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Quasi-complete intersections of monomial curves in projective three-space

by

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Abstract:
We classify completely all quasi-complete intersections of monomial curves in $\mathbb{P}_K^3$, $K$ an infinite field, see Theorem 4.1 and Theorem 4.2. This completes the investigations started in [B,Sch,St].

1. Introduction and preliminaries

The subject matter under discussion are quasi-complete intersections on three surfaces (q.c.i.) of monomial curves $C(n_1, n_2, n_3)$ in $\mathbb{P}_K^3$, $K$ an infinite field, with generic zero $(t_0^{n_3}, t_0^{n_3-n_1}t_1^{n_1}, t_0^{n_3-n_2}t_1^{n_2}, t_1^{n_3})$, $n_1 < n_2 < n_3$ positive integers with $g.c.d.(n_1, n_2, n_3) = 1$. $p(n_1, n_2, n_3) \subseteq K[x_0, \ldots, x_3] =: R$ will denote the prime ideal of $C(n_1, n_2, n_3)$, that is the kernel of the substitution homomorphism $\varphi : K[x_0, \ldots, x_3] \rightarrow K[t_0, t_1]$, $x_i \mapsto t_0^{n_3-n_i}t_1^{n_i}$, $0 \leq i \leq 3$, $n_0 := 0$.

There are several equivalent definitions for $p(n_1, n_2, n_3)$ to be a q.c.i. on $\{F_1, F_2, F_3\}$, $F_i$ homogeneous polynomials, $1 \leq i \leq 3$ (see [St], [P.S], [B,Sch,St]). The definition we will use is:

**Definition 1.1.** $p(n_1, n_2, n_3)$ is a q.c.i. on $\{F_1, F_2, F_3\}$ if $p(n_1, n_2, n_3)|_{x_i=1} = (F_1|_{x_i=1}$, $F_2|_{x_i=1}, F_3|_{x_i=1})R_i$, $0 \leq i \leq 3$. Here $R_i := K[x_0, \ldots, x_i, \ldots, x_3]$ and the image of a set or an element under the substitution homomorphism $R \rightarrow R_i$ given by $x_j \rightarrow x_j$, $j \neq i$, $x_i \rightarrow 1$, $0 \leq i \leq 3$, is denoted by placing $|x_i=1$ behind the set or element.

We list next some prerequisites, needed for an understanding of the proofs and results. Important to our investigation is the algorithm in [B,R] (see also [B,C,F,H]), which obtains a minimal generating set $B$ of binomials for $p(n_1, n_2, n_3)$. It starts from a minimal generating set of binomials for $p(n_1, n_2, n_3)|_{x_0=1}$, which is either two or three in
number [H]. The distinction between $\mu(p(n_1, n_2, n_3)|_{x_0=1})=2$ and $\mu(p(n_1, n_2, n_3)|_{x_0=1})=3$ ($\mu$ denotes the minimum number of generators) can be arithmetically interpreted by considering the so called symmetric property of the numerical semigroup $(n_1, n_2, n_3) := \{z|z := \sum_{i=1}^{3} z_in_i, z_i \text{ nonnegative integers } \}$. There are again several equivalent definitions for a numerical semigroup $(n_1, n_2, n_3)$ to be symmetric. Our definition here is:

**Definition 1.2.** Let $(i, j, h) := \{1, 2, 3\}$, $(n_1, n_2, n_3)$ is $(n_i, n_j)$-symmetric if $n_i = q_i d$, $n_j = q_j d$, $d = g.c.d.(n_i, n_j)$ and $n_h = \alpha_i q_i + \alpha_j q_j \in \langle q_i, q_j \rangle := \{z|z = z_i q_i + z_j q_j, \ z_i \text{ and } z_j \text{ nonnegative integers } \}$. If no specific identification with the integers $n_i, n_j$ is needed, then $(n_1, n_2, n_3)$ is said to be symmetric, if it is $(n_i, n_j)$-symmetric for some $i, j \in \{1, 2, 3\}$, $i \neq j$. If $(n_1, n_2, n_3)$ is $(n_1, n_j)$-symmetric and $(n_3 - n_2, n_3 - n_1, n_3)$ is symmetric, then we say that $(n_1, n_2, n_3)$ and $(n_3 - n_2, n_3 - n_1, n_3)$ are symmetric of the same type, if for $j = 2$ $(n_3 - n_2, n_3 - n_1, n_3)$ is $(n_3 - n_2, n_3 - n_1)$-symmetric, or for $j = 3$ $(n_3 - n_2, n_3 - n_1, n_3)$ is $(n_3 - n_2, n_3)$-symmetric.

It follows from [H], that $p(n_1, n_2, n_3)|_{x_0=1}$ is minimally generated by $G_1 := \{g_{ij} := x_0^{q_i} - x_1^{q_i}, g_h := x_h^{d} - x_i^{\alpha_i} x_j^{\alpha_j} \}$ if $(n_1, n_2, n_3)$ is $(n_i, n_j)$-symmetric and $G_2 := \{f_i := x_i^{\alpha_i} - x_j^{\alpha_j} x_h^{\alpha_h}, f_j := x_j^{\alpha_j} - x_i^{\alpha_i} x_h^{\alpha_h}, f_h := x_h^{\alpha_h} - x_i^{\alpha_i} x_j^{\alpha_j}, \alpha_i, \alpha_j, \alpha_h \text{ minimal and all } exponents \text{ positive, if } (n_1, n_2, n_3) \text{ is not symmetric. We will refer to } g_{ij}, g_h, \text{ respectively } f_i, f_j, f_h, \text{ as the canonical binomial generators of } p(n_1, n_2, n_3)|_{x_0=1}. \text{ They are uniquely determined, up to multiplication by } -1, \text{ provided for } n_i < n_j, \alpha_i < q_j \text{ in } G_1, \text{ which we will assume from now on.}

The algorithm in [B.R] now starts with either the homogenized binomials in $G_1$ or $G_2$ and produces a minimal generating set $B$ of $p(n_1, n_2, n_3)$ as follows.

**Case 1.** $(n_1, n_2, n_3)$ is $(n_i, n_j)$-symmetric

a) $(i, j) = \{2, 3\}$. In this case $p(n_1, n_2, n_3) = (x_0^{q_3} - x_3^{q_3} x_2^{q_2} - x_2^{q_2} x_1^{d} - x_0^{\alpha_0} x_2^{\alpha_2} x_3^{\alpha_3})R$, $d := g.c.d.(n_2, n_3)$.

b) $(i, j) \neq \{2, 3\}$. W.l.o.g. assume $n_i = n_1$. We obtain a sequence of binomials as follows. We start with $B_{j_1} := x_0^{q_i} - x_1^{q_1} x_2^{q_1} =: m_0 - m_1$, $B_{h_1} := x_0^{\alpha_0} x_2^{d} - x_1^{\alpha_1} x_2^{\alpha_1} =: m_0 - m_1'$, $d := g.c.d.(n_1, n_j)$, $n_1 = q_1 d$, $n_j = q_2 d$, $\alpha_{h_1} < q_j$. If $\alpha_{h_0} = 0$, then $h = 2$, if $\alpha_{h_1} = 0$, then $j = 2$. In both cases $p(n_1, n_2, n_3) = (B_{j_1}, B_{h_1})R$. Otherwise we cross multiply the mon-
mial terms of $B_{j_1}$ and $B_{h_1}$ and cancel in the resulting binomial $m_0m'_1-m'_0m_1$ the highest common monomial term in $x_0$ and $x_1$. We obtain a binomial $B_{j_2}$ (with lower $x_1$-exponent than in $B_{j_1}$) in $\mathcal{P}(n_1,n_2,n_3)$. We also will say $B_{h_1}$ acts on $B_{j_1}$ to produce $B_{j_2}$. If $B_{j_2}$ has a pure power in $x_2$ as a monomial term, then the process stops. If this is not the case and the $x_1$-exponents of $B_{j_2}$ and $B_{h_1}$ are not equal, then $B_{h_1}$ acts on $B_{j_2}$ to produce $B_{j_3}$ (in case the $x_1$-exponent in $B_{h_1}$ is smaller than the $x_1$-exponent in $B_{j_2}$) or $B_{j_2}$ acts on $B_{h_1}$ to produce $B_{h_2}$. In case of equal $x_1$-exponents in $B_{j_2}$ and $B_{h_1}$, the above process of cross multiplying the monomial terms of $B_{j_2}$ and $B_{h_1}$ and cancelling the highest common monomial term in $x_0$ and $x_1$, results in a binomial with a pure power term in $x_2$ in $\mathcal{P}(n_1,n_2,n_3)$. We call this the binomial $B_{j_3}$ for $j=2$ and $B_{h_2}$ for $h=2$ and the algorithm terminates. Iterating this procedure results in two well defined sequences $B_{j_1},\ldots,B_{jr_0}$ and $B_{h_1},\ldots,B_{hs_0}$. We set \{B_{j_1},\ldots,B_{jr_0}\} = $\mathcal{B}(j)$ and \{B_{h_1},\ldots,B_{hs_0}\} = $\mathcal{B}(h)$. The binomial with the pure power in $x_2$, which ends the algorithm, is the binomial in $\mathcal{B}(2)$ with largest second subscript. Now $\mathcal{B} = \mathcal{B}(j) \cup \mathcal{B}(h)$.

**Case 2.** $(n_1,n_2,n_3)$ is not symmetric.

a) Assume that the homogenized binomials $B_{11}, B_{21}, B_{31}$ in $G_2$ are $B_{11} := x_1^{a_1} - x_0^{a_{10}}x_2^{a_{12}}x_3^{a_{13}}, B_{21} := x_2^{a_3} - x_0^{a_{20}}x_1^{a_{21}}x_3^{a_{23}}, B_{31} := x_0^{a_{30}}x_3^{a_3} - x_1^{a_{31}}x_2^{a_{32}}$. Then $\mathcal{P}(n_1,n_2,n_3)$ is minimally generated by these elements.

b) Not a). Then $B_{21} = x_0^{a_{20}}x_2^{a_3} - x_1^{a_{21}}x_3^{a_{23}}$ and with $B_{21}$ and $B_{31} = x_0^{a_{30}}x_3^{a_3} - x_1^{a_{31}}x_2^{a_{32}}$ we proceed as in Case 1. to obtain two sets $\{B_{21},\ldots,B_{2r_0}\} = \mathcal{B}(2), \{B_{31},\ldots,B_{3s_0}\} = \mathcal{B}(3)$ with $B_{2r_0}$ the binomial with a pure power in $x_2$ ending the algorithm. Then $\mathcal{B} = \{B_{11}\} \cup \mathcal{B}(2) \cup \mathcal{B}(3)$ is a minimal generating set of $\mathcal{P}(n_1,n_2,n_3)$.

If in the above $B_{2r_0}$ is the binomial with the pure power in $x_2$, then $B_{3s_0}$ and $B_{2r_0}$ are said to be the last two elements of the algorithm, $B_{2,r_0-1}, B_{3s_0}, B_{2r_0}$ the last three elements. An analogous statement is obtained if $B_{2r_0}$ is the binomial with the pure power in $x_2$. We now also have, that if we permute the variables $x_0 \leftrightarrow x_3, x_1 \leftrightarrow x_2$, then the corresponding set $\overline{\mathcal{B}}$ becomes a minimal generating set for $\mathcal{P}(n_3-n_2,n_3-n_1,n_3)$, with the order in which the corresponding binomials appear reversed. This means that the last two, respectively the last three, elements, obtained in the algorithm for $\mathcal{B}$, become, by setting $x_3 = 1$, a minimal
binomial generating set for $p(n_1, n_2, n_3)|_{x_3=1}$.

We collect the above information in

**Lemma 1.3.** (i) If $\{i,j\} = \{2,3\}$ and $(n_1, n_2, n_3)$ is $(n_i, n_j)$-symmetric, then $\mu(p(n_1, n_2, n_3)) = 2$.

(ii) If $j \in \{2,3\}$ and $(n_1, n_2, n_3)$ is $(n_1, n_j)$-symmetric, then $B_{j1} = x_0^{q_1} - x_j^{q_1} - x_1^{q_i}$, $B_{h1} = x_0^{\alpha_0} x_h^d - x_1^{\alpha_1} x_j^{\alpha_j}$, where $d := g.c.d.(n_1, n_j)$.

If $\alpha_0 = 0$, then $h = 2$ and $\mu(p(n_1, n_2, n_3)) = 2$.

If $\alpha_1 = 0$, then $j = 2$ and $\mu(p(n_1, n_2, n_3)) = 2$

(iii) If $j \in \{2,3\}$ and $(n_1, n_2, n_3)$ is $(n_1, n_j)$-symmetric, then the $x_k$-exponents in all binomials constructed above are multiples of $d := g.c.d.(n_1, n_j)$.

(iv) In each of $B(j)$ and $B(h)$ the $x_0$ and $x_1$-exponents are decreasing, the $x_2$ and $x_3$-exponents increasing sequentially with increasing second subscripts. If $(n_1, n_2, n_3)$ is not symmetric, then $B_{11}$ has the largest $x_0$ and $x_1$-exponents in $B$.

**Proof.** All of (i)-(iv) are an immediate consequence of the previous and [H].

Finally of importance is Theorem 1.1, Theorem 1.3 and Lemma 1.1 in [B,Sch,St]. Theorem 1.1 gives three necessary and sufficient conditions for a q.c.i. in terms of height, local cohomology and the degrees of the surfaces to be taken. For our purposes here, Theorem 1.3 says that if $p(n_1, n_2, n_3)$ is a q.c.i. on $\{F_1, F_2, F_3\}$, then, say $F_1, F_2$, can be taken to be binomials in $B$, one of them of lowest degree and the other (as it turns out) of next lowest degree. Lemma 1.1 states that if $\mu(p(n_1, n_2, n_3)) \geq 3$, then $F_1, F_2, F_3$ are irreducible.

If one allows complete intersections to be a special type of q.c.i., then for all $p(n_1, n_2, n_3)$ such that $\mu(p(n_1, n_2, n_3)) \leq 3$, $p(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces). Thus of interest is, if $\mu(p(n_1, n_2, n_3)) \geq 4$. The description of all $p(n_1, n_2, n_3)$ with $\mu(p(n_1, n_2, n_3)) = 4$ and $p(n_1, n_2, n_3)$ a q.c.i. was obtained in [B, Sch, St], where it was also shown that $p(n_1, n_2, n_3)$ is not always a q.c.i. on binomials. In view of [B, Sch, St] therefore the assumption $\mu(p(n_1, n_2, n_3)) \geq 5$ is not restrictive.
2. Prerequisite lemmata and corollaries

We start with

Lemma 2.1. Assume \( (n_1, n_2, n_3) \) is not symmetric and \( p(n_1, n_2, n_3)|_{x_0=1} = (f_i, f_j, f_h)R_0 \), \( \{i,j,h\} = \{1,2,3\} \). If \( \{b, f, g\} \) is a minimal generating set of \( p(n_1, n_2, n_3)|_{x_0=1} \) and \( b \) is a monic binomial, then \( \pm b \in \{f_i, f_j, f_h\} \). An analogous statement is true for \( p(n_1, n_2, n_3)|_{x_0=1} \) if \( (n_2 - n_2, n_3 - n_1, n_3) \) is not symmetric.

Proof. \( \{f_i, f_j, f_h\} \) is a minimal homogeneous generating set of \( a := p(n_1, n_2, n_3)|_{x_0=1} \) with respect to the grading \( \deg(x_i) = n_i \) of \( R_0, 1 \leq i \leq 3 \). Therefore \( \{f_i, f_j, f_h\} \) is a minimal generating set of the extended ideal \( a^e \subseteq K[[x_1, x_2, x_3]] \). Thus \( \{b, f, g\} \) is also a minimal generating set of \( a^e \). Let \( M \) and \( M^* \) be any \( 3 \times 3 \) matrices with polynomial entries and with the property that \( (b, f, g)^T = M(f_i, f_j, f_h)^T \) and \( (f_i, f_j, f_h)^T = M^*(b, f, g)^T \). Then \( (MM^* - E_3)(b, f, g)^T = (0, 0, 0)^T, E_3 \) the \( 3 \times 3 \) identity matrix. Since \( \{b, f, g\} \) is a minimal generating set of \( a^e \subseteq K[[x_1, x_2, x_3]] \), this implies \( (MM^* - E_3)\mod(m := (x_1, x_2, x_3)R_0) \) is the \( 3 \times 3 \) 0-matrix. Hence every row of \( M \) has an element not in \( m \). If \( M = ((r_{ij})) \), since \( b, f, f, f_h \) are homogeneous, we may assume \( r_{11}, r_{12} \) and \( r_{13} \) to be homogeneous. W.l.o.g. assume \( r_{11} \not\in m \), thus \( r_{11} \) is a constant \( c \not= 0 \). By the assumptions on the exponents in \( f_i, f_j, f_h \), the term \( c_1 x_1^{\alpha_1} \) does not cancel. Since \( b \) is assumed monic, we conclude \( \pm x_1^{\alpha_1} \) is a monomial term of \( b \). But the monic binomial in \( a \) with one of its monomial terms \( \pm x_1^{\alpha_1} \) is uniquely determined up to multiplication by \( \pm 1 \) and is equal to \( \pm f_i \). Therefore \( \pm b \in \{f_i, f_j, f_h\} \).

Corollary 2.2. If \( \{b_1, b_2, f\} \) is a minimal generating set of \( p(n_1, n_2, n_3)|_{x_0=1} \), \( b_1, b_2 \) monic binomials, then for some set \( \{i, j\} \subseteq \{i, j, h\} = \{1,2,3\} \), \( \pm b_1 = f_i, \pm b_2 = f_j \).

Proof. The existence of \( \{i, j\} \) such that \( \pm b_1 = f_i, \pm b_2 = f_j \) follows by applying Lemma 2.1. twice.

Corollary 2.3. Assume \( (n_1, n_2, n_3) \) and \( (n_3 - n_2, n_3 - n_1, n_3) \) are not symmetric and \( \mu(p(n_1, n_2, n_3)) \geq 5 \). Then \( p(n_1, n_2, n_3) \) is not a q.c.i.
Proof. Suppose \( p(n_1, n_2, n_3) \) is a q.c.i.. Then by Theorem 1.3 [B, Sch, St] we may assume that two of the elements on which \( p(n_1, n_2, n_3) \) is a q.c.i. are binomials in \( B \), which, by Lemma 2.1, become part of the canonical generating set \( \{ f_i, f_j, f_k \} \) (up to multiplying by \(-1\)) of \( p(n_1, n_2, n_3)_{x_3 = 1} \) and an analogous statement is true for \( p(n_1, n_2, n_3)_{x_3 = 1} \). Thus the first and last three elements in \( B \) have two elements in common, from which \( \mu(p(n_1, n_2, n_3)) \leq 4 \), a contradiction. \( \blacksquare \)

Remark 2.4. If \( (n_1, n_2, n_3) = (n_1, n_2) \)-symmetric and neither \( n_1 \) nor \( n_2 \) equals \( n_1 \), then by Lemma 1.3 \( \mu(p(n_1, n_2, n_3)) = 2 \). Thus w.l.o.g. we may assume \( n_1 = n_2 \) and \( \{ n_2, n_3 \} \).

Lemma 2.5. Assume \( (n_1, n_2, n_3) \) is \( (n_1, n_2) \)-symmetric, \( \mu(p(n_1, n_2, n_3)) \geq 4 \) and \( \{ B_1, B_2, F \} \subseteq p(n_1, n_2, n_3) \) is such that \( B_1, B_2 \) are irreducible monic binomials and \( \{ B_1_{x_3 = 1}, B_2_{x_3 = 1}, F_{x_3 = 1} \} \) is a generating set for \( p(n_1, n_2, n_3)_{x_3 = 1} \). Then one of \( B_1_{x_3 = 1}, B_2_{x_3 = 1} \) is a canonical generator of \( p(n_1, n_2, n_3)_{x_3 = 1} \) (up to multiplication by \(-1\)). An analogous statement holds for \( p(n_1, n_2, n_3)_{x_3 = 1} \) if \( (n_3 - n_2, n_3 - n_1, n_3) \) is symmetric.

Proof. It is convenient and necessary for later considerations to rewrite

\[ B = B^*(j) \cup B^*(h), \quad B^*(j) := \{ x_0^{i-j_1} x_j^{j_1} - x_1^{j_1} =: B_{j_1}, x_0^{(q_j - q_1) - \alpha_h} x_j^{\alpha_h} - x_1^{q_j - \alpha_h} x_k^{\alpha_h} =: B_{j_k}, \ldots, B_{j_{l(j)}} \}, \quad B^*(h) := \{ x_0^{\alpha_h} x_h^{\alpha_h} - x_1^{\alpha_h} x_j^{\alpha_j} =: B_{h_1}(\alpha_h; 0 < \alpha_h < q_j, \alpha_h > 0), B_{h_2}, \ldots, B_{h_{l(h)}} \} \].

where \( B(x_2) \), the binomial with a pure power in \( x_2 \) obtained by our algorithm, is in \( B^*(j) \) for either \( j = 2, h = 3 \) or \( j = 3, h = 2 \). Thus the reassignment of \( B(x_2) \) is possibly the only change in \( B \) as constructed in the previous section. Its purpose is to obtain \( B^*(h) \) as a set of binomials, where each monomial term is divisible by exactly two different variables. Let \( \text{deg} \) be the weighted degree with \( \text{deg}(x_0) = 0, \text{deg}(x_1) = n_1, 1 \leq i \leq 3 \). Since the exponents of \( x_j \) in \( B^*(j) \) and \( x_h \) in \( B^*(h) \) are increasing, \( \text{deg}(B_{j_1}) < \text{deg}(B_{j_{i+1}}), 1 \leq i \leq l(j) - 1, \) in \( B^*(j) \) and \( \text{deg}(B_{h_1}) < \text{deg}(B_{h_{i+1}}), 1 \leq i \leq l(h) - 1, \) in \( B^*(h) \). (This includes \( B(x_2) \) in either case.). Thus \( \min\{\text{deg}(B_{j_1}), \text{deg}(B_{h_1})\} \) is the smallest weighted degree in \( B \).

Also since \( q_j - \alpha_k > 0 \), \( d_n = \text{deg}(B_{j_1}) = \text{deg}(B_{j_2}) = d_n + (q_j - \alpha_k) n_1 < \text{deg}(B_{j_i}), i \geq 3 \).

If \( B_{h_2} \in B^*(h) \), then \( B_{h_2} = x_0^{(\nu + 1)\alpha_h - (q_j - q_1) \alpha_h} x_j^{(\nu + 1) \alpha_j - q_j} = x_1^{\alpha_h} x_j^{\alpha_j}. \) Therefore \( q_1 n_j < \text{deg}(B_{j_1}) < \text{deg}(B_{h_2}) = ((\nu + 1)\alpha_h + q_1) n_j + ((\nu + 1)\alpha_h - q_j)n_1 < \text{deg}(B_{h_1}), i \geq 3. \)
From this \( \hat{\text{deg}}(B_{j1}) < \hat{\text{deg}}(B_{st}) \) for \( s=j, 2 \leq t \leq l(j) \), and for \( s=h, 2 \leq t \leq l(h) \), \( \hat{\text{deg}}(B_{h1}) < \hat{\text{deg}}(B_{st}) \) for \( s=j, 2 \leq t \leq l(j) \), and for \( s=h, 2 \leq t \leq l(h) \), \( \hat{\text{deg}}(B_{h1}) < \hat{\text{deg}}(B_{st}) \) for \( s=j, 2 \leq t \leq l(j) \), and for \( s=h, 2 \leq t \leq l(h) \). Next \( \hat{\text{deg}}(B_{j1}) \neq \hat{\text{deg}}(B_{h1}) \). Otherwise \( q_1 n_j = q_j n_1 = d n_h \).

If \( h=2, j=3 \), then \( x_2^j - x_0^j x_1^j \in \mathfrak{p}(n_1, n_2, n_3) \), if \( h = 3, j = 2 \), then \( x_3^j - x_2^j - x_2^j \in \mathfrak{p}(n_1, n_2, n_3) \). In both cases \( \mu(\mathfrak{p}(n_1, n_2, n_3)) = 2 \), a contradiction. Assume therefore \( \hat{\text{deg}}(B_{r1}) < \hat{\text{deg}}(B_{t1}) \) with \( (r, t) \in \{(j, h), (h, j)\} \). Set \( b_{r1} := B_{r1}|_{x_0=1}, b_{t1} := B_{t1}|_{x_0=1} \). Then \( (b_{r1}, b_{t1}) R_0 = \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} \) by our assumption. Suppose \( \{ \pm B_1, \pm B_2 \} \cap \{ B_{j1}, B_{h1} \} = \emptyset \). Then \( \hat{\text{deg}}(B_1) > \hat{\text{deg}}(B_{t1}), \hat{\text{deg}}(B_2) > \hat{\text{deg}}(B_{t1}) \), since, up to multiplication by \( \pm 1, B_{j1} \) and \( B_{h1} \) are uniquely determined amongst monic irreducible binomials.

Since \( \{ B_1|_{x_0=1} = : b_1, B_2|_{x_0=1} = : b_2, F|_{x_0=1} = : f \} \) generate \( \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} \), at least one generator must have a nonzero component of lowest degree, thus \( f \) must have a nonzero summand \( c_0 b_{r1}, c_0 \neq 0, c_0 \in K \). This means if \( f \) is written as a linear combination of elements in \( B|_{x_0=1} \), then \( f = f' + c_0 b_{r1}, c_0 \neq 0, \hat{\text{deg}}(m') > \hat{\text{deg}}(b_{t1}) \) for all monomial terms \( m' \) of \( f' \). We also must have:

\[(*) \quad b_{t1} := p_1 b_1 + p_2 b_2 + p_3 f.
\]

If \( p_3 \) does not have a nonzero constant term \( c_1 \), then we obtain the weighted homogeneous equation \( b_{t1} = p'_3 b_{r1} \) for a suitable \( p'_3 \), which is impossible since \( \mu(\mathfrak{p}(n_1, n_2, n_3)|_{x_0=1}) = 2 \).

But for \( c_1 \neq 0 \) in \( p_3 \), the right hand side in \( (*) \) always has a noncancelling component of smaller degree than the left side, again a contradiction. Thus \( \{ \pm B_1, \pm B_2 \} \cap \{ B_{j1}, B_{h1} \} \neq \emptyset \), which by the uniqueness property of the canonical binomial generators (up to sign) implies the statement of the lemma.

**Lemma 2.6.** Let \( G^h := \{ B_{11}, B_{21}, B_{31} \} \), where \( B_{11}|_{x_0=1}, B_{21}|_{x_0=1}, B_{31}|_{x_0=1} \), are the canonical binomial generators for \( \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} \) if \( (n_1, n_2, n_3) \) is not symmetric, and let \( G^h := \{ B_{j1}, B_{h1} \}, \{ j, h \} = \{ 2, 3 \} \), where \( B_{j1}|_{x_0=1}, B_{h1}|_{x_0=1} \) are the canonical binomial generators of \( \mathfrak{p}(n_1, n_2, n_3)|_{x_0=1} \) if \( (n_1, n_2, n_3) \) is \( (n_1, n_j) \)-symmetric. Assume \( B := m_0 - m_1 \in G^h \), \( r \in \{ 1, 2 \} \) and \( B' := m_0 - m'_1 \in B \setminus G^h \). Then there exists a variable \( x_s, s \in \{ 1, 2, 3 \} \) such that \( x_s^p | m'_1 \), but \( x_s^p | m_0, p_1 \geq 1 \). The same type relation (perhaps for a different variable) exists between \( m'_1 \) and \( m_1' \), and \( m_0, m'_1 \) and \( m_1 \).

**Proof.** A tedious proof follows by checking different cases. A short proof ensues by ob-
serving that \( \mathcal{B} \) forms a reduced and normalized Gröbner basis for \( \mathfrak{p}(n_1, n_2, n_3) \) with respect to the graded lexicographical term order, where \( x_2 \) is the largest linear term \([B,C]\). This proves the above statement for the monomials not divisible by \( x_0 \). For the other monomials the proof is obtained by observing that the \( x_2 \) and \( x_3 \)-exponents are increasing throughout the algorithm. \( \blacksquare \)

Definition 2.7. Let \( p \in K[x_0, \ldots, x_n] \). Then \( \text{supp}(p) := \\{ (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{n+1} | cx_0^{\alpha_0} \cdots x_n^{\alpha_n} \) is a monomial term of \( p \) with \( c \neq 0 \} \) is called the support of \( p \). For \( S \subseteq K[x_0, \ldots, x_n] \) we define \( \text{supp} \ S := \bigcup_{p \in S} \text{supp} (p) \).

Corollary 2.8. Let \( \mathcal{B} \) be as in the statement of Lemma 2.6. Then \( \text{supp} \ \{ x_0^{\rho_0} B | \rho_0 \geq 0 \} \cap \text{supp} \ \{ (\mathcal{B} \setminus G^*_k) R \} = \emptyset. \)

Proof. This is immediate by Lemma 2.6. \( \blacksquare \)

Corollary 2.9. Assume \( \{ B_1, B_2, B_3 \} \subseteq \mathcal{B} := \{ B_1, \ldots, B_l \} \) and \( F \in \mathfrak{p}(n_1, n_2, n_3) \) are such that \( (B_1, B_2, F) R |_{x_0=1} = \mathfrak{p}(n_1, n_2, n_3) |_{x_0=1} = (B_1, B_2, B_3) R |_{x_0=1} = (B_1, B_2, B_3) R |_{x_0=1} \subseteq \mathfrak{p}(n_1, n_2, n_3) |_{x_0=1} \) and w.l.o.g. \( F := \sum_{i=3}^l P_i B_i \), where \( P_i, 3 \leq i \leq l \), is homogeneous. Then \( P_3 = P_3' + c_0 x_0^{\rho_0}, c_0 \neq 0, P_3' \in (x_1, x_2, x_3) R \) and \( \text{supp} \ (c_0 x_0^{\rho_0} B_3) \cap \text{supp} \ (P_3' B_3, P_3 B_4, \ldots, P_l B_l) = \emptyset. \) If the above hypothesis holds for \( x_3=1 \), with binomials \( B_1, B_2 \) as before, \( B_s, s > 3, \) in \( \mathcal{B} \), and \( F \) as before, then \( P_3 = P_3' + c_3 x_3^{\rho_3}, c_3 \neq 0, \) with an analogous statement about the support.

Proof. That \( P_3 = P_3' + c_0 x_0^{\rho_0}, c_0 \neq 0, \) follows from Lemma 2.6 and the fact that \( B_3 |_{x_0=1} \) is the unique binomial which contains at least one minimal pure power of the variables \( x_1, x_2, x_3 \). It is easily checked that \( \text{supp} \ (c_0 x_0^{\rho_0} B_3) \cap \text{supp} \ (P_3' B_3) = \emptyset. \) Therefore Corollary 2.8 implies the statement about the support. The last statement now also follows from Corollary 2.8. \( \blacksquare \)

3. Two types of q.c.i. for monomial curves in \( \mathbb{P}_K^3 \).

Throughout this paragraph if \( \mathfrak{p}(n_1, n_2, n_3) \) is a q.c.i. on three surfaces \( \{ B_1, B_2, F \} \), then \( B_1, B_2 \) will denote binomials in \( \mathcal{B} \), in accordance with Theorem 3.1 in [B, Sch, St]. Also
always $\mu(p(n_1, n_2, n_3)) \geq 4$. Assume first $(n_1, n_2, n_3)$ is $(n_1, n_j)$-symmetric (i.e. $p(n_1, n_2, n_3)|_{x_0=1} = (x_0^{q_i-q_j} x_j^{q_j} - x_0^{q_j} x_j^{q_i})|_{x_0=1} = x_0^{\alpha_0} x_h^{\alpha_1} x_0^{\alpha_1} x_j^{\alpha_1}$) and $(n_3-n_2, n_3-n_1, n_3)$ is symmetric. We then have:

**Theorem 3.1.** $p(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces) iff $p(n_1, n_2, n_3)$ is a q.c.i. on three binomials in $B$. Also in this case if $p(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces), then

i) $(n_1, n_2, n_3)$ and $(n_3-n_2, n_3-n_1, n_3)$ are symmetric of the same type, and

ii) $B^*(h) = \{ x_0^{\alpha_0} x_h^{\alpha_1} x_0^{\alpha_1} x_j^{\alpha_1} \}$. 

**Proof.** $\iff$. This follows trivially.

$\Rightarrow$. Assume $p(n_1, n_2, n_3)$ is a q.c.i. on $\{B_1, B_2, F\}$. Let $B := \{B_1, B_2\}, B^0 := \{B_1^0, B_2^0\}$ be such that $\{B_1^0|_{x_0=1}, B_2^0|_{x_0=1}\}$ is a set of canonical generators of $p(n_1, n_2, n_3)|_{x_0=1}$ and define $B^3 := \{B_1^3, B_2^3\}$ analogously for $p(n_1, n_2, n_3)|_{x_3=1}$.

I. Assume $B^0 \cap B^3 \neq \emptyset$. For $B^0 = \{B_1^0 = B_{1h}, B_2^0 = B_{j1}\}$ (in our previous notation) $= x_0^{\alpha_0} x_h^{\alpha_1} x_0^{\alpha_1} x_j^{\alpha_1}$, $B_2^0 = B_{j1}$ (in our previous notation) $= x_j^{q_j} - x_i^{q_i} - x_1^{q_1} - x_i^{q_i}$, $\{h, j\} = \{2, 3\}$, $n_1 = q_i d, n_j = q_j d, d := \text{g.c.d.}(n_1, n_j)$, $x_0^{q_0} x_j^{q_j} - x_0^{q_0} x_j^{q_j} \not\in B^3$ (since by the algorithm of [B,R] this binomial always has a successor ($B_{j2}$ in the proof of Lemma 2.5), therefore is not one of the last two binomials in $B$). Therefore $B^0 \cap B^3 = \{ x_0^{\alpha_0} x_h^{\alpha_1} x_0^{\alpha_1} x_j^{\alpha_1} = B_1^3 \}$, where all exponents in $B_1^0$ are positive since $\mu(p(n_1, n_2, n_3)) \neq 2$. By Lemma 2.5, $B_1^0$ is one of $B_1$ or $B_2$. Let $B_1 := B_1^0$. It now follows:

$p(n_1, n_2, n_3)$ is a q.c.i. on $\{B_2 = B_2^0, B_1 = B_1^0 = B_1^3, B_2^3\}$, since $p(n_1, n_2, n_3)|_{x_1=1} = (B_2|x_1=1 = B_2^0|x_1=1, B_1|x_1=1 = B_1^0|x_1=1 = B_1^3|x_1=1) R_1$, respectively $p(n_1, n_2, n_3)|_{x_3=1} = (B_1|x_2=1 = B_1^0|x_2=1, B_2|x_2=1 = B_2^0|x_2=1) R_2$ (see $[B, C]$).

(i) All exponents in $B_1 = B_1^0 = B_1^3$ are positive. Thus if $j=2, h=3$, then $(n_3-n_2, n_3-n_1, n_3)$ is $(n_3-n_2, n_3-n_1)$-symmetric. If $j=3, h=2$, then $B(x_2) = x_0^{q_0} x_3^{q_3} x_2^{q_2}$, thus $(n_3-n_2, n_3-n_1, n_3)$ is $(n_3-n_2, n_3-n_1)$-symmetric. Therefore $(n_1, n_2, n_3)$ and $(n_3-n_2, n_3-n_1, n_3)$ are symmetric of the same type.

(ii) $B^*(h) = \{ B_1 = B_1^0 = B_1^3 \}$.

II. Suppose $B^0 \cap B^3 = \emptyset$. Then $p(n_1, n_2, n_3)$ is not a q.c.i. on three binomials, since if this
was the case, binomials being homogeneous in the weighted grading and since every homogeneous generating set contains a minimal generating set, this would imply that the set of binomials (on which \(p(n_1, n_2, n_3)\) is a q.c.i.) for \(x_0 = 1\) and \(x_3 = 1\) would contain a minimal set of generators, thus \(B^0 \cap B^3 \neq \emptyset\). We now show, that under the stated hypothesis, II cannot happen. We do this by obtaining:

1. The pure power binomials \(B(x_1)\) and \(B(x_2)\) in \(B\) are of equal degree.
2. \(p(n_1, n_2, n_3)\) is a q.c.i. on \(\{B_1, B_2, B(x_1) + cB(x_2)\}\) for suitable \(c \in K^*\).
3. \((n_1 - n_2, n_3 - n_1, n_3)\) are symmetric of the same type.
4. The assumption \(B^0 \cap B^3 = \emptyset\) entails a contradiction.

Since by Lemma 2.5 \(\{B_1, B_2\} \cap B^0 \neq \emptyset\) and \(\{B_1, B_2\} \cap B^3 \neq \emptyset\), assume \(B_1 =: B^0_1 \in B^0, B_2 =: B^3_1 \in B^3\). Then by Corollary 2.8., \(F\) (after possibly multiplying by a nonzero constant) has a nonzero summand \(c x_0^{n_0} B^0_1 + x_3^{n_3} B^3_1, c \neq 0\). Thus \(\deg(F) \geq \deg(B^0_1)\) and \(\deg(F) \geq \deg(B^3_1)\). If \(B^0_1 \not\subset q\) for all \(q \in \text{Ass}(R/(B_1, B_2)R), q \neq p(n_1, n_2, n_3)\), then by Theorem 1.1 in [B, Sch, St] and since \(p(n_1, n_2, n_3)|_{x_0 = 1} = (B_1, B_2, B^0_1)R|_{x_0 = 1}, p(n_1, n_2, n_3)\) is a q.c.i. on \(\{B_1, B_2, L^a B^0_1\}\), where \(L\) is a generic linear form and \(\rho := \deg(F) - \deg(B^0_1)\). By Lemma 1.1 in [B, Sch, St], we must have \(\rho = 0\), a contradiction since \(p(n_1, n_2, n_3)\) is not a q.c.i. on three binomials. Assume \(B^0_2 \in q, q \in \text{Ass}(R/(B_1 = B^0_1, B_2 = B^3_2)R), q \neq p(n_1, n_2, n_3)\). Since \((B_1 = B^0_1, B_2 = B^3_1)R\) is a complete intersection, \(q\) is minimal over \((B_1 = B^0_1, B_2 = B^3_1)R\), thus minimal over \((B_1 = B^0_1, B_2 = B^3_1, B^0_2)R\), hence \(q \in \text{Ass}(R/(B_1, B_2, B^0_1)R)\). Since \((B_1|_{x_0 = 1} = B^0_1|_{x_0 = 1}, B_2|_{x_0 = 1}, B^0_2|_{x_0 = 1})R_0 = p(n_1, n_2, n_3)|_{x_0 = 1, x_1 \in q ([Z, S], \text{Vol. II, Ch. VII, Theorem 18})}. From \(B^0 \subset q, x_0^{n_0} x_1^{n_1} - x_1^{n_1} \in q\), thus \(x_1 \in q\), hence \(q = (x_0, x_1)R\) since \((B_1 = B^0_1, B_2 = B^3_1)R\) is unmixed. Since \(B^3_1 \in q, x_0\) and \(x_1\) divide different monomial terms in \(B^3_1\) and therefore all four variables appear in \(B^3_1\) (a general property of the binomials in \(B\) except perhaps \(B(x_1)\)). Hence \(B^0_2\) is a binomial in exactly three variables, \(x_0\) and \(x_1\) divide the same monomial term of \(B^0_2\), thus \(B^0_2 \not\subset q\) and \(B^0_2 = B(x_2)\). Analogously we obtain \(x_2\) and \(x_3\) divide different monomial terms of \(B^0_2\), thus all four variables divide a monomial term of \(B^0_1\), hence \(B^0_2\) (up to sign) = \(x_0^{n_0} x_1^{n_1} - x_1^{n_1} = B(x_1)\), a binomial in exactly three variables. Assume first that \(B^0_2\) and \(B^3_2\) are of equal degree. Let \(q_i, 1 \leq i \leq n\), be the prime ideals of \(\text{Ass}(R/(B_1 = B^0_1, B_2 = B^3_1)R)\) with \(q_i \neq p(n_1, n_2, n_3)\) and \(B^0_2 \not\subset q_i\). There exist then
ζ_i ∈ \mathbb{P}_K^3, K the algebraic closure of K, ζ_i a zero of q_i, such that B^0_2(ζ_i) ≠ 0. Let c ∈ K such that c ≠ 0 and \frac{-B^0_2(ζ_i)}{B^1_2(ζ_i)} ≠ c, 1 ≤ i ≤ n, a possible choice since K is infinite. Then cB^0_2(ζ_i) + B^1_2(ζ_i) ≠ 0, thus cB^0_2 + B^1_2 \not∈ q_i, 1 ≤ i ≤ n. Thus cB^0_2 + B^1_2 \not∈ q for all q ∈ \text{Ass}(R/(B_1 = B^0_1, B_2 = B^1_2)R), q ≠ p(n_1, n_2, n_3). Let b = p = p(n_1, n_2, n_3) - primary component of p^2 + (B^0_1, B^1_1)R. We have B^0_2 \not∈ b if b ⊂ p, since then bR_p ⊂ pR_p and pR_p = (B^0_1, B^1_1)R_p. Let C^∗ = \{c^∗ ∈ L \mid c^∗B^0_2 + B^1_2 ∈ b\} and suppose c^∗_1 ∈ C^∗, c^∗_2 ∈ C^∗, c^∗_1 ≠ c^∗_2. This implies c^∗_1(c^∗_1B^0_2 + B^1_2) - c^∗_1(c^∗_2B^0_2 + B^1_2) = (c^∗_1 - c^∗_2)B^0_2 ∈ b, thus B^0_2 ∈ b and therefore B^0_2 ∈ b, a contradiction. Thus the cardinality of C^∗ is equal to or less than 1. Assume additionally to the above restrictions on c, c \not∈ C^∗. Also deg(F) ≥ cB^0_2 + B^1_2. By Theorem 1.1 in [B, Sch, St] therefore p(n_1, n_2, n_3) is a q.c.i. on \{B_1 = B^0_1, B_2 = B^1_2, L^0(cB^0_2 + B^1_2)\}, where:

a) L is a linear form which is "generic enough","n
b) B_1 = B^0_1 and B_2 = B^1_2 are such that all four variables divide some monomial term,

c) B^0_2 = B(x_1), B^1_2 = B(x_2) are binomials in exactly three variables,

d) c ≠ 0.

But then p(n_1, n_2, n_3) is also a q.c.i. on \{B_1 = B^0_1, B_2 = B^1_2, cB^0_2 + B^1_2\} (since (B_1 = B^0_1, B_2 = B^1_2, L^0(cB^0_2 + B^1_2))R ⊆ (B_1 = B^0_1, B_2 = B^1_2, cB^0_2 + B^1_2)R), which by the degree requirement of Theorem 1.1 in [B, Sch, St] is only possible if actually deg(F) = deg(cB^0_2 + B^1_2). Thus in this case p(n_1, n_2, n_3) is a q.c.i. on \{B_1 = B^0_1, B_2 = B^1_2, cB^0_2 + B^1_2\}. We show next that B^0_2 and B^1_2 must be of equal degree. Thus suppose B^0_2 and B^1_2 are of unequal degree. Assume first deg(B^0_2) < deg(B^1_2) and cB^0_2 + x_3^{p_3}B^3_2 is homogeneous with p_3 ≥ 1. As before if B^0_2 \in q, q ∈ \text{Ass}(R/(B_1 = B^0_1, B_2 = B^1_2)R), q ≠ p(n_1, n_2, n_3), then q = (x_0, x_1)R, thus x_3^{p_3}B^3_2 \not∈ q. Also, again as above, let c ∈ K, c ≠ 0, be such that cB^0_2 + x_3^{p_3}B^3_2 \not∈ q, q ∈ \text{Ass}(R/(B_1 = B^0_1, B_2 = B^1_2)R), q ≠ p(n_1, n_2, n_3), and cB^0_2 + x_3^{p_3}B^3_2 \not∈ b if b ⊂ p := p(n_1, n_2, n_3), where b is defined as above. Since deg(F) ≥ deg(B^0_2), we have again p(n_1, n_2, n_3) is a q.c.i. on \{B_1 = B^0_1, B_2 = B^1_2, cB^0_2 + x_3^{p_3}B^3_2\} (after first perhaps multiplying cB^0_2 + x_3^{p_3}B^3_2 by L^0, where L is a "generic enough" linear form). But this is impossible, since p(n_1, n_2, n_3)|_{x_2=1} has a binomial with a constant nonzero term, whereas the ideal \{B_1|_{x_2=1} = B^0_1|_{x_2=1}, B_2|_{x_2=1} = B^1_2|_{x_2=1}, cB^0_2|_{x_2=1} + x_3^{p_3}B^3_2|_{x_2=1}\}R_2 does not since p_3 ≥ 1 (c.f. [B, C]). The proof for deg(B^0_2) < deg(B^1_2) is analogous, since also x_0^{p_0}B^0_2 \not∈ b
if \( b \subset p = p(n_1, n_2, n_3) \). This proves 1. and 2.

Suppose next that \( (n_1, n_2, n_3) \) and \( (n_3 - n_2, n_3 - n_1, n_3) \) are not symmetric of the same type. W.l.o.g. assume \( (n_1, n_2, n_3) \) is \( (n_1, n_2) \)-symmetric and \( (n_3 - n_2, n_3 - n_1, n_3) \) is \( (n_3 - n_2, n_3) \)-symmetric. We have \( B_2^0 = x_0^{2 - q_1} x_1^{q_1} - x_1^{q_1}, B_0^1 = x_0^{\alpha_1} x_1^d - x_1^{\alpha_1} x_1^d, B_3^1 = x_0^{\beta_1} x_1^d - x_1^{\beta_1} x_1^d \), \( n_1 = q_1, n_2 = q_2, d = q.c.d.(n_1, n_2) \) and \( \text{deg}(B_2^0) = \text{deg}(B_2^1) = q_2 \). But then, since \( (n_3 - n_2, n_3 - n_1, n_3) \) is \( (n_3 - n_2, n_3) \)-symmetric, \( q_2 | n_3 \), say \( n_3 = q_2 d^* \). From \( B_0^0 |_{x_0 = 1} = 1 \), we obtain \( dq_2 d^* = d n_3 = \alpha_3 n_1 + \alpha_3 n_2 \). On the other hand \( dq_2 d^* = d^* n_2 \), which gives \( (d^* - \alpha_3) n_2 = \alpha_3 n_1 \), a contradiction since \( 0 < \alpha_3 < q_2 \). Therefore \( (n_1, n_2, n_3) \) and \( (n_3 - n_2, n_3 - n_1, n_3) \) must be symmetric of the same type, which proves 3.

We have now \( B_2^0 = x_0^{q_1 - q_1} x_1^{q_1} - x_1^{q_1}, B_0^1 = x_0^{\alpha_1} x_1^d - x_1^{\alpha_1} x_1^d, B_3^1 = x_0^{\beta_1} x_1^d - x_1^{\beta_1} x_1^d \) with \( \nu > 1 \) since \( B_0^0 \cap B_3^1 = \emptyset \).

\[
B_3^1 = \begin{cases} 
  x_0^{q_1} - x_1^{q_1} & x_1^{\alpha_1} = \mu d \\
  x_0^{q_1} - x_1^{q_1} & x_1^{\alpha_1} = \mu d 
\end{cases} \quad \text{if} \quad j = 2, h = 3, h = 2.
\]

(We note again that by Lemma 1.3 (iii) all \( x_k \)-exponents are multiples of \( d \).

Observe that \( \{ B_0^0 | x_0 = 1, B_0^0 | x_0 = 1 \} \) forms a Gröbner basis of \( p(n_1, n_2, n_3) | x_0 = 1 \) with respect to the lexicographical term order with \( x_1 \) as largest linear term. Since \( p(n_1, n_2, n_3) \) is a q.c.i. on \( \{ B_0^0, B_0^1, B_0^2 - c B_3^1 \} \). \( c \neq 0 \), we must have \( (x_1^d - x_1^{\alpha_1} x_1^{\beta_1} - x_1^{q_1} - x_1^{q_1})R_0 = p(n_1, n_2, n_3) | x_0 = 1 = (x_1^d - x_1^{\alpha_1} x_1^{\beta_1} - x_1^{q_1} - x_1^{q_1} + c B_3^1 | x_0 = 1)R_0 \). Let \( G = \{ x_1^d - x_1^{\alpha_1} x_1^{\beta_1} - x_1^{q_1} - x_1^{q_1} + c B_3^1 | x_0 = 1 \} \). The set \( G \) can be changed into a generating set \( \{ x_1^d - x_1^{\alpha_1} x_1^{\beta_1} - x_1^{q_1} - x_1^{q_1} + c g_1 \} \) where \( g_1 = x_1^d - x_1^{\alpha_1} x_1^{\beta_1} - x_1^{q_1} - x_1^{q_1} \), \( P, P \in K[x_1, x_1] \), \( \gamma_1 > 0, \gamma_2 > 0, g_2 := x_1^d - x_1^{\beta_1} + c g_2 \) for \( j = 2, h = 3 \); \( g := x_1^d - x_1^{\beta_1} + c g_1 \). Since \( B_0^0 | x_0 = 1, B_2^1 | x_0 = 1 \) is a Gröbner basis for \( p(n_1, n_2, n_3) \) in the specified term order and since the leading terms in \( g_1, g_2 \) are relatively prime, the Gröbner algorithm implies that we must have \( J_1 := (x_1^{q_1} x_1^{q_1} P, 1 + c x_1^{\alpha_1} Q_1)R_0 = R_0, i \in \{ 1, 2 \} \). But this is impossible since \( c \neq 0 \) and \( 1 + c x_1^{\alpha_1} Q_1 | x_0 = 0 \) is a polynomial in \( K[x_1] \) of degree > 0, thus has a zero \( \zeta_j \in K \). Therefore \( J_1 \) has a zero \( x_1 = 0, x_j = \zeta_j \) in \( K^2 \), thus \( J_1 \neq R_0 \).

This proves that \( \Pi \) as stated cannot happen, which completes the proof of Theorem 3.1. \( \blacksquare \)
Now we assume \((n_1, n_2, n_3)\) is \((n_1, n_j)\)-symmetric and \((n_3 - n_2, n_3 - n_1, n_3)\) is not symmetric. Thus again \(B^0 := \{B^0_1 = x^0_0 x^d_1 - x^0_1 x^d_0, B^0_2 = x^q_0 x^d_0 - x^d_0 x^q_1 \} \subseteq \mathcal{B}\) is such that \(\{B^0_1|_{x_0 = 1}, B^0_2|_{x_0 = 1}\}\) is a canonical set of binomial generators for \(\mathfrak{p}(n_1, n_2, n_3)|_{x_0 = 1}\). Similarly define \(B^3 := \{B^3_1, B^3_2, B^3_3\} \subseteq \mathcal{B}\) such that \(B^3_i|_{x_3 = 1}, 1 \leq i \leq 3\), are the canonical generators of \(\mathfrak{p}(n_1, n_2, n_3)|_{x_3 = 1}\). W.l.o.g. assume \(B^3_3 = B(x_2)\) has the pure power in \(x_2\).

We have:

**Theorem 3.2.** \(\mathfrak{p}(n_1, n_2, n_3)\) is a q.c.i. on \(\{B_1, B_2, F\} = B \cup \{F\}\), where \(B := \{B_1, B_2\}\) iff

(i) \(\text{deg}(B^0_2) = \text{deg}(B^3_2)\)

(ii) \(B^*(h) = \{B^0_1 = x^0_0 x^d_1 - x^0_1 x^d_0, B^3_2\}\).

In this case \(F\) can be taken to be \(cB^0_2 + B^3_3, c \neq 0\).

**Proof.** \(\Rightarrow\). Assume \(\mathfrak{p}(n_1, n_2, n_3)\) is a q.c.i. on \(\{B_1, B_2, F\}\), \(B_1, B_2\) binomials in \(\mathcal{B}\) in accordance with Theorem 3.1 in [B, Sch, St]. By Lemma 2.5 \(B \cap B^0 \neq \emptyset\). \(B^0_2 \notin B \cap B^0\), since by Corollary 2.2 \(B \subseteq B^3\) and no binomial in \(B^3\) has a pure power in \(x_1\) (since \(\mu(\mathfrak{p}(n_1, n_2, n_3)) \geq 4\)). Therefore \(B \cap B^0 = \{B^0_1 = x^0_0 x^d_1 - x^0_1 x^d_0\}\) thus \(B^0_1 \in B \subseteq B^3\).

W.l.o.g. \(B^0_1 =: B_1 =: B^3_1\). Note that \(F\) cannot be a binomial. For if \(F\) were a binomial, then since \(\{B_1 = B^0_1 = B^3_1|_{x_0 = 1}, B_2|_{x_3 = 1}, F|_{x_3 = 1}\}\) generates \(\mathfrak{p}(n_1, n_2, n_3)|_{x_0 = 1}, F\) would have to be \(B^0_2\). Here, after setting \(x_0 = 1\), we reason by way of homogeneous elements in the weighted grading and assume \(F\) to be irreducible by Lemma 1.1 in [B, Sch, St]. Since also \(\{B_1|_{x_3 = 1}, B_2|_{x_3 = 1}, F|_{x_3 = 1}\}\) is a minimal generating set of binomials for \(\mathfrak{p}(n_1, n_2, n_3)|_{x_0 = 1}\), also \(F \in B^3\), a contradiction as already observed. By Corollary 2.8 and 2.9 \(F\) has nonzero summand \(c x^{n_i}_0 B^0_2 + x^{n_3}_3 B^3_3, c \neq 0, i \in \{2, 3\}\). (Here \(B^3_1 \notin B_0\),) Therefore \(\text{deg}(F) \geq \text{deg}(B^0_2)\) and \(\text{deg}(F) \geq \text{deg}(B^3_3)\).

Let \(B_2 := B^3_s, s \in \{2, 3\}, s \neq i\). Assume \(B^0_2 \in q, q \in \text{Ass}(R/(B_1 = B^0_1, B_2 = B^3_3)R)\), \(q \neq \mathfrak{p}(n_1, n_2, n_3)\). Then since \(\{B^0_1|_{x_0 = 1}, B^0_1|_{x_0 = 1} = B_1|_{x_0 = 1}, B_2|_{x_0 = 1} = B^3_3|_{x_0 = 1}\}R_0 = \mathfrak{p}(n_1, n_2, n_3)|_{x_0 = 1}\), as in the proof of Theorem 3.1. \(x_0 \in q\), and since \(B^0_2 \in q, x_1 \in q\), thus \(q = (x_0, x_1)R\). Since \(B_2 = B^3_s \in q, x_0\) and \(x_1\) divide different monomials in \(B^3_s\), thus \(s = 2, i = 3\) and \(B^3_3 \notin q\) with

\[
B^3_3 = \begin{cases} 
 x^{2j}_j - x^{2j}_0 x^j_1, x^{2h}_h = \mu d \in B^*(j) \text{ if } j = 2, h = 3, \\
 x^{0n}_0 x^j_1 x^{hj}_j - x^{n_3}_3 \in B^*(j) \text{ if } j = 3, h = 2.
\end{cases}
\]
As in the proof of Theorem 3.1 we obtain \( \deg(B_0^2) = \deg(B_3^2) = \deg(F) \) and \( p(n_1, n_2, n_3) \) is a q.c.i. on \( \{B_1 = B_4, B_2 = B_3^2, cB_2^2 + B_3^3, c \neq 0\} \).

Suppose next \( B_2 = B_3^2 = x_0^{\sigma_0} x_h^d - x_1^{\sigma_1} x_j^{\sigma_j}, \nu > 1 \). We then obtain a contradiction to \( B_1|_{x_3 = 1} B_1^1|_{x_3 = 1} \) being a canonical binomial generator of \( p(n_1, n_2, n_3)|_{x_3 = 1} \), since for \( h = 2, j = 3 \) the \( x_1 \)-exponent in \( B_2 = B_3^2 \) is smaller than the \( x_1 \)-exponent in \( B_1 = B_1^1 = x_0^{\sigma_0} x_h^d - x_1^{\sigma_1} x_j^{\sigma_j} \) by Lemma 1.3 (iv), and for \( h = 3, j = 2 \) the same is true for the \( x_0 \)-exponents of the two binomials. We note here that for \( h = 2, j = 3 \) or \( h = 3, j = 2 \), the "missing" canonical generator for \( p(n_1, n_2, n_3)|_{x_3 = 1} \), i.e. the canonical generator whose homogenization is not used for the q.c.i., is obtained from \( B_3^3 \notin \mathcal{B} \). Thus \( B_2^2 \in \mathcal{B}^*(h) \) and \( \mathcal{B}^*(h) = \{B_1 = B_1^1 = B_1^2\} \).

\[ \iff \] We show that for \( p(n_1, n_2, n_3) \), with \( \langle n_1, n_2, n_3 \rangle (n_1, n_2) \)-symmetric and \( \langle n_3 - n_1, n_3 - n_3 - n_1, n_3 \rangle \) not symmetric, with \( \mathcal{B}^*(h) = \{B_1^0 = x_0^{\sigma_0} x_h^d - x_1^{\sigma_1} x_j^{\sigma_j} = B_3^1 \} \) and \( \deg(B_0^2 = B(x_1)) = \deg(B_3^2 = B(x_2)), p(n_1, n_2, n_3) \) is a q.c.i. on \( \{B_1^0, B_2^2, B_3^2 + B_3^3\} \). (For convenience we use \( c = 1 \), but the proof is valid for any \( c \neq 0 \).) Note that there is no ambiguity as to the binomial \( B_3^2 \). We make this clear by explicitly listing \( B \) subject to our hypothesis. We have: \( \mathcal{B}^*(h) = \{B_1^1 = x_0^{\sigma_0} x_h^d - x_1^{\sigma_1} x_j^{\sigma_j} = B_3^1 \} = B_{h1} \) (previously), \( \mathcal{B}^*(j) = \{x_0^{g_j} x_j^{d_j} = B_0^2 = B(0) = B_{h1} \) (previously), \( x_0^{(g_j - g_1)} - x_1^{(g_j - g_1)} \)\( x_j^{(g_j - g_1)} = B_3^2 \) =: B(1) = B_{h2} \) (previously), etc. 

\[ B_3^3 = \begin{cases} x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} - x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} x_1^{(e_1 - g_1)} & \text{for } j = 2, h = 3, \\ x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} x_1^{(e_1 - g_1)} x_1^{(e_1 - g_1)} - x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} x_1^{(e_1 - g_1)} & \text{for } j = 3, h = 2. \end{cases} \]

We show next that for \( \mathcal{G}^* := \{B_1^0 = B_3^1, B_2^2, B_3^2 + B_3^3\}, \mathcal{G}^*|_{x_i = 1} \) generates \( p(n_1, n_2, n_3)|_{x_i = 1} \), for all \( i \) such that \( 0 \leq i \leq 3 \), by showing that the needed canonical binomial generators are obtainable from the given elements in \( \mathcal{G}^*|_{x_i = 1} \). For this we need to obtain certain syzygies of elements in \( \mathcal{B} \). We do this by eliminating monomial terms of two binomials in \( \mathcal{B} \) (as listed above). These syzygies allow us to show that one of the two binomials in \( B_2^2|_{x_i = 1} + B_3^3|_{x_i = 1} \) is in \( \mathcal{G}^*|_{x_i = 1} \), thus defining a generating set.

1. Let \( j = 2, h = 3 \). Then \( x_j^{(e_j - e_1)} B_3^2 = x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} x_1^{(e_1 - g_1)} B_3^2 = x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} x_1^{(e_1 - g_1)} B_3^2 \). Similarly for \( j = 3, h = 2 \), \( x_1^{(e_1 - g_1)} x_j^{(e_j - e_1)} B_3^3 = x_0^{(g_j - g_1)} x_j^{(e_j - e_1)} x_1^{(e_1 - g_1)} B_3^3 \). This shows that \( (\mathcal{G}^*|_{x_0 = 1})R_0 = \)
\( p(n_1, n_2, n_3)|_{x_1=1} \).

2. Let \( j = 2, h = 3 \). Then \( x_0^{(\nu+1)\alpha \delta -(\eta - \eta_1)} x_1^d B_2^3 - x_1^{\alpha \delta_1} B_3^3 = x_0^{\gamma_1 + \nu \alpha \delta_1} B_1^0 \). For \( j = 3, h=2, x_0^d B_2^3 - x_1^{\nu \alpha \delta_1} B_3^3 = x_0^{(\eta - \eta_1) -(\nu+1)\alpha \delta -(\nu(\nu+1)) \alpha \delta_1} x_1^{\gamma_1 + \nu \alpha \delta_1} B_1^0 \). This means \((G^*)|_{x_1=1}) R_1 = p(n_1, n_2, n_3)|_{x_1=1}\) by [B,C].

3. Let \((j, h) = (2, 3)\) or \((j, h) = (3, 2)\). Let \( B(\nu - 1) := x_0^{(\eta - \eta_1)- (\nu-1) \alpha \delta} x_1^{\gamma_1 + \nu \alpha \delta_1} - x_1^{\nu \alpha \delta} x_{j-1}^{\nu(\nu+1) \alpha \delta_1} x_{j-1}^{\nu \alpha \delta_1} B_0^0 \) be the predecessor of \( B(\nu) = B_2^0 \) in \( B \). Then \( x_0^d B(\nu - 1) - x_0^{\alpha \delta_1} B_2^3 = x_0^{(\eta - \eta_1) -(\nu-1) \alpha \delta} x_1^{\gamma_1 + \nu \alpha \delta_1} B_0^0 \). Thus \( x_0^d B(\nu - 1) \in (G^*)R \). Inductively \( x_0^d B_2^0 \in (G^*)R \), \( B_2^0 = B(0) \), thus \((G^*)|_{x_1=1}) R_1 = p(n_1, n_2, n_3)|_{x_1=1} \). Finally \( x_1^{\alpha \delta_1} B(\nu - 1) - x_0^{\alpha \delta_1} B_2^3 = x_1^{\nu \alpha \delta_1} x_{j-1}^{\nu(\nu+1) \alpha \delta_1} \). Therefore \( x_1^{\alpha \delta_1} B(\nu - 1) \in (G^*)R \). Inductively \( x_1^{\alpha \delta_1} B_2^0 \in (G^*)R \), thus \((G^*)|_{x_1=1}) R_2 = p(n_1, n_2, n_3)|_{x_1=1} \). (Here \((G^*)|_{x_2=1}) R_2 \). This completes the proof of Theorem 3.2. qed

Example 3.3. Examples for the monomial curves with \((n_1, n_2, n_3) = (1, n-1, n)\), \(n \geq 4, \mu(p(1, n-1, n)) = n\). For Theorem 3.2 we give examples with \(5 \leq \mu(p(n_1, n_2, n_3)) < \infty \) (examples for Theorem 3.2 with \(\mu(p(n_1, n_2, n_3)) = 4\) are given in [B,Sch,St]). In 1. below \(\mu(p(n_1, n_2, n_3))\) assumes every value \(\geq 5\), in 2. this is not the case, by \(\mu(p(n_1, n_2, n_3))\) increases beyond any finite bound.

1. Consider the binomials \(x_0^d x_1^d - x_1^a + b, x_0 x_1^d - x_2^e x_1^c x_2^e, \) i.e. \(d = e - 1\). If \(\sigma\) denotes the number of binomials in the algorithm of [B,R] after the first two binomials, then \(\sigma \geq 3\) is needed. To obtain equality of degree for the pure power binomials, \(\sigma(c-1) = a\) is required. As for the algorithm to end properly, we need \(\sigma c > \sigma(c-1) = a > (\sigma-1)c\).

2. For \((n_1 - n_2, n_2 - n_1, n_3)\) not to be symmetric, \(a + b > \sigma e\) is required. These conditions reduce to \(\sigma(c-1) = a, c > \sigma, b > \sigma(e-c+1)\). In particular for \(c = 4, \sigma = 3, a\) is equal to 9. Let \(d = 1, e = 2, b = 3\). We obtain \(p(n_1, n_2, n_3) = p(1, 10, 32)\) the minimal generating set \(\{x_0^4 x_3 - x_1^d x_2, x_0^2 x_2 - x_0^4 x_3, x_0 x_2^2 - x_0^4 x_3, x_0^4 x_2 - x_0^4 x_3\}\). Similarly for \(\sigma = n \geq 3, c = n+1, a = n^2, d = 1, e = 2, b \geq n(2-n-1+1) = n(2-n), n_1 = 1, n_2 = n^2 + 1, n_3 = 2 + n(n^2 + 1)\), we obtain \(\mu(p(n_1, n_2, n_3)) = n + 2\) and \(p(n_1, n_2, n_3)\) is as in Theorem 3.2 with \(j = 2, h = 3\).

2. We start now with \(x_0^d x_1^a - x_1^{a+b}, x_0 x_2^d - x_1^a x_2^e, \) i.e. \(d = e - 1\). With \(\sigma\) as in
1., we need to satisfy $\sigma d = a + b$, $e > \sigma$, $a > \sigma c$. Let $\sigma = n$, $n$ an odd prime, $e = n + 1$, $d = n$, $a + b = n^2$, $c = 2$, $b = 2$, $a = n^2 - 2$. From the matrix
\[
\begin{pmatrix}
  n^2 & 0 & -2 \\
  -(n + 1) & n & -1 
\end{pmatrix}
\] we obtain $n_1 = 2n < n_2 = n^2 + 2n + 2 < n_3 = n^3$ (since $n \geq 3$). Also $g.c.d.(n_1, n_2) = 1$ since $n$ is an odd prime. Now $p(2n, n^2 + 2n + 2, n^3)$ satisfies Theorem 3.2 with $j = 3, h = 2$ and $p(2n, n^2 + 2n + 2, n^3) = n + 2$. In particular for $\sigma = 3, e = 4, d = 3, a + b = 9, c = 2, a = 7, b = 2$. We have $n_1 = 6, n_2 = 17, n_3 = 27$ and $p(6, 17, 27)$ has a minimal generating set \( \{x_0^3 x_2^3 - x_1^4 x_3, x_0^7 x_3^2 - x_1^5 x_2^2, x_0^5 x_3^3 - x_1^6 x_2^3, x_0^3 x_3^4 - x_1 x_2^6, x_0 x_3^5 x_3^3 - x_2^9 \} \).

4. Conclusion

We combine here Theorem 3.1 and Theorem 3.2 with the results obtained in [B, Sch, St].

Our notation for $B$, $B^*(j), B^*(h), \{j, h\} = \{2, 3\}, B(x_1), B(x_2)$ is as before and if $(n_1, n_2, n_3)$ is $(n_1, n_2)$-symmetric, then the binomials $x_0^q - x_1^q, x_0 x_1^q - x^q, x_0 x_1 x_2^q - x^q, x_1 x_2 x_3^q - x^q$ are in the previous paragraphs. For the convenience of the reader we restate:

**Theorem 4.1 [B, Sch, St].** Assume $\mu(p(n_1, n_2, n_3)) = 4$. Then $p(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces) iff either

(i) $(n_1, n_2, n_3)$ and $(n_3 - n_2, n_3 - n_1, n_3)$ are symmetric of the same type with $B^*(h) = \{x_0^{\alpha h} x_1^{\alpha h} - x_1^{\alpha h} x_2^{\alpha h}\}, h \in \{2, 3\}$, or

(ii) $(n_1, n_2, n_3)$ and $(n_3 - n_2, n_3 - n_1, n_3)$ are both not symmetric, but $\deg(B(x_1)) = \deg(B(x_2))$.

In case (i) $p(n_1, n_2, n_3)$ is a q.c.i. on binomials in $B$, in case (ii) it is not, but is a q.c.i. on two binomials and $B(x_1) + B(x_2)$.

Combining Theorems 3.1, 3.2 and 4.1 we have:

**Theorem 4.2.** Assume $\mu(p(n_1, n_2, n_3)) \geq 4$. Then $p(n_1, n_2, n_3)$ is a q.c.i. (on three surfaces) iff either

(i) $(n_1, n_2, n_3)$ and $(n_3 - n_2, n_3 - n_1, n_3)$ are symmetric of the same type and $B^*(h) = \{x_0^{\alpha h} x_1^{\alpha h} - x_1^{\alpha h} x_2^{\alpha h}\}, h \in \{2, 3\}$, or

(ii) exactly one of $(n_1, n_2, n_3)$ and $(n_3 - n_2, n_3 - n_1, n_3)$ is symmetric, (w.l.o.g. assume $(n_1, n_2, n_3)$ is $(n_1, n_3)$-symmetric), $B^*(h) = \{x_0^{\alpha h} x_1^{\alpha h} - x_1^{\alpha h} x_2^{\alpha h}\}, h \in \{2, 3\}$, and $\deg(B(x_1)) = \deg(B(x_2))$, or
(iii) \( \mu(p(n_1, n_2, n_3)) = 4. \) \( (n_1, n_2, n_3) \) and \( (n_3 - n_2, n_3 - n_1, n_3) \) are both not symmetric, but \( \text{deg}(B(x_1)) = \text{deg}B((x_2)) \).

In case (i) \( p(n_1, n_2, n_3) \) is a q.c.i. on binomials, in cases (ii) and (iii) it is not, but is a q.c.i. on two binomials and \( B(x_1) + B(x_2) \).

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