbb{R}^1$ is a two-dimensional submanifold in $T^2 \times T^2$ and this case can be analyzed in style of Bott generalization of Morse theory, see [Bo]). (This is a usual phenomenon, compare the function $f = (x + y)^2$ on $\mathbb{R}^2$ which is not Morse, but the restriction of f on any straight line parallel to one of the coordinate axes is Morse).

II. DISTANCE FUNCTIONS on SURFACES

It is not simple to describe location of distance functions inside the space of Min-type functions. In particular, we cannot deduce the statement, that for a given $p \in M_n$ the set of non-positively curved metrics, such that d_p is a Morse distance functions are everywhere dense in the set of non-positively curved metrics, from the corresponding results for Min-type functions. We prove this result for non-positively curved surfaces. We study two dimensional case since ideas here can be presented in a very clear form. The multidimensional case we discuss in forthcoming papers.

We prove the following theorem in this Chapter.

THEOREM II.1. *For a generic non-positively curved metric on a surface M_2 distance functions d_p , $p \in M_2$ are Morse for almost all $p \in M_2$.*

8. Degenerate points on surfaces

Definition 1.8 of non-degenerate points includes two conditions. The following lemma shows that the second condition holds for all distance functions on non-positively curved surfaces and all points with multiplicity ≤ 3 .

LEMMA 8.1 ([EI1]). *Let (M_2, ρ) be a Riemannian surface; $q, x, y, z \in M_2$ are such that $\rho(q, x) = \rho(q, y) = \rho(q, z)$, and $x \neq y$, $y \neq z$, $x \neq z$. Then $(\text{grad } d_x - \text{grad } d_y)$ and $(\text{grad } d_y - \text{grad } d_z)$ are linearly independent at any point $q \in M_n$.*

PROOF ([EI1])

The tangent vectors $\text{grad } d_x(q), \text{grad } d_y(q), \text{grad } d_z(q)$ have unit norm and then cannot belong to a one-dimensional affine space. $\odot$

This gives the following description of degenerate regular points of the distance functions on non-positively curved surfaces.

PROPOSITION 8.1 (degenerate regular points). *Let (M_2, ρ) be a non-positively curved compact surface, $p, q \in M_2$. A point q is a degenerate regular point of d_p iff $m_p(q) \geq 4$ and the shortest geodesics $\gamma_1, \dots, \gamma_{m_p(q)}$ connecting p and q satisfy the following property:*

There exists an open half-space $T_q^+ M_2 \subset T_q M_2$ such that $\dot{\gamma}_1(q), \dots, \dot{\gamma}_{m_p(q)}(q) \in T_q^+ M_2$.

There exist the following two types of degenerate critical points. Denote $\text{Conv}\{\dots\}$ the convex hull.

DEFINITION 8.2 (degenerate critical points with high multiplicity). Let (M_2, ρ) be a non-positively curved compact surface, $p, q \in M_2$. A point q is a degenerate critical point of the **first type** for d_p if $m_p(q) \geq 4$ and the shortest geodesics $\gamma_1, \dots, \gamma_{m_p(q)}$ connecting p and q satisfy the following property:
 $\text{Conv} \{ \dot{\gamma}_1(q), \dots, \dot{\gamma}_{m_p(q)}(q) \} = T_q M_2$.

Degenerate critical points of the second type are critical points with multiplicity three. A point q with $m_p(q) = 3$ is a degenerate critical point if the shortest geodesics $\gamma = \gamma_1, \gamma_2, \gamma_3$ connecting p and q (after a suitable renumeration) satisfy the following property:

$\dot{\gamma}_2(q)$ and $\dot{\gamma}_3(q)$ are linearly dependent; more precisely $-\dot{\gamma}_2(q) = \dot{\gamma}_3(q)$. Then γ_2 and γ_3 are two segments of the same geodesic κ which is either self-intersecting at p or periodic. Denote $d = \rho(p, q)$. Then we have (we omit the subscript for the first geodesic)

$$\kappa(0) = \kappa(2d) = \gamma(0) = p \text{ and } \kappa(d) = \gamma(d) = q, \quad (6)$$

and we obtain the following description.

DEFINITION 8.3 (degenerate critical points with multiplicity three). Let (M_2, ρ) be a non-positively curved compact surface, $p, q \in M_2$. A point q is said to be a degenerate critical point of the **second type** for d_p if $m_p(q) = 3$ and there exist geodesics γ, κ satisfying (6), and such that $\dot{\kappa}(d)$ and $\dot{\gamma}(d)$ are not perpendicular.

REMARK. We include an additional “generic condition that $\dot{\kappa}(d)$ and $\dot{\gamma}(d)$ are not perpendicular. (The set of geodesics which has degenerate critical point of index 3, such that κ and γ are perpendicular at q has codimension 2. The decomposition of such a singularity is more complicate than decomposition for degenerate points of the second type).

9. Generic distance functions on surfaces (multiplicity)

Fix a point $p \in M_2$. We demonstrate that $m_p(q) \leq 3$ for all q , for a generic metric on M_2 .

DEFINITION 9.1 (multiple points). Let (M_2, ρ) be a non-positively curved surface, $p \in M_2$. A point q is said to be p -multiple if $m_p(q) \geq 3$.

PROPOSITION 9.1. Let (M_2, ρ) be a surface with a non-positively curved metric. Then there exists only a finite number of p -multiple points.

PROOF

A point q is p -multiple iff

$$d_p^1(q) = d_p^2(q) = d_p^3(q). \quad (7)$$

that is $\tilde{d}_{\tilde{p}_1}(\tilde{q}) = \tilde{d}_{\tilde{p}_2}(\tilde{q}) = \tilde{d}_{\tilde{p}_3}(\tilde{q})$. In accordance with [E], equations (7) define a finite set of vertices of the Dirichlet domain; (these vertices are p -multiple points on M_2). $\odot$

REMARK. Eberlein’s result holds for any metric ρ without conjugate points on M_2 . However it may happen that $m_p^\rho(q) > 3$ for some points. The rest of this section is devoted to a construction of a small perturbation $\bar{\rho}$ of the metric ρ to obtain $m_{\bar{p}}^{\bar{\rho}}(q) \leq 3$ for all $q \in M_2$. (Each point q with $m_p(q) \geq 4$ can be decomposed into several multiple points $q_1, \dots, q_L$ with $m_p(q_i) = 3$, $i = 1, \dots, L$; (new multiple points arise only due to this splitting).

PROPOSITION 9.2. Let M_2 be a compact surface (of a positive genus); $p \in M_2$, ρ be a non-positively curved generic metric. Then for all $q \in M_n$ we have $m_p(q) \leq 3$.

The proposition follows from the next three lemmas.

Denote $Q^p = \{q_1, \dots, q_l\}$ the set of multiple points for p and the ϵ -neighbourhood of Q^p as Q_ϵ^p .

LEMMA 9.1. *Let (M_2, ρ) be a non-positively curved compact 2-surface, $p \in M_2$. Then any small enough perturbation $\bar{\rho}$ of ρ does not create multiple points outside Q_ϵ^p .*

PROOF

We have $d_p^3 - d_p^1 > \delta(\epsilon) > 0$ on $M_2 \setminus Q_\epsilon$, and then the same inequality is correct for any metric $\bar{\rho}$ close enough to ρ .

Let $\gamma_1^{p,q_1}, \dots, \gamma_{m_p(q_1)}^{p,q_1}$ be the shortest geodesics connecting p and q_i , $i = 1, \dots, l$. Define $\mathcal{G}$ the union of all shortest geodesics connecting p and all q_i , $i = 1, \dots, l$ but γ_1^1 , and $\mathcal{G}_\epsilon$ the ϵ -neighbourhood of $\{\mathcal{G}\}$.

Then $\gamma_1 \not\subset \mathcal{G}_\epsilon$, for small enough ϵ and one can choose $x \in \gamma_1^1 \setminus \mathcal{G}_\epsilon$. Let $\bar{g}$ be a small perturbation of the metric tensor localized in a $D_\epsilon(x)$. More precisely we choose $\bar{g}$ which satisfies the following four conditions:

- 1) $\text{supp}(g - \bar{g}) \subset D_{\frac{\epsilon}{2}}(x)$; (we denote $\text{supp}(g - \bar{g})$ the support of the tensor $(g - \bar{g})$);
- 2) C^∞ -norm of $(g - \bar{g})$ is less than ϵ ;
- 3) $(g - \bar{g})$ -is positively semi-definite at each point;
- 4) $\bar{g}(x) = (1 - \frac{\epsilon}{2}) g(x)$.

⊙

LEMMA 9.2. *The perturbation $g \rightarrow \bar{g}$ does not create new multiple critical points outside a small neighbourhood $U_\epsilon(q_1)$.*

PROOF

- 1) New multiple points do not arise outside Q_ϵ (see Lemma 9.1).
- 2) Any shortest ρ -geodesic connecting p with a point $y \in D_\epsilon(q_j)$, $j \geq 2$ does not intersects $D_\epsilon(x)$ for small enough positive ϵ .
- 3) Then any shortest $\bar{\rho}$ geodesic connecting p and q_i , $i = 2, \dots, l$ (for the metric $\bar{\rho}$ close enough to ρ) does not intersect the ball $D_{\frac{\epsilon}{2}}(x)$ for small enough perturbation $g \rightarrow \bar{g}$. ⊙

In fact we proved also the following result.

LEMMA 9.3. *The p -multiple points $q_2, \dots, q_l$ do not move, disappear, or change their multiplicities under the perturbation $\rho \rightarrow \bar{\rho}$.*

LEMMA 9.4. *The p -multiplicity $m_p^{\bar{\rho}}(y)$ of any point $y \in U_\epsilon(q_1)$ is less or equal to $m_p^\rho(q_1) - 1$.*

PROOF

- 1) For the metric ρ and small enough δ we have

$$d_p^{m_p(q_1)+1}(y) - d_p^1(y) > \delta > 0$$

for all $y \in U_\delta(q)$. Then the $\bar{\rho}$ -multiplicity of any point $y \in U_\delta(q)$ cannot be more than $m_p^\rho(q_1)$.

- 2) $d_{p_1}^{\bar{\rho}}(q_1) > d_{p_2}^{\bar{\rho}}(q_1) = \dots = d_{p_{m_p(q_1)}}^{\bar{\rho}}(q_1)$, hence $m_p^{\bar{\rho}}(q_1) = m_p^\rho(q_1) - 1$.

3) The p -multiple point q_1 is isolated, see [E], then for all q around q_1 we have $m_p^\rho(q) \leq 2$. For any point $q \in U_\delta(q_1) \setminus q_1$ we have $m_p^{\bar{\rho}}(q) \leq m_p^\rho(q) + 1$, since we change only one shortest geodesic. Then $m_p^{\bar{\rho}}(q) \leq 3$ for $q \in U_\delta(q_1) \setminus q_1$. ⊙

REMARKS.

1) The number of non-degenerate critical points, obtained as a result of such splitting, can be roughly estimated; the process of splitting can be presented in the following form: a point q_1 with multiplicity $m_p(q_1)$ can be splitted into not more than $m_p(q_1)$ points of multiplicity $(m_p(q_1) - 1)$, some of these points are splitted into not more than $m_p(q_1) - 1$ points with multiplicities $m_p(q_1) - 2$ and so on.

2) Precise estimates must take into account indexes of non-degenerate multiple points obtained (finally) as a result of splitting.

3) When we start with a degenerate regular point we obtain a collection of non-degenerate regular points, compare [GR]. Generally, we obtain a number of non-degenerate regular points and non-degenerate critical points; some of non-degenerate critical points form pairs of non-degenerate critical points with neighbour indexes; such pairs do not make any contribution in changing of the topology type of $f^{-1}(-\infty, c)$. This leads to the concept of "generalized" index for degenerate critical points and clarify relations between the classical and Gromov's definition for degenerate points.

10. Generic distance functions (degenerate points of multiplicity three)

The results of the previous section allow us to construct a small perturbation of the Riemannian metric such that for a given p all points $q \in M_2$ have p -multiplicities ≤ 3 . To construct a small perturbation of the metric which eliminates degenerate critical points with multiplicity three, we need two simple statements related to self-intersecting and periodic geodesics for generic metrics on non-positively curved manifolds.

PROPOSITION 10.1. *Let M_2 be a surface of genus $g \geq 2$, $p \in M_2$. For a non-positively curved metric ρ on M_2 the set of geodesics self-intersecting at p is discrete (that is finite).*

PROOF

Let $\{\kappa_i\}_{i=1}^\infty$ is an infinite series of geodesics on M_2 self-intersecting at p and converging to κ_∞ . Choose their parametrization in such a way that $\kappa_i(0) = p$ and $|\dot{\kappa}_i| \equiv 1$. Denote d_∞ the length of κ_∞ . Then the point p is self-conjugative along κ_∞ , (that is there exists a non-zero Jacobi field $J(t)$ along $\kappa_\infty(t)$ vanishing for $t = 0$ and for $t = d_\infty$). Since the surface is non-positively curved this is impossible. $\odot$

In a similar way one can prove the following statement.

PROPOSITION 10.2. *For a non-positively curved metric on M_2 the set of periodic geodesics γ such that $\gamma(0) = p$ is discrete.*

LEMMA 10.1. *Let $p, q \in M_2$, $m_p(q) = 3$ and q be a degenerate critical point for d_p^ρ . Then there exists a small perturbation $\rho \rightarrow \bar{\rho}$ of the Riemannian metric such that this degenerate point disappears (and this perturbation does not create new p -multiple points).*

PROOF

1) Since q is a degenerate critical point, there exists a geodesic κ , such that $\kappa(0) = \kappa(2d) = p$, $\kappa(d) = q$ and $|\dot{\kappa}| \equiv 1$; κ is either self-intersecting at p or periodic; there exists also a shortest geodesic γ connecting p and q , such that $\gamma(0) = p$, $\gamma(d) = q$ and $|\dot{\gamma}| \equiv 1$.

2) Let κ_ϵ be an ϵ -neighbourhood of κ . Define $\mathcal{G}_\epsilon = \kappa_\epsilon \cup \bigcup_{j \geq 2} \Gamma_\epsilon^{p, q_j}$; one can choose a point $x \in \gamma \setminus \mathcal{G}_\epsilon$.

3) There are no other geodesics $\kappa, \bar{\kappa}(0) = p$ self-intersecting at p (or periodic) in some neighbourhood of κ .

4) Let us choose a point $x \in \gamma \setminus \mathcal{G}_\epsilon$ and take a C^∞ -small perturbation of the metric tensor $g \rightarrow \bar{g}$ with a support of $(g - \bar{g})$ inside an ϵ -ball $D_\epsilon(x)$; we assume that the perturbation of the metric tensor satisfies all conditions pointed out in lemma 9.2).

5) New multiple points cannot arise on any other geodesic self-intersecting at p . $\odot$

When κ and γ are not orthogonal at q we obtain the following decomposition of the degenerate critical point.

LEMMA 10.2 Let d_p has a non-degenerate critical point of the second type at q . Then there exists a small perturbation of the metric tensor $g \rightarrow \bar{g}$, localized in a small neighbourhood of a point $x \in \gamma$, such that one obtains a non-degenerate critical point q of index one for $d_p^{\bar{g}}$ and a new regular point q_ϵ with $m_p^{\bar{g}}(q_\epsilon) = 2$; (q_ϵ is the intersection of geodesics κ and the perturbed geodesic $\bar{\gamma}$).

11. Proof of the theorem II.1

A simple version of the theorem is the following one.

THEOREM 11.1. Let M_n be a surface, ρ be a non-positively curved metric on M_2 . Then there exists a small perturbation $\rho \rightarrow \bar{\rho}$ such the the set $\mathcal{P} = \{p \in M_2 \mid d_p \text{ is Morse}\}$ is open and everywhere dense in M_2 .

REMARK. When we considered perturbations of the metric tensor $g \rightarrow \bar{g}$ (to make a distance function d_p Morse), these perturbations were concentrate around several points on geodesics connecting p with p -multiple points. It is easy to check that if we fix a small positive δ , we can choose all perturbations of the metric tensor in such a way that $\text{supp}(\bar{g} - g) \subset D_\delta(p)$.

PROOF

Denote the initial metric tensor as g_0 .

- 1) Choose a countable everywhere dense subset $\{x_1, \dots, x_m, \dots\} \subset M_2$ and a small positive ϵ .
- 2) There exists a small a C^∞ -small perturbation of the metric tensor $g_0 \rightarrow g_1$, such that $(\bar{g} - g)$ localises inside $D_\epsilon(x_1)$, which makes d_{x_1} Morse.
- 3) Replace $\epsilon \rightarrow \frac{\epsilon}{2}$ and continue this process in such a way that we make d_{x_2} Morse using ϵ -small C^∞ -perturbation $g_1 \rightarrow g_2$ such that $\text{supp}(g_2 - g_1) \subset D_{\frac{\epsilon}{2}}(x_2)$ and we do not destroy Morse property of d_{x_1} (to conserve Morse properties of d_{x_1} we decrease ϵ if necessary).
- 4) Continue this process, as a result we obtain C^∞ -metric tensor g_∞ such that d_{x_i} are Morse at all x_i . $\odot$

A more interesting version of this theorem is the following.

THEOREM 11.2. Let M_n be a surface, ρ be a non-positively curved metric on M_2 . Then there exists a small perturbation $\rho \rightarrow \bar{\rho}$ such the the set $\mathcal{Q} = \{p \in M_2 \mid d_p \text{ is not Morse}\}$ has codimension one in M_2 .

A proof of this theorem is based on the technique developed in the next Chapter.

III. SINGULARITY THEORY for DISTANCE FUNCTIONS on SURFACES

We discuss two connected problems in this Chapter:

- 1) Metrics ρ with minimal possible degenerations of distance functions d_p at all points $p \in M_2$.

2) Morse metrics on $M_2 \times M_2$.

12. Morse homogeneous metrics on T^2

DEFINITION 12.1 (Morse homogeneous metrics). A metric on M_n is said to be Morse homogeneous if distance functions d_p^ρ are Morse for all $p \in M_n$.

REMARK. When a metric ρ is Morse homogeneous, the distance functions d_p have the same number of critical points for all p in M_n ; denote $\{x_1^p, \dots, x_l^p\}$ the set of critical points for d_p . The numeration of these points can be chosen in such a way that all maps $p \rightarrow x_i^p$ are continuous, and $\text{Ind}_{x_i^p} d_p = \text{Ind}_{x_i^q} d_q$ for all $p, q \in M_n$.

Let us present T^2 as a rectangle $[0, 1] \times [0, 1]$ with identified opposite sides; choose $p = (0, 0)$ and $q = (1/2, 1/2)$. For the standard Riemannian metric on T^2 we have $m_p(q) = 4$. Note that the distance function d_p is Morse on $T^2 \setminus \{q\}$ for the standard flat metric on T^2 . We show that there exists a small perturbation $\tilde{\rho}$ of ρ such that $\tilde{\rho}$ is Morse homogeneous and $\max_{p, q \in T^2} m_p(q) = 3$.

Consider the universal covering $(\mathbb{R}^2, \tilde{\rho}) \rightarrow (T^2, \rho)$, where $\tilde{\rho}$ is the Euclidean metric on $\mathbb{R}^2$. Choose an orthogonal basis $\xi_1 = (1, 1)$ and $\xi_2 = (-1, 1)$ of $\mathbb{R}^2$. Choose a small positive δ and perturb the metric $\tilde{\rho} \rightarrow \tilde{\rho}^\delta$ on $\mathbb{R}^2$ in such a way that (ξ_1^δ, ξ_2) , where $\xi_1^\delta = (1 - \delta)\xi_1$ is an orthonormal basis on $\mathbb{R}^2$.

PROPOSITION 12.1. ρ^δ is a Morse homogeneous metric on T^2 and $\max_{p, q \in T^2} m_p(q) = 3$.

PROOF

The function d_p on T^2 has six critical points. The point o has index 0; Points $a = (0, 1/2), b = (1/2, 0), c = (1/2, 1/2)$ are non-degenerate critical points with indexes 1. There are two non-degenerate critical points with indexes two $d = ((1 - \delta)/2, (1 - \delta)/2)$ and $e = ((1 + \delta)/2, (1 + \delta)/2)$.

All distance functions d_q has six non-degenerate critical points (obtained by a shift by $q - p$ from the corresponding critical points of the function d_p). $\odot$

13. Generic metrics on negatively curved surfaces

Properties of generic metrics negatively curved manifolds are described by the following statements.

PROPOSITION 13.1. Let M_2 be a surface of genus ≥ 2 . Then $\left(\max_{p \in M_2} m_p(q)\right) = 5$ for a generic negatively curved metric on M_2 .

COMMENTS. We demonstrate in following sections that a generic negatively curved metric is Morse (as a function on $M_2 \times M_2$). Points of local maxima of this function have index 4 and then are connected by five shortest geodesics.

PROOF of the Proposition

The proof is similar to the proof of corresponding results in Chapter II. We present a scheme of the proof.

1) Points (p, q) with $m_p(q) \geq 5$ are isolated in $M_2 \times M_2$, see [H].

2) Some of these points can have multiplicity ≥ 6 . Let $m_p(q_1) \geq 6$ and $\gamma_1^1, \dots, \gamma_{m_p(q_1)}^1$ be the shortest geodesics connecting p and q_1 .

3) Consider all other points $q_2, \dots, q_l$ with $m_p(q_j) \geq 5$ and all shortest geodesics $\gamma_1^j, \dots, \gamma_{m_p(q_j)}^j, j = 2, \dots, l$ connecting points q_j with p . Take the union of all these geodesics but γ_1^1 as $\mathcal{G}$ and denote the ϵ -neighbourhood of $\mathcal{G}$ as $\mathcal{G}_\epsilon$.

4) Choose a point $x \in \gamma_1^1 \setminus \mathcal{G}_\epsilon$. Then a small perturbation of the metric tensor inside $D_\epsilon(x)$ decreases the p -multiplicity at q_1 . $\odot$

Horja, [H], proved that points $p, q \in M_2$ such that (p, q) is a local maximum of the metric $\rho : M_2 \times M_2 \rightarrow \mathbb{R}^1$ are connected by at least 5 shortest geodesics. The previous theorem shows that for a generic metric such points are connected by exactly five shortest geodesics.

For saddle points of ρ we obtain the following statements, see section 4.

PROPOSITION 13.2 (index four). *Let M_2 be a compact surface, ρ be a Morse negatively curved metric on M_2 . Then there exist points $p, q \in M_2$ connected by four shortest geodesics and such that p is a local maximum for d_q and q is a local maximum for d_p (but (p, q) is not a local maximum for ρ).*

In the same way one can prove the following result.

PROPOSITION 13.3 (index three). *Let M_2 be a compact surface, ρ be a Morse negatively curved metric on M_2 . Then there exist points $p, q \in M_2$ connected by three shortest geodesics and such that p is a local maximum for d_q and q is a local maximum for d_p (but (p, q) is not a local maximum for ρ).*

These propositions are based on the fact that a generic negatively curved metric is Morse.

THEOREM 13.1. *The set of Morse negatively curved metrics on M_2 is open and everywhere dense in the space of all negatively curved metrics on M_2 .*

CONJECTURE. *The same result is correct for any almost non-positively curved metric, in particular for small perturbations of the flat metric on T^2 .*

REMARK. *The author believe that this result is correct, in fact, in far more general situation, namely, for an arbitrary dimension and without any restrictions on the sectional curvature.*

PROOF of the Proposition

1) There exists a critical point (p, q) of index three for any Morse function on $M_2 \times M_2$ (where the genus of M_2 is positive).

2) There exists four geodesics $\gamma_1, \dots, \gamma_4$ connecting p and q ; denote them $\gamma_1, \dots, \gamma_4$.

3) We have

$$\text{Convex } \{\dot{\gamma}_1(p), \dots, \dot{\gamma}_4(p)\} = T_p M_2;$$

and

$$\text{Convex } \{\dot{\gamma}_1(q), \dots, \dot{\gamma}_4(q)\} = T_q M_2.$$

Then p is a local maximum for d_q , and q is a local maximum for d_p .

4) (p, q) is not a local maximum for ρ since the index of (p, q) is three. $\odot$

Theorem 13.1 states that there are no points of multiplicities ≥ 6 for generic metrics. There are no generate points of multiplicity two. To obtain the necessary result we have to prove following three statements.

LEMMA 13.1. *There are no degenerate points of multiplicity three for generic metrics.*

LEMMA 13.2. *There are no degenerate points of multiplicity four for generic metrics.*

LEMMA 13.3. *There are no degenerate points of multiplicity five for generic metrics.*

The rest of the paper is devoted to a proof of these statements.

14. Points of multiplicity three

PROOF of LEMMA 13.1

Let $\gamma_1, \gamma_2, \gamma_3$ are the shortest geodesics connecting p and q .

1) Only possible degeneration for a point of multiplicity three is linear dependence of the tangent vectors $(\dot{\gamma}_1(p), \dot{\gamma}_1(q))$ and $(\dot{\gamma}_2(p), \dot{\gamma}_2(q))$ in $T_{p,q}(M_2 \times M_2)$ (after a suitable renumeration of geodesics). The second type of degeneration is impossible due to theorem 8.1.

2) Then there exists a periodic geodesics κ (consisting on two segments γ_1 and γ_2) such that.

$$\kappa(0) = \kappa(2d) = p, \kappa(d) = q, |\dot{\kappa}| = 1$$

and the shortest geodesic γ connecting p and q

$$\gamma(0) = p, \gamma(d) = q, |\dot{\gamma}| = 1.$$

3) The periodic geodesic κ is isolated (there exists only a finite number of periodic geodesics on a negatively curved manifold whose lengths are bounded from above by a given constant (in particular, with the length less or equal to the diameter of M_2) is finite. Then C^∞ -small perturbation of the metric tensor inside a small ball around a point $x \in \gamma \setminus \kappa$ eliminates this degenerate point and do not create new degenerate points of multiplicity three. $\odot$

We prove several statements related to points of multiplicity three which are interesting by themselves and useful in our following struggle with degenerate critical points of multiplicity four in the next section.

DEFINITION 14.1 ($\mathcal{P}_3$). Define the set $\mathcal{P}_3$ of points $(p, q) \in M_2 \times M_2$ such that p and q are connected by three shortest geodesics $\gamma_1^{p,q}, \gamma_2^{p,q}, \gamma_3^{p,q}$, which satisfy the following conditions:

- 1) any two vectors among $\dot{\gamma}_1^{p,q}(p), \dot{\gamma}_2^{p,q}(p), \dot{\gamma}_3^{p,q}(p)$ are linearly independent;
- 2) any two vectors among $\dot{\gamma}_1^{p,q}(q), \dot{\gamma}_2^{p,q}(q), \dot{\gamma}_3^{p,q}(q)$ are linearly independent;

PROPOSITION 14.1. $\mathcal{P}_3$ is a two dimensional submanifold in $M_2 \times M_2$.

PROOF

1) Let $(p, q) \in \mathcal{P}_3$; we show that for any x close enough to p there exists y close to q such that x and y are connected by three shortest geodesics.

2) Choose $\tilde{q} \in \pi^{-1}(q)$; let $\pi^{-1}p = \{\tilde{p}_1, \tilde{p}_2, \dots\}$; perturb slightly $p \rightarrow o$. For r close enough to q choose $\tilde{r} \in \pi^{-1}r$ close to $\tilde{q}$, then we have $\pi^{-1}(o) = \{\tilde{o}_1, \dots, \tilde{o}_m, \dots\}$, where $\tilde{o}_i$ is close to $\tilde{p}_i$ for $i = 1, 2, 3$.

3) Denote $\Omega(\tilde{p}_1, \tilde{p}_2) \subset \tilde{M}_2$ the equidistant set of points, (such that for any $\tilde{z} \in \Omega(\tilde{p}_1, \tilde{p}_2)$ one has $\tilde{\rho}(\tilde{z}, \tilde{p}_1) = \tilde{\rho}(\tilde{z}, \tilde{p}_2)$).

4) $\Omega(\tilde{p}_1, \tilde{p}_2)$ and $\Omega(\tilde{p}_2, \tilde{p}_3)$ have a transversal intersection at $\tilde{q}$, hence the sets $\Omega(\tilde{o}_1, \tilde{o}_2)$ and $\Omega(\tilde{o}_2, \tilde{o}_3)$ has a transversal intersection at a unique point $\tilde{r}$ near $\tilde{q}$.

This gives a map $U_\epsilon(p) \rightarrow U_\epsilon(q)$; the graph of this map gives a germ of $\mathcal{P}_3$ at (p, q) . $\odot$

DEFINITION 14.2. We separate four types of points in $\mathcal{P}_3$:

- 1) $\mathcal{P}_3^{cc} = \{(p, q) \in \mathcal{P}_3 \mid p \text{ is critical for } q, \text{ and } q \text{ is critical for } p\}$;
- 2) $\mathcal{P}_3^{cr} = \{(p, q) \in \mathcal{P}_3 \mid p \text{ is critical for } q, \text{ and } q \text{ is regular for } p\}$;
- 3) $\mathcal{P}_3^{rc} = \{(p, q) \in \mathcal{P}_3 \mid p \text{ is regular for } q, \text{ and } q \text{ is critical for } p\}$;
- 4) $\mathcal{P}_3^{rr} = \{(p, q) \in \mathcal{P}_3 \mid p \text{ is regular for } q, \text{ and } q \text{ is regular for } p\}$.

PROPOSITION 14.2. Each of four sets $\mathcal{P}_3^{cc}, \mathcal{P}_3^{cr}, \mathcal{P}_3^{rc}, \mathcal{P}_3^{rr}$ consists of several connected components of $\mathcal{P}_3$.

PROOF

We prove this for $\mathcal{P}_3^{ss}$, for the other sets the proof is similar.

Let $(p, q) \in \mathcal{P}_3^{sc}$ then q be a local maximum for d_p ; hence for $\bar{p}$ close enough to p , the function $d_{\bar{p}}$ has a unique non-degenerate local maximum $\bar{q}$ in a small neighbourhood of q , and visa versa. Then any $(\bar{p}, \bar{q})$ close enough to (p, q) is in $\mathcal{P}_3^{sc}$. $\odot$

For Morse homogeneous metrics we obtain the following result.

PROPOSITION 14.3. *Let ρ be a Morse homogeneous metric on M_2 . Then $\mathcal{P}_3^{ss} \subset M_2 \times M_2$ is a closed submanifold, and each of the projection $(p, q) \rightarrow p$ and $(p, q) \rightarrow q$ defines a covering $\mathcal{P}_3^{ss} \rightarrow M_2$.*

Critical points of ρ of multiplicity three are inside $\mathcal{P}_3^{sc}$; let us describe such points. Let $(p, q) \in \mathcal{P}_3^{ss}$; denote $\gamma_1^{p,q}, \gamma_2^{p,q}, \gamma_3^{p,q}$ the shortest geodesics connecting p and q . Consider a map $\Phi : \mathcal{P}_3^{ss} \rightarrow T^2$ defined as follows. Denote angles on T^2 as $\phi_{1,2}$ and $\phi_{2,3}$; define

$$\phi_{i,i+1}(p) = \text{angle} \left(\dot{\gamma}_i(p), \dot{\gamma}_{i+1}(p) \right), \quad \phi_{i,i+1}(q) = \text{angle} \left(\dot{\gamma}_i(q), \dot{\gamma}_{i+1}(q) \right), \quad \phi_{i,i+1} = \phi_{i,i+1}(p) - \phi_{i,i+1}(q)$$

for $i = 1, 2$. (The angles are assumed to be oriented and their orientation is defined by the orientation of the surface M_2). Finally, define $\Phi(p, q) = (\phi_{1,2}, \phi_{2,3})$. Denote $O = (0, 0)$. Comparison with the definition of critical points gives the following result.

LEMMA 14.1. *The points $(p, q) \in \mathcal{P}_3^{ss}$ with $\Phi(p, q) = O$ are non-degenerate critical points of index two for the metric ρ .*

15. Technical lemmas

We need several technical lemmas which we use in our study of points of multiplicity four and five in the next two sections.

LEMMA 15.1. *Let M_n be a negatively curved manifold. Suppose there exist two curves $\alpha, \beta : (-\delta, \delta) \rightarrow M_n$ such that for each τ the points $\alpha(\tau), \beta(\tau)$ are connected by exactly m shortest geodesics. Then such geodesics form m continuous families, that is, there exist m continuous maps $\Gamma_i : (-\delta, \delta) \times [0, d] \rightarrow M_2$, $i = 1, \dots, m$ such that for each $\tau \in (-\delta, \delta)$ the curve $\gamma_i(t) = \Gamma_i(\tau, t)$ is a shortest geodesic connecting $\alpha(\tau)$ and $\beta(\tau)$.*

PROOF

Consider the universal covering $\pi : (\tilde{M}_n, \tilde{\rho}_n) \rightarrow (M_n, \rho_n)$, $\tilde{q}^\tau \in \pi^{-1}(\beta(\tau))$ and $\{\tilde{p}_1^\tau, \dots, \tilde{p}_k^\tau, \dots\} = \pi^{-1}(\alpha(\tau))$. Any pair of points $\alpha(\tau), \beta(\tau)$ are connected by exactly m shortest geodesics, hence the maps $\tau \rightarrow \tilde{p}_i^\tau$ can be chosen in such a way that they define continuous curves for $i = 1, \dots, m$. (This defines continuous families Γ_i , for $i = 1, \dots, m$). $\odot$

We recall Caratheodory theorem

THEOREM 15.1 (Caratheodory). *Let $a_1, \dots, a_m \in \mathbb{R}^n$, $n < m$ are such that $\text{Convex} \{a_1, \dots, a_m\} \ni O$. Then there exist $a_{i_1}, \dots, a_{i_{n+1}}$ such that*

$$\text{Convex} \{a_{i_1}, \dots, a_{i_{n+1}}\} \ni O.$$

We need two evident lemmas in similar style.

LEMMA 15.2. *Let L be a linear space, $\nu_1, \dots, \nu_m \in L$ satisfy the following two conditions:*

- a) $\nu_1, \dots, \nu_{m-1}$ are linearly dependent;
- b) $\text{Convex}(\nu_1, \dots, \nu_m) \ni O$;

Then there exist $(m - 1)$ vectors $\nu_{i_1}, \dots, \nu_{i_{m-1}}$ such that

$$\text{Convex } \{\nu_{i_1}, \dots, \nu_{i_{m-1}}\} \ni O.$$

LEMMA 15.3. Let L be a linear space; $\nu_1, \dots, \nu_m \in L$ are on an affine hyperplane and

$$\text{Convex } \{\nu_1, \dots, \nu_m\} \ni O.$$

Then there exist $\nu_{i_1}, \dots, \nu_{i_{m-1}}$ which are linearly dependent.

16. Points of multiplicity four

Denote $\mathcal{P}_4$ the set of points $(p, q) \in M_2 \times M_2$ such that p and q are connected by (at least) four shortest geodesics.

There are two types of degeneration for critical points of multiplicity four, see section 4.

a) A point with the first type of degeneration is a point $(p, q) \in M_2 \times M_2$ where for three among four shortest geodesics connecting p and q (denote these geodesics $\gamma_1, \gamma_2, \gamma_3$) vectors

$$\begin{pmatrix} \dot{\gamma}_1(p), \dot{\gamma}_1(q) \end{pmatrix} \begin{pmatrix} \dot{\gamma}_2(p), \dot{\gamma}_2(q) \end{pmatrix} \begin{pmatrix} \dot{\gamma}_3(p), \dot{\gamma}_3(q) \end{pmatrix} \in T_{p,q}(M_2 \times M_2)$$

are linearly dependent.

b) A point with the second type of degeneration is a point (p, q) where vectors

$$\begin{pmatrix} \dot{\gamma}_1(p), \dot{\gamma}_1(q) \end{pmatrix} \begin{pmatrix} \dot{\gamma}_2(p), \dot{\gamma}_2(q) \end{pmatrix} \begin{pmatrix} \dot{\gamma}_3(p), \dot{\gamma}_3(q) \end{pmatrix} \begin{pmatrix} \dot{\gamma}_4(p), \dot{\gamma}_4(q) \end{pmatrix}$$

are in a two dimensional affine plane in $T_{p,q}(M_2 \times M_2)$.

Similar to the section 14 one can prove the following result.

LEMMA 16.1. Points $(p, q) \in \mathcal{P}_4$ such that

a) any two vectors $\{\dot{\gamma}_i(p), \dot{\gamma}_i(q)\}, \{\dot{\gamma}_j(p), \dot{\gamma}_j(q)\} \in T_{p,q}(M_2 \times M_2)$ are linearly independent.

b) $\text{Convex } \{\dot{\gamma}_i(p), \dot{\gamma}_i(q)\}_{i=1}^3 \ni O$

are isolated.

Such a degenerate critical point can be eliminated by a small perturbation of a riemannian metric tensor inside a small ball centered at a point of forth shortest geodesic connecting p and q . Simple exercises in linear algebra, presented in the previous section allows to obtain the necessary result for all degenerate critical points of multiplicity four. Lemma 15.2 gives the following result.

LEMMA 16.2. Let (p, q) be a degenerate critical point of the first type of multiplicity four. Then either one can numerate shortest geodesics connecting (p, q) in such a way that they satisfy the condition of Lemma 16.1 or two of these geodesics are segments of a periodic geodesic.

PROPOSITION 16.1. Let (M_2, ρ) be a negatively curved surface such that there are no degenerate critical points for ρ with multiplicity three. Then there exists a small perturbation $\rho \rightarrow \bar{\rho}$ which eliminates all degenerate critical points of the first type with multiplicity four.

PROOF

1) If two of shortest geodesics are segments of a periodic geodesic, one can decrease the number of shortest geodesics connecting p and q as in section 14.

2) If the shortest geodesics satisfy the condition of Lemma 16.1, it is sufficient to take a small perturbation of the metric tensor at a point of the forth geodesic. $\odot$

REMARK. A perturbation in the previous statement can be chosen so small that it does not create new degenerate critical points with multiplicity three and four.

The second type of degenerations connects with critical points where

$$\{\dot{\gamma}_i^{p,q}(p), \dot{\gamma}_i^{p,q}(q)\} - \{\dot{\gamma}_4^{p,q}(p), \dot{\gamma}_4^{p,q}(q)\}, i = 1, 2, 3$$

are linearly dependent.

The lemma (which follows from Lemma 15.3) shows that this case can be reduced, in fact, to the previous one.

LEMMA 16.3. Let (p, q) be a degenerate critical point of the second type of multiplicity four. Then one can numerate shortest geodesics connecting (p, q) in such a way that they satisfy the condition of Lemma 16.1.

17. Points of multiplicity five

PROPOSITION 17.1. Let (M_2, ρ) be a negatively curved surface. Suppose $\rho : M_2 \times M_2$ has not degenerate critical points with multiplicity $\neq 5$. Then there exists a small perturbation of the metric tensor (localized around a finite number of points of M_2) which eliminates all degenerate critical points of multiplicity five (and does not create new multiple points).

The scheme of the proof is the same as in the previous section.

REMARK. The proof for degenerate critical points of multiplicity five can be simplified if one uses results of [H], which state that points connected by five shortest geodesics are isolated. The presented scheme works in fact in more general situation.

18. Concluding remarks

We discuss possible generalizations of this theory and its applications.

More precisely one can discuss generalizations and applications of two connected theories presented in this paper.

I.1 Morse theory and singularity theory for Min-type functions

Natural classes of non-smooth functions arise in “smooth” problems of optimization, calculus of variations, mathematical economics and so on. Some (but not all) classes are connected with operations of Maximum and Minimum (and their superpositions). The Morse theory for Min-type functions which can be obtained as minima of finite families of smooth functions was constructed in [GR].

Stable singularities for functions obtained by Max and Min operations from continuous families of smooth functions were classified in [Br1], [Br2], [Ma1], [Ma2]. (The Morse theory was not constructed for this class, but it can be done easily).

Singularity theory for the class of Max-Min functions (which arose in this paper) is far more difficult, see a discussion in [CGR1], but it seems can be constructed.

Applications of these results are in non-smooth optimization, [CGR1], mathematical economics in S. Smale style, variational problems, see [A].

1.2 Morse theory for Riemannian distance function

This theory must be (and it seems can be) generalized to general Riemannian manifolds; it would be interesting to extend this theory also for some other categories of manifolds, in the first turn, to sub-Riemannian manifolds.

The most natural and clear applications relate to classification of stable singularities of spheres (and geodesic spheres) in Riemannian manifolds (when the the radius of a sphere is greater than the injectivity radius at the center) and singularities of sub-Riemannian spheres (with all positive radii, the injectivity radius here is zero at all points, see [Ge1]).

In sub-Riemannian geometry this theory can also be useful in some open problems such the problem of contractibility of small balls in sub-Riemannian manifolds.

More problematic but potentially very interesting is to use this theory for a deeper understanding of the difference between positively and non-negatively curved manifolds (this includes a number of problems such as Hopf's problem of existence of a positively curved metric on $S^2 \times S^2$, and Gromov's conjecture that the maximum of sums of Betti numbers for non-negatively curved n -manifolds is reached on the flat n -torus (the last problem is pointed out by M. Gromov)).

The following conjecture can clarify connection of these problems with Morse theory for Riemannian distance functions.

Gromov proved, [Gr], that there exists a constant c_n depending only on dimension, such that the sum of Betti numbers for any positively curved n -manifold M_n is bounded from above by c_n . For such a manifold denote $C(M_n)$ the minimal possible number of critical points of a Morse distance function on M_n for positively curved Riemannian metric.

If there exists C_n such that $C(M_n) \leq C_n$ (for all M_n admitting a positively curved metric).

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