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UNIVERSITÄT LEIPZIG

Naturwissenschaftlich-Theoretisches

Zentrum

Quasi-complete intersections of
monomial curves in projective three-space

by

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Abstract:

We classify completely all quasi-complete intersections of monomial curves in $\mathbb{P}_K^3$, K an infinite field, see Theorem 4.1 and Theorem 4.2. This completes the investigations started in [B,Sch,St].

1. Introduction and preliminaries

The subjectmatter under discussion are quasi-complete intersections on three surfaces (q.c.i.) of monomial curves $C(n_1, n_2, n_3)$ in $\mathbb{P}_K^3$, K an infinite field, with generic zero $(t_0^{n_3}, t_0^{n_3-n_1}t_1^{n_1}, t_0^{n_3-n_2}t_1^{n_2}, t_1^{n_3})$, $n_1 < n_2 < n_3$ positive integers with $\text{g.c.d.}(n_1, n_2, n_3) = 1$. $\mathfrak{p}(n_1, n_2, n_3) \subseteq K[x_0, \dots, x_3] =: R$ will denote the prime ideal of $C(n_1, n_2, n_3)$, that is the kernel of the substitution homomorphism $\varphi : K[x_0, \dots, x_3] \longrightarrow K[t_0, t_1]$, $x_i \xrightarrow{\varphi} t_0^{n_3-n_i}t_1^{n_i}$, $0 \leq i \leq 3$, $n_0 := 0$.

There are several equivalent definitions for $\mathfrak{p}(n_1, n_2, n_3)$ to be a q.c.i. on $\{F_1, F_2, F_3\}$, F_i homogeneous polynomials, $1 \leq i \leq 3$ (see [St], [P,S], [B,Sch,St]). The definition we will use is:

Definition 1.1. $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{F_1, F_2, F_3\}$ if $\mathfrak{p}(n_1, n_2, n_3)|_{x_i=1} = (F_1|_{x_i=1}, F_2|_{x_i=1}, F_3|_{x_i=1})R_i$, $0 \leq i \leq 3$. Here $R_i := K[x_0, \dots, \hat{x}_i, \dots, x_3]$ and the image of a set or an element under the substitution homomorphism $R \longrightarrow R_i$ given by $x_j \longmapsto x_j$, $j \neq i$, $x_i \longmapsto 1$, $0 \leq i \leq 3$, is denoted by placing $|_{x_i=1}$ behind the set or element.

We list next some prerequisites, needed for an understanding of the proofs and results. Important to our investigation is the algorithm in [B,R] (see also [B,C,F,H]), which obtains a minimal generating set $\mathcal{B}$ of binomials for $\mathfrak{p}(n_1, n_2, n_3)$. It starts from a minimal generating set of binomials for $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$, which is either two or three in

number [H]. The distinction between $\mu(\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1})=2$ and $\mu(\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1})=3$ (μ denotes the minimum number of generators) can be arithmetically interpreted by considering the so called symmetric property of the numerical semigroup $\langle n_1, n_2, n_3 \rangle := \{z | z := \sum_{i=1}^3 z_i n_i, z_i \text{ nonnegative integers}\}$. There are again several equivalent definitions for a numerical semigroup $\langle n_1, n_2, n_3 \rangle$ to be symmetric. Our definition here is:

Definition 1.2. Let $\{i, j, h\} := \{1, 2, 3\}$. $\langle n_1, n_2, n_3 \rangle$ is (n_i, n_j) -symmetric if $n_i = q_i d$, $n_j = q_j d$, $d := \text{g.c.d.}(n_i, n_j)$ and $n_h = \alpha_{hi} q_i + \alpha_{hj} q_j \in \langle q_i, q_j \rangle := \{z | z = z_i q_i + z_j q_j, z_i \text{ and } z_j \text{ nonnegative integers}\}$. If no specific identification with the integers n_i, n_j is needed, then $\langle n_1, n_2, n_3 \rangle$ is said to be symmetric, if it is (n_i, n_j) -symmetric for some $i, j \in \{1, 2, 3\}$, $i \neq j$. If $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is symmetric, then we say that $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are symmetric of the same type, if for $j = 2$ $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is $(n_3 - n_2, n_3 - n_1)$ -symmetric, or for $j = 3$ $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is $(n_3 - n_2, n_3)$ -symmetric.

It follows from [H], that $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ is minimally generated by $G_1 := \{g_{ij} := x_j^{q_i} - x_i^{q_j}, g_h := x_h^d - x_i^{\alpha_{hi}} x_j^{\alpha_{hj}}\}$ if $\langle n_1, n_2, n_3 \rangle$ is (n_i, n_j) -symmetric and $G_2 := \{f_i := x_i^{\alpha_i} - x_j^{\alpha_{ij}} x_h^{\alpha_{ih}}, f_j := x_j^{\alpha_j} - x_i^{\alpha_{ji}} x_h^{\alpha_{jh}}, f_h := x_h^{\alpha_h} - x_i^{\alpha_{hi}} x_j^{\alpha_{hj}}\}$, $\alpha_i, \alpha_j, \alpha_h$ minimal and all exponents positive, if $\langle n_1, n_2, n_3 \rangle$ is not symmetric. We will refer to g_{ij}, g_h , respectively f_i, f_j, f_h , as the canonical binomial generators of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$. They are uniquely determined, up to multiplication by -1 , provided for $n_i < n_j$, $\alpha_{hi} < q_j$ in G_1 , which we will assume from now on.

The algorithm in [B,R] now starts with either the homogenized binomials in G_1 or G_2 and produces a minimal generating set $\mathcal{B}$ of $\mathfrak{p}(n_1, n_2, n_3)$ as follows.

Case 1. $\langle n_1, n_2, n_3 \rangle$ is (n_i, n_j) -symmetric

a) $\{i, j\} = \{2, 3\}$. In this case $\mathfrak{p}(n_1, n_2, n_3) = (x_0^{q_3 - q_2} x_3^{q_2} - x_2^{q_3}, x_1^d - x_0^{\alpha_{10}} x_2^{\alpha_{12}} x_3^{\alpha_{13}})R$, $d := \text{g.c.d.}(n_2, n_3)$.

b) $\{i, j\} \neq \{2, 3\}$. W.l.o.g. assume $n_i = n_1$. We obtain a sequence of binomials as follows. We start with $B_{j1} := x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} =: m_0 - m_1$, $B_{h1} := x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} =: m'_0 - m'_1$, $d := \text{g.c.d.}(n_1, n_j)$, $n_1 = q_1 d$, $n_j = q_j d$, $\alpha_{h1} < q_j$. If $\alpha_{h0} = 0$, then $h = 2$, if $\alpha_{h1} = 0$, then $j = 2$. In both cases $\mathfrak{p}(n_1, n_2, n_3) = (B_{j1}, B_{h1})R$. Otherwise we cross multiply the mono-

omial terms of B_{j_1} and B_{h_1} and cancel in the resulting binomial $m_0 m'_1 - m'_0 m_1$ the highest common monomial term in x_0 and x_1 . We obtain a binomial B_{j_2} (with lower x_1 -exponent than in B_{j_1}) in $\mathfrak{p}(n_1, n_2, n_3)$. We also will say B_{h_1} acts on B_{j_1} to produce B_{j_2} . If B_{j_2} has a pure power in x_2 as a monomial term, then the process stops. If this is not the case and the x_1 -exponents of B_{j_2} and B_{h_1} are not equal, then B_{h_1} acts on B_{j_2} to produce B_{j_3} (in case the x_1 -exponent in B_{h_1} is smaller than the x_1 -exponent in B_{j_2}) or B_{j_2} acts on B_{h_1} to produce B_{h_2} . In case of equal x_1 -exponents in B_{j_2} and B_{h_1} , the above process of cross multiplying the monomial terms of B_{j_2} and B_{h_1} and cancelling the highest common monomial term in x_0 and x_1 , results in a binomial with a pure power term in x_2 in $\mathfrak{p}(n_1, n_2, n_3)$. We call this the binomial B_{j_3} for $j=2$ and B_{h_2} for $h=2$ and the algorithm terminates. Iterating this procedure results in two well defined sequences $B_{j_1}, \dots, B_{j_{r_0}}$ and $B_{h_1}, \dots, B_{h_{s_0}}$. We set $\{B_{j_1}, \dots, B_{j_{r_0}}\} =: \mathcal{B}(j)$ and $\{B_{h_1}, \dots, B_{h_{s_0}}\} =: \mathcal{B}(h)$. The binomial with the pure power in x_2 , which ends the algorithm, is the binomial in $\mathcal{B}(2)$ with largest second subscript. Now $\mathcal{B} = \mathcal{B}(j) \cup \mathcal{B}(h)$.

Case 2. $\langle n_1, n_2, n_3 \rangle$ is not symmetric.

a) Assume that the homogenized binomials B_{11}, B_{21}, B_{31} in G_2 are $B_{11} := x_1^{\alpha_1} - x_0^{\alpha_{10}} x_2^{\alpha_{12}} x_3^{\alpha_{13}}, B_{21} := x_2^{\alpha_2} - x_0^{\alpha_{20}} x_1^{\alpha_{21}} x_3^{\alpha_{23}}, B_{31} := x_0^{\alpha_{30}} x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}$. Then $\mathfrak{p}(n_1, n_2, n_3)$ is minimally generated by these elements.

b) Not a). Then $B_{21} = x_0^{\alpha_{20}} x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}$ and with B_{21} and $B_{31} = x_0^{\alpha_{30}} x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}$ we proceed as in Case 1. to obtain two sets $\{B_{21}, \dots, B_{2_{r_0}}\} =: \mathcal{B}(2), \{B_{31}, \dots, B_{3_{s_0}}\} =: \mathcal{B}(3)$ with $B_{2_{r_0}}$ the binomial with a pure power in x_2 ending the algorithm. Then $\mathcal{B} = \{B_{11}\} \cup \mathcal{B}(2) \cup \mathcal{B}(3)$ is a minimal generating set of $\mathfrak{p}(n_1, n_2, n_3)$.

If in the above $B_{2_{r_0}}$ is the binomial with the pure power in x_2 , then $B_{3_{s_0}}$ and $B_{2_{r_0}}$ are said to be the last two elements of the algorithm, $B_{2_{r_0-1}}, B_{3_{s_0}}, B_{2_{r_0}}$ the last three elements. An analogous statement is obtained if $B_{3_{s_0}}$ is the binomial with the pure power in x_2 . We now also have, that if we permute the variables $x_0 \leftrightarrow x_3, x_1 \leftrightarrow x_2$, then the corresponding set $\bar{\mathcal{B}}$ becomes a minimal generating set for $\mathfrak{p}(n_3 - n_2, n_3 - n_1, n_3)$, with the order in which the corresponding binomials appear reversed. This means that the last two, respectively the last three, elements, obtained in the algorithm for $\mathcal{B}$, become, by setting $x_3=1$, a minimal

binomial generating set for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$.

We collect the above information in

Lemma 1.3. (i) If $\{i, j\} = \{2, 3\}$ and $\langle n_1, n_2, n_3 \rangle$ is (n_i, n_j) -symmetric, then $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 2$.

(ii) If $j \in \{2, 3\}$ and $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric, then $B_{j1} = x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j}$, $B_{h1} = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}$, where $d := g.c.d.(n_1, n_j)$.

If $\alpha_{h0} = 0$, then $h = 2$ and $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 2$.

If $\alpha_{h1} = 0$, then $j = 2$ and $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 2$

(iii) If $j \in \{2, 3\}$ and $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric, then the x_h -exponents in all binomials constructed above are multiples of $d := g.c.d.(n_1, n_j)$.

(iv) In each of $\mathcal{B}(j)$ and $\mathcal{B}(h)$ the x_0 and x_1 -exponents are decreasing, the x_2 and x_3 -exponents increasing sequentially with increasing second subscripts. If $\langle n_1, n_2, n_3 \rangle$ is not symmetric, then B_{11} has the largest x_0 and x_1 -exponents in $\mathcal{B}$.

Proof. All of (i)-(iv) are an immediate consequence of the previous and [H]. ■

Finally of importance is Theorem 1.1, Theorem 1.3 and Lemma 1.1 in [B, Sch, St]. Theorem 1.1 gives three necessary and sufficient conditions for a q.c.i. in terms of height, local cohomology and the degrees of the surfaces to be taken. For our purposes here, Theorem 1.3 says that if $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{F_1, F_2, F_3\}$, then, say F_1, F_2 , can be taken to be binomials in $\mathcal{B}$, one of them of lowest degree and the other (as it turns out) of next lowest degree. Lemma 1.1 states that if $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 3$, then F_1, F_2, F_3 are irreducible.

If one allows complete intersections to be a special type of q.c.i., then for all $\mathfrak{p}(n_1, n_2, n_3)$ such that $\mu(\mathfrak{p}(n_1, n_2, n_3)) \leq 3$, $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces). Thus of interest is, if $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 4$. The description of all $\mathfrak{p}(n_1, n_2, n_3)$ with $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 4$ and $\mathfrak{p}(n_1, n_2, n_3)$ a q.c.i. was obtained in [B, Sch, St], where it was also shown that $\mathfrak{p}(n_1, n_2, n_3)$ is not always a q.c.i. on binomials. In view of [B, Sch, St] therefore the assumption $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 5$ is not restrictive.

2. Prerequisite lemmata and corollaries

We start with

Lemma 2.1. Assume $\langle n_1, n_2, n_3 \rangle$ is not symmetric and $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} = (f_i, f_j, f_h)R_0$, $\{i, j, h\} = \{1, 2, 3\}$. If $\{b, f, g\}$ is a minimal generating set of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ and b is a monic binomial, then $\pm b \in \{f_i, f_j, f_h\}$. An analogous statement is true for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$ if $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is not symmetric.

Proof. $\{f_i, f_j, f_h\}$ is a minimal homogeneous generating set of $\mathfrak{a} := \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ with respect to the grading $\widetilde{\deg}(x_i) = n_i$ of R_0 , $1 \leq i \leq 3$. Therefore $\{f_i, f_j, f_h\}$ is a minimal generating set of the extended ideal $\mathfrak{a}^e \subseteq K[[x_1, x_2, x_3]]$. Thus $\{b, f, g\}$ is also a minimal generating set of $\mathfrak{a}^e$. Let M and M^* be any 3×3 matrices with polynomial entries and with the property that $(b, f, g)^T = M(f_i, f_j, f_h)^T$ and $(f_i, f_j, f_h)^T = M^*(b, f, g)^T$. Then $(MM^* - E_3)(b, f, g)^T = (0, 0, 0)^T$, E_3 the 3×3 identity matrix. Since $\{b, f, g\}$ is a minimal generating set of $\mathfrak{a}^e \subseteq K[[x_1, x_2, x_3]]$, this implies $(MM^* - E_3) \bmod (\mathfrak{m} := (x_1, x_2, x_3)R_0)$ is the 3×3 0-matrix. Hence every row of M has an element not in $\mathfrak{m}$. If $M = ((r_{ij}))$, since b, f_i, f_j, f_h are homogeneous, we may assume r_{11}, r_{12} and r_{13} to be homogeneous. W.l.o.g. assume $r_{11} \notin \mathfrak{m}$, thus r_{11} is a constant $c_1 \neq 0$. By the assumptions on the exponents in f_i, f_j, f_h , the term $c_1 x_i^{\alpha_i}$ does not cancel. Since b is assumed monic, we conclude $\pm x_i^{\alpha_i}$ is a monomial term of b . But the monic binomial in $\mathfrak{a}$ with one of its monomial terms $\pm x_i^{\alpha_i}$ is uniquely determined up to multiplication by ± 1 and is equal to $\pm f_i$. Therefore $\pm b \in \{f_i, f_j, f_h\}$. ■

Corollary 2.2. If $\{b_1, b_2, f\}$ is a minimal generating set of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$, b_1, b_2 monic binomials, then for some set $\{i, j\} \subseteq \{i, j, h\} = \{1, 2, 3\}$, $\pm b_1 = f_i$, $\pm b_2 = f_j$.

Proof. The existence of $\{i, j\}$ such that $\pm b_1 = f_i, \pm b_2 = f_j$ follows by applying Lemma 2.1. twice. ■

Corollary 2.3. Assume $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are not symmetric and $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 5$. Then $\mathfrak{p}(n_1, n_2, n_3)$ is not a q.c.i.

Proof. Suppose $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i.. Then by Theorem 1.3 [B, Sch, St] we may assume that two of the elements on which $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. are binomials in $\mathcal{B}$, which, by Lemma 2.1, become part of the canonical generating set $\{f_i, f_j, f_h\}$ (up to multiplying by -1) of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ and an analogous statement is true for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$. Thus the first and last three elements in $\mathcal{B}$ have two elements in common, from which $\mu(\mathfrak{p}(n_1, n_2, n_3)) \leq 4$, a contradiction. ■

Remark 2.4. If $\langle n_1, n_2, n_3 \rangle$ is (n_i, n_j) -symmetric and neither n_i nor n_j equals n_1 , then by Lemma 1.3 $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 2$. Thus w.l.o.g. we may assume $n_i = n_1$ and $\{n_j, n_h\} = \{n_2, n_3\}$.

Lemma 2.5. Assume $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric, $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 4$ and $\{B_1, B_2, F\} \subseteq \mathfrak{p}(n_1, n_2, n_3)$ is such that B_1, B_2 are irreducible monic binomials and $\{B_1|_{x_0=1}, B_2|_{x_0=1}, F|_{x_0=1}\}$ is a generating set for $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$. Then one of $B_1|_{x_0=1}, B_2|_{x_0=1}$ is a canonical generator of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ (up to multiplication by -1). An analogous statement holds for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$ if $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is symmetric.

Proof. It is convenient and necessary for later considerations to rewrite

$\mathcal{B} = \mathcal{B}^*(j) \cup \mathcal{B}^*(h)$, $\mathcal{B}^*(j) := \left\{ x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} =: B_{j1}, x_0^{(q_j - q_1) - \alpha_{h0}} x_j^{q_1 + \alpha_{hj}} - x_1^{q_j - \alpha_{h1}} x_h^d =: B_{j2}, \dots, B_{j, l(j)} \right\}$, $\mathcal{B}^*(h) := \left\{ x_0^{\alpha_{h0}} x_h^{\alpha_h} - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} =: B_{h1}(\alpha_{h0} > 0, \alpha_{h1} < q_j, \alpha_{hj} > 0), B_{h2}, \dots, B_{h, l(h)} \right\}$, $\alpha_h = d$, where $B(x_2)$, the binomial with a pure power in x_2 obtained by our algorithm, is in $\mathcal{B}^*(j)$ for either $j=2, h=3$ or $j=3, h=2$. Thus the reassignment of $B(x_2)$ is possibly the only change in $\mathcal{B}$ as constructed in the previous section. Its purpose is to obtain $\mathcal{B}^*(h)$ as a set of binomials, where each monomial term is divisible by exactly two different variables. Let $\widetilde{\deg}$ be the weighted degree with $\widetilde{\deg}(x_0)=0, \widetilde{\deg}(x_i)=n_i, 1 \leq i \leq 3$. Since the exponents of x_j in $\mathcal{B}^*(j)$ and x_h in $\mathcal{B}^*(h)$ are increasing, $\widetilde{\deg}(B_{ji}) < \widetilde{\deg}(B_{j, i+1}), 1 \leq i \leq l(j)-1$, in $\mathcal{B}^*(j)$ and $\widetilde{\deg}(B_{hi}) < \widetilde{\deg}(B_{h, i+1}), 1 \leq i \leq l(h)-1$, in $\mathcal{B}^*(h)$. (This includes $B(x_2)$ in either case.). Thus $\min\{\widetilde{\deg}(B_{j1}), \widetilde{\deg}(B_{h1})\}$ is the smallest weighted degree in $\mathcal{B}$. Also since $q_j - \alpha_{h1} > 0, d n_h = \widetilde{\deg}(B_{h1}) < \widetilde{\deg}(B_{j2}) = d n_h + (q_j - \alpha_{h1}) n_1 < \widetilde{\deg}(B_{ji}), i \geq 3$. If $B_{h2} \in \mathcal{B}^*(h)$, then $B_{h2} = x_0^{(\nu+1)\alpha_{h0} - (q_j - q_1)} x_h^{(\nu+1)d} - x_1^{(\nu+1)\alpha_{h1} - q_j} x_j^{(\nu+1)\alpha_{hj} + q_1}$. Therefore $q_1 n_j = \widetilde{\deg}(B_{j1}) < \widetilde{\deg}(B_{h2}) = ((\nu+1)\alpha_{hj} + q_1) n_j + ((\nu+1)\alpha_{h1} - q_j) n_1 < \widetilde{\deg}(B_{hi}), i \geq 3$.

From this $\widetilde{\deg}(B_{j1}) < \widetilde{\deg}(B_{st})$ for $s=j$, $2 \leq t \leq l(j)$, and for $s=h$, $2 \leq t \leq l(h)$, $\widetilde{\deg}(B_{h1}) < \widetilde{\deg}(B_{st})$ for $s=j$, $2 \leq t \leq l(j)$, and for $s=h$, $2 \leq t \leq l(h)$, $\widetilde{\deg}(B_{h1}) < \widetilde{\deg}(B_{st})$ for $s=j$, $2 \leq t \leq l(j)$, and for $s=h$, $2 \leq t \leq l(h)$. Next $\widetilde{\deg}(B_{j1}) \neq \widetilde{\deg}(B_{h1})$. Otherwise $q_1 n_j = q_j n_1 = d n_h$. If $h=2$, $j=3$, then $x_2^d - x_0^{d-q_1} x_3^{q_1} \in \mathfrak{p}(n_1, n_2, n_3)$, if $h=3$, $j=2$, then $x_0^{q_1-d} x_3^d - x_2^{q_1} \in \mathfrak{p}(n_1, n_2, n_3)$. In both cases $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 2$, a contradiction. Assume therefore $\widetilde{\deg}(B_{r1}) < \widetilde{\deg}(B_{t1})$ with $(r, t) \in \{(j, h), (h, j)\}$. Set $b_{r1} := B_{r1}|_{x_0=1}$, $b_{t1} := B_{t1}|_{x_0=1}$. Then $(b_{r1}, b_{t1})R_0 = \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ by our assumption. Suppose $\{\pm B_1, \pm B_2\} \cap \{B_{j1}, B_{h1}\} = \emptyset$. Then $\widetilde{\deg}(B_1) > \widetilde{\deg}(B_{t1})$, $\widetilde{\deg}(B_2) > \widetilde{\deg}(B_{t1})$, since, up to multiplication by ± 1 , B_{j1} and B_{h1} are uniquely determined amongst monic irreducible binomials. Since $\{B_1|_{x_0=1} =: b_1, B_2|_{x_0=1} =: b_2, F|_{x_0=1} =: f\}$ generate $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$, at least one generator must have a nonzero component of lowest degree, thus f must have a nonzero summand $c_0 b_{r1}$, $c_0 \neq 0$, $c_0 \in K$. This means if f is written as a linear combination of elements in $\mathcal{B}|_{x_0=1}$, then $f = f' + c_0 b_{r1}$, $c_0 \neq 0$, $\widetilde{\deg}(m') > \widetilde{\deg}(b_{r1})$ for all monomial terms m' of f' . We also must have:

$$(*) \quad b_{t1} := p_1 b_1 + p_2 b_2 + p_3 f.$$

If p_3 does not have a nonzero constant term c_1 , then we obtain the weighted homogeneous equation $b_{t1} = p'_3 b_{r1}$ for a suitable p'_3 , which is impossible since $\mu(\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1})=2$. But for $c_1 \neq 0$ in p_3 , the right hand side in $(*)$ always has a noncancelling component of smaller degree than the left side, again a contradiction. Thus $\{\pm B_1, \pm B_2\} \cap \{B_{j1}, B_{h1}\} \neq \emptyset$, which by the uniqueness property of the canonical binomial generators (up to sign) implies the statement of the lemma. ■

Lemma 2.6. Let $G_1^h := \{B_{11}, B_{21}, B_{31}\}$, where $B_{11}|_{x_0=1}, B_{21}|_{x_0=1}, B_{31}|_{x_0=1}$, are the canonical binomial generators for $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ if $\langle n_1, n_2, n_3 \rangle$ is not symmetric, and let $G_2^h := \{B_{j1}, B_{h1}\}$, $\{j, h\} = \{2, 3\}$, where $B_{j1}|_{x_0=1}, B_{h1}|_{x_0=1}$ are the canonical binomial generators of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ if $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric. Assume $B := m_0 - m_1 \in G_r^h$, $r \in \{1, 2\}$ and $B' := m'_0 - m'_1 \in \mathcal{B} \setminus G_r^h$. Then there exists a variable x_s , $s \in \{1, 2, 3\}$ such that $x_s^{\rho_s} | m'_0$, but $x_s^{\rho_s} \nmid m_0$, $\rho_s \geq 1$. The same type relation (perhaps for a different variable) exists between m'_0 and m_1, m'_1 and m_0, m'_1 and m_1 .

Proof. A tedious proof follows by checking different cases. A short proof ensues by ob-

serving that $\mathcal{B}$ forms a reduced and normalized Gröbner basis for $\mathfrak{p}(n_1, n_2, n_3)$ with respect to the graded lexicographical term order, where x_2 is the largest linear term $[B, C]$. This proves the above statement for the monomials not divisible by x_0 . For the other monomials the proof is obtained by observing that the x_2 and x_3 -exponents are increasing throughout the algorithm. ■

Definition 2.7. Let $p \in K[x_0, \dots, x_n]$. Then $\text{supp}(p) := \{(\alpha_0, \dots, \alpha_n) \in \mathbb{Z}^{n+1} \mid cx_0^{\alpha_0} \dots x_n^{\alpha_n} \text{ is a monomial term of } p \text{ with } c \neq 0\}$ is called the support of p . For $S \subseteq K[x_0, \dots, x_n]$ we define $\text{supp } S := \cup_{p \in S} \text{supp}(p)$.

Corollary 2.8. Let B be as in the statement of Lemma 2.6. Then $\text{supp } \{x_0^{\rho_0} B \mid \rho_0 \geq 0\} \cap \text{supp } \{(\mathcal{B} \setminus G_r^h)R\} = \emptyset$.

Proof. This is immediate by Lemma 2.6. ■

Corollary 2.9. Assume $\{B_1, B_2, B_3\} \subseteq \mathcal{B} := \{B_1, \dots, B_l\}$ and $F \in \mathfrak{p}(n_1, n_2, n_3)$ are such that $(B_1, B_2, F)R|_{x_0=1} = \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} = (B_1, B_2, B_3)R|_{x_0=1}, (B_1, B_2)R|_{x_0=1} \subset \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ and w.l.o.g. $F := \sum_{i=3}^l P_i B_i$, where $P_i, 3 \leq i \leq l$, is homogeneous. Then $P_3 = P'_3 + c_0 x_0^{\rho_0}, c_0 \neq 0, P'_3 \in (x_1, x_2, x_3)R$ and $\text{supp}(c_0 x_0^{\rho_0} B_3) \cap \text{supp}\{P'_3 B_3, P_4 B_4, \dots, P_l B_l\} = \emptyset$. If the above hypothesis holds for $x_3=1$, with binomials B_1, B_2 as before, $B_s, s > 3$, in $\mathcal{B}$, and F as before, then $P_s = P'_s + c_3 x_3^{\rho_3}, c_3 \neq 0$, with an analogous statement about the support.

Proof. That $P_3 = P'_3 + c_0 x_0^{\rho_0}, c_0 \neq 0$, follows from Lemma 2.6 and the fact that $B_3|_{x_0=1}$ is the unique binomial which contains at least one minimal pure power of the variables x_1, x_2, x_3 . It is easily checked that $\text{supp}(c_0 x_0^{\rho_0} B_3) \cap \text{supp}(P'_3 B_3) = \emptyset$. Therefore Corollary 2.8 implies the statement about the support. The last statement now also follows from Corollary 2.8. ■

3. Two types of q.c.i. for monomial curves in $\mathbb{P}_K^3$.

Throughout this paragraph if $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on three surfaces $\{B_1, B_2, F\}$, then B_1, B_2 will denote binomials in $\mathcal{B}$, in accordance with Theorem 3.1 in [B,Sch,St]. Also

always $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 4$. Assume first $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric (i.e. $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} = (x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} |_{x_0=1}, x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} |_{x_0=1}) R_0$) and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is symmetric. We then have:

Theorem 3.1. $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces) iff $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on three binomials in $\mathcal{B}$. Also in this case if $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces), then

- i) $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are symmetric of the same type, and
- ii) $\mathcal{B}^*(h) = \{x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}\}$.

Proof. $\Leftarrow$. This follows trivially.

$\Rightarrow$. Assume $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1, B_2, F\}$. Let $B := \{B_1, B_2\}$, $B^0 := \{B_1^0, B_2^0\}$ be such that $\{B_1^0|_{x_0=1}, B_2^0|_{x_0=1}\}$ is a set of canonical generators of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ and define $B^3 := \{B_1^3, B_2^3\}$ analogously for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$.

I. Assume $B^0 \cap B^3 \neq \emptyset$. For $B^0 = \{B_1^0 := B_{h1} \text{ (in our previous notation)} = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}, B_2^0 := B_{j1} \text{ (in our previous notation)} = x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j}, \{h, j\} = \{2, 3\}, n_1 = q_1 d, n_j = q_j d, d := g.c.d.(n_1, n_j)\}$, $x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} \notin B^3$ (since by the algorithm of [B,R] this binomial always has a successor (B_{j2} in the proof of Lemma 2.5), therefore is not one of the last two binomials in $\mathcal{B}$). Therefore $B^0 \cap B^3 = \{x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} = B_1^0\}$, where all exponents in B_1^0 are positive since $\mu(\mathfrak{p}(n_1, n_2, n_3)) \neq 2$. By Lemma 2.5, B_1^0 is one of B_1 or B_2 . Let $B_1 := B_1^0$. It now follows:

$\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_2 = B_2^0, B_1 = B_1^0 = B_1^3, B_2^3\}$, since $\mathfrak{p}(n_1, n_2, n_3)|_{x_1=1} = (B_2|_{x_1=1} = B_2^0|_{x_1=1}, B_1|_{x_1=1} = B_1^0|_{x_1=1} = B_1^3|_{x_1=1}) R_1$, respectively $\mathfrak{p}(n_1, n_2, n_3)|_{x_2=1} = (B_1|_{x_2=1} = B_1^0|_{x_2=1} = B_1^3|_{x_2=1}, B_2^3|_{x_2=1}) R_2$ (see [B,C]).

(i) All exponents in $B_1 = B_1^0 = B_1^3$ are positive, thus if $j=2, h=3$, then $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is $(n_3 - n_2, n_3 - n_1)$ -symmetric, if $j=3, h=2$, then $B(x_2) = x_0^{\beta_{20}} x_3^{\beta_{23}} - x_2^{\beta_2}$, thus $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is $(n_3 - n_2, n_3)$ -symmetric. Therefore $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are symmetric of the same type.

(ii) $\mathcal{B}^*(h) = \{B_1 = B_1^0 = B_1^3\}$.

II. Suppose $B^0 \cap B^3 = \emptyset$. Then $\mathfrak{p}(n_1, n_2, n_3)$ is not a q.c.i. on three binomials, since if this

was the case, binomials being homogeneous in the weighted grading and since every homogeneous generating set contains a minimal generating set, this would imply that the set of binomials (on which $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i.) for $x_0 = 1$ and $x_3 = 1$ would contain a minimal set of generators, thus $B^0 \cap B^3 \neq \emptyset$. We now show, that under the stated hypothesis, II cannot happen. We do this by obtaining:

1. The pure power binomials $B(x_1)$ and $B(x_2)$ in $\mathcal{B}$ are of equal degree.
2. $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1, B_2, B(x_1) + cB(x_2)\}$ for suitable $c \in K^*$.
3. $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are symmetric of the same type.
4. The assumption $B^0 \cap B^3 = \emptyset$ entails a contradiction.

Since by Lemma 2.5 $\{B_1, B_2\} \cap B^0 \neq \emptyset$ and $\{B_1, B_2\} \cap B^3 \neq \emptyset$, assume $B_1 =: B_1^0 \in B^0, B_2 =: B_1^3 \in B^3$. Then by Corollary 2.8., F (after possibly multiplying by a nonzero constant) has a nonzero summand $cx_0^{\rho_0}B_2^0 + x_3^{\rho_3}B_2^3, c \neq 0$. Thus $\deg(F) \geq \deg(B_2^0)$ and $\deg(F) \geq \deg(B_2^3)$. If $B_2^0 \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(R/(B_1, B_2)R)$, $\mathfrak{q} \neq \mathfrak{p}(n_1, n_2, n_3)$, then by Theorem 1.1 in [B, Sch, St] and since $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} = (B_1, B_2, B_2^0)R|_{x_0=1}, \mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1, B_2, L^\rho B_2^0\}$, where L is a generic linear form and $\rho := \deg(F) - \deg(B_2^0)$. By Lemma 1.1 in [B, Sch, St], we must have $\rho = 0$, a contradiction since $\mathfrak{p}(n_1, n_2, n_3)$ is not a q.c.i. on three binomials. Assume $B_2^0 \in \mathfrak{q}, \mathfrak{q} \in \text{Ass}(R/(B_1 = B_1^0, B_2 = B_1^3)R), \mathfrak{q} \neq \mathfrak{p}(n_1, n_2, n_3)$. Since $(B_1 = B_1^0, B_2 = B_1^3)R$ is a complete intersection, $\mathfrak{q}$ is minimal over $(B_1 = B_1^0, B_2 = B_1^3)R$, thus minimal over $(B_1 = B_1^0, B_2 = B_1^3, B_2^0)R$, hence $\mathfrak{q} \in \text{Ass}(R/(B_1, B_2, B_2^0)R)$. Since $(B_1|_{x_0=1}=B_1^0|_{x_0=1}, B_2|_{x_0=1}, B_2^0|_{x_0=1})R_0 = \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}, x_0 \in \mathfrak{q}$ ([Z, S], Vol. II, Ch. VII, Theorem 18). From $B^0 \subseteq \mathfrak{q}$, $x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} \in \mathfrak{q}$, thus $x_1 \in \mathfrak{q}$, hence $\mathfrak{q} = (x_0, x_1)R$ since $(B_1 = B_1^0, B_2 = B_1^3)R$ is unmixed. Since $B_1^3 \in \mathfrak{q}, x_0$ and x_1 divide different monomial terms in B_1^3 and therefore all four variables appear in B_1^3 (a general property of the binomials in $\mathcal{B}$ except perhaps $B(x_1)$). Hence B_2^3 is a binomial in exactly three variables, x_0 and x_1 divide the same monomial term of B_2^3 , thus $B_2^3 \notin \mathfrak{q}$ and $B_2^3 = B(x_2)$. Analogously we obtain x_2 and x_3 divide different monomial terms of B_1^0 , thus all four variables divide a monomial term of B_1^0 , hence B_2^0 (up to sign) $= x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} = B(x_1)$, a binomial in exactly three variables. Assume first that B_2^0 and B_2^3 are of equal degree. Let $\mathfrak{q}_i, 1 \leq i \leq n$, be the prime ideals of $\text{Ass}(R/(B_1 = B_1^0, B_2 = B_1^3)R)$ with $\mathfrak{q}_i \neq \mathfrak{p}(n_1, n_2, n_3)$ and $B_2^0 \notin \mathfrak{q}_i$. There exist then

$\zeta_i \in \mathbb{P}_K^3, \overline{K}$ the algebraic closure of K , ζ_i a zero of q_i , such that $B_2^0(\zeta_i) \neq 0$. Let $c \in K$ such that $c \neq 0$ and $\frac{-B_2^3(\zeta_i)}{B_2^0(\zeta_i)} \neq c, 1 \leq i \leq n$, a possible choice since K is infinite. Then $cB_2^0(\zeta_i) + B_2^3(\zeta_i) \neq 0$, thus $cB_2^0 + B_2^3 \notin q_i, 1 \leq i \leq n$. Thus $cB_2^0 + B_2^3 \notin q$ for all $q \in \text{Ass}(R/(B_1 = B_1^0, B_2 = B_1^3)R), q \neq \mathfrak{p}(n_1, n_2, n_3)$. Let $\mathfrak{b}$ be the $\mathfrak{p} := \mathfrak{p}(n_1, n_2, n_3)$ -primary component of $\mathfrak{p}^2 + (B_1^0, B_1^3)R$. We have $B_2^0 \notin \mathfrak{b}$ if $\mathfrak{b} \subset \mathfrak{p}$, since then $\mathfrak{b}R_{\mathfrak{p}} \subset \mathfrak{p}R_{\mathfrak{p}}$ and $\mathfrak{p}R_{\mathfrak{p}} = (B_1^0, B_2^0)R_{\mathfrak{p}}$. Let $C^* = \{c^* \in L^* | c^*B_2^0 + B_2^3 \in \mathfrak{b}\}$ and suppose $c_1^* \in C^*, c_2^* \in C^*, c_1^* \neq c_2^*$. This implies $c_2^*(c_1^*B_2^0 + B_2^3) - c_1^*(c_2^*B_2^0 + B_2^3) = (c_2^* - c_1^*)B_2^3 \in \mathfrak{b}$, thus $B_2^3 \in \mathfrak{b}$ and therefore $B_2^0 \in \mathfrak{b}$, a contradiction. Thus the cardinality of C^* is equal to or less than 1. Assume additionally to the above restrictions on $c, c \notin C^*$. Also $\deg(F) \geq cB_2^0 + B_2^3$. By Theorem 1.1 in [B, Sch, St] therefore $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1 = B_1^0, B_2 = B_1^3, L^\rho(cB_2^0 + B_2^3)\}$, where:

- a) L is a linear form which is "generic enough",
- b) $B_1 = B_1^0$ and $B_2 = B_1^3$ are such that all four variables divide some monomial term,
- c) $B_2^0 = B(x_1), B_2^3 = B(x_2)$ are binomials in exactly three variables,
- d) $c \neq 0$.

But then $\mathfrak{p}(n_1, n_2, n_3)$ is also a q.c.i. on $\{B_1 = B_1^0, B_2 = B_1^3, cB_2^0 + B_2^3\}$ (since $(B_1 = B_1^0, B_2 = B_1^3, L^\rho(cB_2^0 + B_2^3))R \subseteq (B_1 = B_1^0, B_2 = B_1^3, cB_2^0 + B_2^3)R$), which by the degree requirement of Theorem 1.1 in [B, Sch, St] is only possible if actually $\deg(F) = \deg(cB_2^0 + B_2^3)$. Thus in this case $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1 = B_1^0, B_2 = B_1^3, cB_2^0 + B_2^3\}$. We show next that B_2^0 and B_2^3 must be of equal degree. Thus suppose B_2^0 and B_2^3 are of unequal degree. Assume first $\deg(B_2^3) < \deg(B_2^0)$ and $cB_2^0 + x_3^{\rho_3}B_2^3$ is homogeneous with $\rho_3 \geq 1$. As before if $B_2^0 \in q, q \in \text{Ass}(R/(B_1 = B_1^0, B_2 = B_1^3)R), q \neq \mathfrak{p}(n_1, n_2, n_3)$, then $q = (x_0, x_1)R$, thus $x_3^{\rho_3}B_2^3 \notin q$. Also, again as above, let $c \in K, c \neq 0$, be such that $cB_2^0 + x_3^{\rho_3}B_2^3 \notin q, q \in \text{Ass}(R/(B_1 = B_1^0, B_2 = B_1^3)R), q \neq \mathfrak{p}(n_1, n_2, n_3)$, and $cB_2^0 + x_3^{\rho_3}B_2^3 \notin \mathfrak{b}$ if $\mathfrak{b} \subset \mathfrak{p} := \mathfrak{p}(n_1, n_2, n_3)$, where $\mathfrak{b}$ is defined as above. Since $\deg(F) \geq \deg(B_2^0)$, we have again $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1 = B_1^0, B_2 = B_1^3, cB_2^0 + x_3^{\rho_3}B_2^3\}$ (after first perhaps multiplying $cB_2^0 + x_3^{\rho_3}B_2^3$ by L^ρ , where L is a "generic enough" linear form). But this is impossible, since $\mathfrak{p}(n_1, n_2, n_3)|_{x_2=1}$ has a binomial with a constant nonzero term, whereas the ideal $(B_1|_{x_2=1} = B_1^0|_{x_2=1}, B_2|_{x_2=1} = B_1^3|_{x_2=1}, cB_2^0|_{x_2=1} + x_3^{\rho_3}B_2^3|_{x_2=1})R_2$ does not since $\rho_3 \geq 1$ (c.f. [B, C]). The proof for $\deg(B_2^0) < \deg(B_2^3)$ is analogous, since also $x_0^{\rho_0}B_2^0 \notin \mathfrak{b}$

if $\mathfrak{b} \subset \mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$. This proves 1. and 2.

Suppose next that $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are not symmetric of the same type. W.l.o.g. assume $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_2) -symmetric and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is $(n_3 - n_2, n_3)$ -symmetric. We have $B_2^0 = x_0^{q_2 - q_1} x_2^{q_1} - x_1^{q_2}, B_1^0 = x_0^{\alpha_{30}} x_3^d - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, n_1 = q_1 d, n_2 = q_2 d, d = \text{q.c.d.}(n_1, n_2)$ and $\deg(B_2^0) = \deg(B_2^3) = q_2$. But then, since $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is $(n_3 - n_2, n_3)$ -symmetric, $q_2 | n_3$, say $n_3 = q_2 d^*$. From $B_1^0|_{x_0=1}$ we obtain $d q_2 d^* = d n_3 = \alpha_{31} n_1 + \alpha_{32} n_2$. On the other hand $d q_2 d^* = d^* n_2$, which gives $(d^* - \alpha_{32}) n_2 = \alpha_{31} n_1$, a contradiction since $0 < \alpha_{31} < q_2$. Therefore $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ must be symmetric of the same type, which proves 3.

We have now $B_2^0 = x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j}, B_1^0 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}, B_1^3 = x_0^{\beta_{h0}} x_h^{\nu d} - x_1^{\beta_{h1}} x_j^{\beta_{hj}}$ with $\nu > 1$ since $B^0 \cap B^3 = \emptyset$.

$$B_2^3 = \begin{cases} x_j^{q_j} - x_1^{\gamma_1} x_h^{\gamma_h = \mu d} & \text{if } j = 2, h = 3. \\ x_0^{\delta_0} x_j^{\delta_j} - x_h^{q_j = \mu d} & \text{if } j = 3, h = 2. \end{cases}$$

(We note again that by Lemma 1.3 (iii) all x_h -exponents are multiples of d .)

Observe that $\{B_2^0|_{x_0=1}, B_1^0|_{x_0=1}\}$ forms a Gröbner basis of $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$ with respect to the lexicographical term order with x_h as largest linear term. Since $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1^0, B_1^3, B_2^0 + c B_2^3\}$, $c \neq 0$, we must have $(x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}, x_j^{q_1} - x_1^{q_j}) R_0 = \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} = (x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}, x_h^{\nu d} - x_1^{\beta_{h1}} x_j^{\beta_{hj}}, x_j^{q_1} - x_1^{q_j} + c B_2^3|_{x_0=1}) R_0$. Let $G = \{x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}, x_h^{\nu d} - x_1^{\beta_{h1}} x_j^{\beta_{hj}}, x_j^{q_1} - x_1^{q_j} + c B_2^3|_{x_0=1}\}$. The set G can be changed into a generating set $\{x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} = B_1^0|_{x_0=1} =: g_0, g_1 := x_1^{\nu \alpha_{h1}} x_j^{\nu \alpha_{hj}} - x_1^{\beta_{h1}} x_j^{\beta_{hj}} =: x_1^{\varepsilon_1} x_j^{\varepsilon_j} (x_j^{q_1} - x_1^{q_j}) P, P \in K[x_1, x_j], \varepsilon_1 > 0, \varepsilon_j > 0, g_2 := x_j^{q_1} - x_1^{q_j} + c g\}$ where for $j = 2, h = 3$ $g := x_j^{q_j} - x_1^{\mu \alpha_{h1} + \gamma_1} x_j^{\mu \alpha_{hj}} = x_j^{\mu \alpha_{hj}} (x_j^{q_j - \mu \alpha_{hj}} - x_1^{\mu \alpha_{hj} + \gamma_1}) =: x_j^{\mu \alpha_{hj}} (x_j^{q_1} - x_1^{q_j}) Q_1, Q_1 \in K[x_1, x_j], Q_1 \notin (x_1) R_0$ (compare the right and left side of the last equation) and for $j = 3, h = 2$ $g := x_j^{\delta_j} - x_1^{\mu \alpha_{hj}} x_j^{\mu \alpha_{hj}} = x_j^{\mu \alpha_{hj}} (x_j^{\delta_j - \mu \alpha_{hj}} - x_1^{\mu \alpha_{hj}}) =: x_j^{\mu \alpha_{hj}} (x_j^{q_1} - x_1^{q_j}) Q_2, Q_2 \in K[x_1, x_j], Q_2 \notin (x_1) R_0$. Since $\{B_1^0|_{x_0=1}, B_2^0|_{x_0=1}\}$ is a Gröbner basis for $\mathfrak{p}(n_1, n_2, n_3)$ in the specified term order and since the leading terms in g_0, g_1 and g_0, g_2 are relatively prime, the Gröbner algorithm implies that we must have $\mathfrak{J}_i := (x_1^{\varepsilon_1} x_j^{\varepsilon_j} P, 1 + c x_j^{\mu \alpha_{hj}} Q_i) R_0 = R_0, i \in \{1, 2\}$. But this is impossible since $c \neq 0$ and $1 + c x_j^{\mu \alpha_{hj}} Q_i|_{x_1=0}$ is a polynomial in $K[x_j]$ of degree > 0 , thus has a zero $\zeta_j \in \overline{K}$, therefore $\mathfrak{J}_i$ has a zero $x_1 = 0, x_j = \zeta_j$ in $\overline{K}^2$, thus $\mathfrak{J}_i \neq R_0$. This proves that II as stated cannot happen, which completes the proof of Theorem 3.1. ■

Now we assume $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is not symmetric. Thus again $B^0 := \{B_1^0 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} \in \mathcal{B}^*(h), B_2^0 = x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j}\} \subseteq \mathcal{B}$ is such that $\{B_1^0|_{x_0=1}, B_2^0|_{x_0=1}\}$ is a canonical set of binomial generators for $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$. Similarly define $B^3 := \{B_1^3, B_2^3, B_3^3\} \subseteq \mathcal{B}$ such that $B_i^3|_{x_3=1}, 1 \leq i \leq 3$, are the canonical generators of $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$. W.l.o.g. assume $B_3^3 = B(x_2)$ has the pure power in x_2 . We have:

Theorem 3.2. $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1, B_2, F\} = B \cup \{F\}$, where $B := \{B_1, B_2\}$ iff

- (i) $\deg(B_2^0) = \deg(B_3^3)$ and
- (ii) $\mathcal{B}^*(h) = \{B_1^0 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}\}$.

In this case F can be taken to be $cB_2^0 + B_3^3, c \neq 0$.

Proof. $\implies$. Assume $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1, B_2, F\}$, B_1, B_2 binomials in $\mathcal{B}$ in accordance with Theorem 3.1 in [B, Sch, St]. By Lemma 2.5 $B \cap B^0 \neq \emptyset$. $B_2^0 \notin B \cap B^0$, since by Corollary 2.2 $B \subseteq B^3$ and no binomial in B^3 has a pure power in x_1 (since $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 4$). Therefore $B \cap B^0 = \{B_1^0 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}\}$ thus $B_1^0 \in B \subseteq B^3$. W.l.o.g. $B_1^0 = B_1 = B_1^3$. Note that F cannot be a binomial. For if F were a binomial, then since $\{B_1 = B_1^0 = B_1^3|_{x_0=1}, B_2|_{x_0=1}, F|_{x_0=1}\}$ generates $\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$, F would have to be B_2^0 . Here, after setting $x_0 = 1$, we reason by way of homogeneous elements in the weighted grading and assume F to be irreducible by Lemma 1.1 in [B, Sch, St]. Since also $\{B_1|_{x_3=1}, B_2|_{x_3=1}, F|_{x_3=1}\}$ is a minimal generating set of binomials for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$, also $F \in B^3$, a contradiction as already observed. By Corollary 2.8 and 2.9 F has nonzero summand $cx_0^{\rho_0} B_2^0 + x_3^{\rho_3} B_i^3, c \neq 0, i \in \{2, 3\}$. (Here $B_i^3 \notin B$.) Therefore $\deg(F) \geq \deg(B_2^0)$ and $\deg(F) \geq \deg(B_i^3)$.

Let $B_2 := B_s^3, s \in \{2, 3\}, s \neq i$. Assume $B_2^0 \in \mathfrak{q}, \mathfrak{q} \in \text{Ass}(R/(B_1 = B_1^0, B_2 = B_s^3)R)$, $\mathfrak{q} \neq \mathfrak{p}(n_1, n_2, n_3)$. Then since $(B_2^0|_{x_0=1}, B_1^0|_{x_0=1} = B_1|_{x_0=1}, B_2|_{x_0=1} = B_s^3|_{x_0=1})R_0 = \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$, as in the proof of Theorem 3.1, $x_0 \in \mathfrak{q}$, and since $B_2^0 \in \mathfrak{q}, x_1 \in \mathfrak{q}$, thus $\mathfrak{q} = (x_0, x_1)R$. Since $B_2 = B_s^3 \in \mathfrak{q}$, x_0 and x_1 divide different monomials in B_s^3 , thus $s = 2, i = 3$ and $B_3^3 \notin \mathfrak{q}$ with

$$B_3^3 = \begin{cases} x_j^{\beta_j} - x_0^{\beta_{j0}} x_1^{\beta_{j1}} x_h^{\beta_{jh} = \mu d} \in \mathcal{B}^*(j) \text{ if } j = 2, h = 3, \\ x_0^{\beta_{h0}} x_1^{\beta_{h1}} x_j^{\beta_{hj}} - x_h^{\beta_h = \mu d} \in \mathcal{B}^*(j) \text{ if } j = 3, h = 2. \end{cases}$$

As in the proof of Theorem 3.1 we obtain $\deg(B_2^0) = \deg(B_3^3) = \deg(F)$ and $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1 = B_1^0, B_2 = B_2^3, cB_2^0 + B_3^3\}, c \neq 0$.

Suppose next $B_2 = B_3^3 = x_0^{\gamma_{h0}} x_h^{\nu d} - x_1^{\gamma_{h1}} x_j^{\gamma_{hj}}, \nu > 1$. We then obtain a contradiction to $B_1|_{x_3=1} B_1^0|_{x_3=1} = B_1^3|_{x_3=1}$ being a canonical binomial generator of $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$, since for $h = 2, j = 3$ the x_1 -exponent in $B_2 = B_2^3$ is smaller than the x_1 -exponent in $B_1 = B_1^0 = B_1^3 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}$ by Lemma 1.3 (iv), and for $h = 3, j = 2$ the same is true for the x_0 -exponents of the two binomials. We note here that for $h = 2, j = 3$ or $h = 3, j = 2$, the "missing" canonical generator for $\mathfrak{p}(n_1, n_2, n_3)|_{x_3=1}$, i.e. the canonical generator whose homogenization is not used for the q.c.i., is obtained from $B_3^3 \notin \mathcal{B}$. Thus $B_2^3 \in \mathcal{B}^*(j)$ and $\mathcal{B}^*(h) = \{B_1 = B_1^0 = B_1^3\}$.

$\Leftarrow$. We show that for $\mathfrak{p}(n_1, n_2, n_3)$, with $\langle n_1, n_2, n_3 \rangle$ (n_1, n_j) -symmetric and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ not symmetric, with $\mathcal{B}^*(h) = \{B_1^0 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} = B_1^3\}$ and $\deg(B_2^0 = B(x_1)) = \deg(B_3^3 = B(x_2))$, $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1^0, B_2^3, B_2^0 + B_3^3\}$. (For convenience we use $c = 1$, but the proof is valid for any $c \neq 0$.) Note that there is no ambiguity as to the binomial B_2^3 . We make this clear by explicitly listing $\mathcal{B}$ subject to our hypothesis. We have: $\mathcal{B}^*(h) = \{B_1^0 = x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}} = B_1^3 = B_{h1} \text{ (previously) } \}$, $\mathcal{B}^*(j) = \{x_0^{q_j - q_1} x_j^{q_1} - x_1^{q_j} = B_2^0 =: B(0) = B_{j1} \text{ (previously) }, x_0^{(q_j - q_1) - \alpha_{h0}} x_j^{q_1 + \alpha_{hj}} - x_1^{q_j - \alpha_{h1}} x_h^d =: B(1) = B_{j2} \text{ (previously) }, \dots, x_0^{(q_j - q_1) - \nu \alpha_{h0}} x_j^{q_1 + \nu \alpha_{hj}} - x_1^{q_j - \nu \alpha_{h1}} x_h^{\nu d} = B_2^3 =: B(\nu), B_3^3 = B_{j, l(j)} \text{ (previously) } \}$, where

$$B_3^3 = \begin{cases} x_j^{q_1 + (\nu+1)\alpha_{hj}} - x_0^{(\nu+1)\alpha_{h0} - (q_j - q_1)} x_1^{q_j - (\nu+1)\alpha_{h1}} x_h^{(\nu+1)d} & \text{for } j = 2, h = 3, \\ x_0^{(q_j - q_1) - (\nu+1)\alpha_{h0}} x_j^{q_1 + (\nu+1)\alpha_{hj}} x_1^{(\nu+1)\alpha_{h1} - q_j} - x_h^{(\nu+1)d} & \text{for } j = 3, h = 2. \end{cases}$$

We show next that for $G^* := \{B_1^0 = B_1^3, B_2^3, B_2^0 + B_3^3\}$, $G^*|_{x_i=1}$ generates $\mathfrak{p}(n_1, n_2, n_3)|_{x_i=1}$, for all i such that $0 \leq i \leq 3$, by showing that the needed canonical binomial generators are obtainable from the given elements in $G^*|_{x_i=1}$. For this we need to obtain certain syzygies of elements in $\mathcal{B}$. We do this by eliminating monomial terms of two binomials in $\mathcal{B}$ (as listed above). These syzygies allow us to show that one of the two binomials in $B_2^0|_{x_i=1} + B_3^3|_{x_i=1}$ is in $G^*|_{x_i=1}$, thus defining a generating set.

1. Let $j=2, h=3$. Then $x_j^{\alpha_{hj}} B_2^3 - x_0^{(q_j - q_1) - \nu \alpha_{h0}} B_3^3 = x_1^{q_j - (\nu+1)\alpha_{h1}} x_h^{\nu d} B_1^0$. Similarly for $j=3, h=2$, $x_1^{(\nu+1)\alpha_{h1}} x_j^{\alpha_{hj}} B_2^3 - x_0^{\alpha_{h0}} B_3^3 = x_h^{\nu d} B_1^0$. This shows that $(G^*|_{x_0=1})R_0 =$

$\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}$.

2. Let $j = 2, h = 3$. Then $x_0^{(\nu+1)\alpha_{h0}-(q_j-q_1)}x_h^dB_2^3 - x_1^{\alpha_{h1}}B_3^3 = x^{q_1+\nu\alpha_{hj}}B_1^0$. For $j = 3, h=2$, $x_h^dB_2^3 - x_1^{q_j-\nu\alpha_{h1}}B_3^3 = x_0^{(q_j-q_1)-(\nu+1)\alpha_{h0}}x_j^{q_1+\nu\alpha_{hj}}B_1^0$. This means $(G^*|_{x_1=1})R_1 = \mathfrak{p}(n_1, n_2, n_3)|_{x_1=1}$ by [B,C].

3. Let $(j, h) = (2, 3)$ or $(j, h) = (3, 2)$. Let $B(\nu - 1) := x_0^{(q_j-q_1)-(\nu-1)\alpha_{h0}}x_j^{q_1+(\nu-1)\alpha_{hj}} - x_1^{q_j-(\nu-1)\alpha_{hj}}x_h^{(\nu-1)d}$ be the predecessor of $B(\nu) = B_2^3$ in $\mathcal{B}$. Then $x_h^dB(\nu - 1) - x_1^{\alpha_{h1}}B_2^3 = x_0^{(q_j-q_1)-\nu\alpha_{h0}}x_j^{q_1+(\nu-1)\alpha_{hj}}B_1^0$. Thus $x_h^dB(\nu - 1) \in (G^*)R$. Inductively $x_h^{\nu d}B_2^0 \in (G^*)R$, $B_2^0 = B(0)$, thus $(G^*|_{x_h=1})R_h = \mathfrak{p}(n_1, n_2, n_3)|_{x_h=1}$. Finally $x_j^{\alpha_{hj}}B(\nu-1) - x_0^{\alpha_{h0}}B_2^3 = x_1^{q_j-\nu\alpha_{h1}}x_h^{(\nu-1)d}B_1^0$. Therefore $x_j^{\alpha_{hj}}B(\nu - 1) \in (G^*)R$. Inductively $x_j^{\nu\alpha_{hj}}B_2^0 \in (G^*)R$, thus $(G^*|_{x_j=1})R_j = \mathfrak{p}(n_1, n_2, n_3)|_{x_j=1}$. (Here $(G^*|_{x_2=1})R_2 = \mathfrak{p}(n_1, n_2, n_3)|_{x_2=1}$ follows again from [B,C].) This completes the proof of Theorem 3.2. qed

Example 3.3. Examples for Theorem 3.1 are the monomial curves with $\langle n_1, n_2, n_3 \rangle = \langle 1, n-1, n \rangle$, $n \geq 4$, $\mu(\mathfrak{p}(1, n-1, n)) = n$. For Theorem 3.2 we give examples with $5 \leq \mu(\mathfrak{p}(n_1, n_2, n_3)) < \infty$ (examples for Theorem 3.2 with $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 4$ are given in [B,Sch,St]). In 1. below $\mu(\mathfrak{p}(n_1, n_2, n_3))$ assumes every value ≥ 5 , in 2. this is not the case, by $\mu(\mathfrak{p}(n_1, n_2, n_3))$ increases beyond any finite bound.

1. Consider the binomials $x_0^ax_2^b - x_1^{a+b}, x_0^cx_3^d - x_1^ex_2^{c-1}$, i.e. $d = e - 1$. If σ denotes the number of binomials in the algorithm of [B.R] after the first two binomials, then $\sigma \geq 3$ is needed. To obtain equality of degree for the pure power binomials, $\sigma(c-1) = a$ is required, and for the algorithm to end properly, we need $\sigma c > \sigma(c-1) = a > (\sigma-1)c$. For $\langle n_3-n_2, n_3-n_1, n_3 \rangle$ not to be symmetric, $a+b > \sigma e$ is required. These conditions reduce to $\sigma(c-1) = a$, $c > \sigma$, $b > \sigma(e-c+1)$. In particular for $c = 4$, $\sigma = 3$, a is equal to 9. Let $d = 1$, then $e = 2$ and $3(2-4+1) = -3 < b$ for $b = 1$. For $n_1 = 1$, $n_2 = a+b = 10$, $n_3 = 2+3 \cdot 10 = 32$ we obtain for $\mathfrak{p}(n_1, n_2, n_3) = \mathfrak{p}(1, 10, 32)$ the minimal generating set $\{x_0^4x_3 - x_1^2x_3^2, x_0^9x_2 - x_1^{10}, x_0^5x_2^4 - x_1^8x_3, x_0x_2^7 - x_1^6x_3^2, x_2^{10} - x_0^3x_1^4x_3^3\}$. Similarly for $\sigma = n \geq 3$, $c = n+1$, $a = n^2$, $d = 1$, $e = 2$, $1 = b > n(2-n-1+1) = n(2-n)$, $n_1 = 1$, $n_2 = n^2 + 1$, $n_3 = 2 + n(n^2 + 1)$, we obtain $\mu(\mathfrak{p}(n_1, n_2, n_3)) = n+2$ and $\mathfrak{p}(n_1, n_2, n_3)$ is as in Theorem 3.2 with $j = 2$, $h = 3$.

2. We start now with $x_0^ax_3^b - x_1^{a+b}, x_0^cx_2^d - x_1^ex_3^{c-1}$, i.e. $d = e - 1$. With σ as in

1., we need to satisfy $\sigma d = a + b$, $e > \sigma$, $a > \sigma c$. Let $\sigma = n$, n an odd prime, $e = n + 1$, $d = n$, $a + b = n^2$, $c = 2$, $b = 2$, $a = n^2 - 2$. From the matrix $\begin{pmatrix} n^2 & 0 & -2 \\ -(n+1) & n & -1 \end{pmatrix}$ we obtain $n_1 = 2n < n_2 = n^2 + 2n + 2 < n_3 = n^3$ (since $n \geq 3$). Also $\text{g.c.d.}(n_1, n_2) = 1$ since n is an odd prime. Now $\mathfrak{p}(2n, n^2 + 2n + 2, n^3)$ satisfies Theorem 3.2 with $j=3, h=2$ and $\mu(\mathfrak{p}(2n, n^2 + 2n + 2, n^3)) = n + 2$. In particular for $\sigma=3, e=4, d=3, a+b=9, c=2, a=7, b=2$. We have $n_1=6, n_2=17, n_3=27$ and $\mathfrak{p}(6, 17, 27)$ has a minimal generating set $\{x_0^2 x_2^3 - x_1^4 x_3, x_0^7 x_3^2 - x_1^9, x_0^5 x_3^3 - x_1^5 x_2^3, x_0^3 x_3^4 - x_1 x_2^6, x_0 x_3^5 x_1^3 - x_2^9\}$.

4. Conclusion

We combine here Theorem 3.1 and Theorem 3.2 with the results obtained in [B,Sch,St]. Our notation for $\mathcal{B}, \mathcal{B}^*(j), \mathcal{B}^*(h), \{j, h\} = \{2, 3\}, B(x_1), B(x_2)$ is as before and if $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric, then the binomials $x_0^{q_j - q_1} x_j^{q_1} - x_1^q, x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}}, x_j^{\alpha_{hj}}$ are as in the previous paragraphs. For the convenience of the reader we restate:

Theorem 4.1 [B,Sch,St]. Assume $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 4$. Then $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces) iff either

- (1) $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are symmetric of the same type with $\mathcal{B}^*(h) = \{x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}\}, h \in \{2, 3\}$, or
- (ii) $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are both not symmetric, but $\deg(B(x_1)) = \deg(B(x_2))$.

In case (i) $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on binomials in $\mathcal{B}$, in case (ii) it is not, but is a q.c.i. on two binomials and $B(x_1) + B(x_2)$.

Combining Theorems 3.1, 3.2 and 4.1 we have:

Theorem 4.2. Assume $\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 4$. Then $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces) iff either

- (i) $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are symmetric of the same type and $\mathcal{B}^*(h) = \{x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}\}, h \in \{2, 3\}$, or
- (ii) exactly one of $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ is symmetric, (w.l.o.g. assume $\langle n_1, n_2, n_3 \rangle$ is (n_1, n_j) -symmetric), $\mathcal{B}^*(h) = \{x_0^{\alpha_{h0}} x_h^d - x_1^{\alpha_{h1}} x_j^{\alpha_{hj}}\}, h \in \{2, 3\}$, and $\deg(B(x_1)) = \deg(B(x_2))$, or

(iii) $\mu(\mathfrak{p}(n_1, n_2, n_3)) = 4$, $\langle n_1, n_2, n_3 \rangle$ and $\langle n_3 - n_2, n_3 - n_1, n_3 \rangle$ are both not symmetric, but $\deg(B(x_1)) = \deg B((x_2))$.

In case (i) $\mathfrak{p}(n_1, n_2, n_3)$ is a q.c.i. on binomials, in cases (ii) and (iii) it is not, but is a q.c.i. on two binomials and $B(x_1) + B(x_2)$.

Acknowledgement. The first named author was supported by the "Naturwissenschaftlich-Theoretisches Zentrum" at the University of Leipzig. For this he wishes to express his thanks and appreciation.

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