Entropy of Extremal Dyonic Black Holes

A. Ghosh\textsuperscript{1} and P. Mitra\textsuperscript{2}

\textit{Saha Institute of Nuclear Physics,}
\textit{1/AF Bidhannagar, Calcutta 700 064, INDIA}

February 14, 1996

Abstract

For extremal charged black holes, the thermodynamic entropy is proportional not to the area but to the mass or charges. This is demonstrated here afresh for dyonic extremal black hole solutions of string theory. By combining the general form of the entropy allowed by thermodynamics with recent observations in the literature it is possible to fix the entropy almost completely.

\textsuperscript{1}electronic address: amit@tnp.saha.ernet.in
\textsuperscript{2}electronic address: mitra@tnp.saha.ernet.in
Black hole thermodynamics has been an intriguing subject for many years. The laws of classical black hole physics suggested definitions of temperature and entropy purely by analogy with the laws of thermodynamics, but the scale of these quantities could not be determined that way [1]. It was only with the introduction of quantum theoretical, or more precisely semiclassical, ideas that the scale could be set in terms of Planck’s constant [2]. The temperature defined in this way, related to surface gravity, was later rederived in a euclidean approach where there is a requirement of periodicity on the euclidean time coordinate if conical singularities are to be avoided.

Apart from the obvious question about the origin of a nonzero entropy in this context, the expression for the entropy has itself been a cause for wonder. For ordinary, or what are now called non-extremal black holes, the entropy is proportional to the area of the horizon. Explanations have been sought to be given for this dependence. For instance [3] the statistical entropy of matter outside the black hole is proportional to the area of the horizon.

There has been a lot of interest lately in the special case of extremal black holes [4, 5, 6, 7]. The temperature and the entropy behave differently from the case of nonextremal black holes. Thus, when the temperature defined through the surface gravity is zero or infinity, it is found that there is no conical singularity, so that the temperature may really be arbitrary. Again, the thermodynamical entropy fails to be proportional to the area of the horizon.

Another direction which recent research has taken involves black hole solutions of string theory. It has been argued that massive string states may be identified with extremal black hole solutions [8, 9, 10]. This presents an opportunity of reaching a better understanding of the entropy of black holes in terms of the underlying theory. The entropy has indeed been calculated [10] from the density of string states. The result is sometimes nonzero even when the area of the horizon vanishes, but a new interpretation of the word “horizon” can be developed to match the area with the nonzero entropy. However, as mentioned earlier, the thermodynamic entropy of extremal black holes is in general not proportional to the area. Instead of seeking an area interpretation, the string result can be shown [11] to be consistent with the correct thermodynamical formula. The expression proportional to the mass that we advocated earlier [6] (see also [12]) for the thermodynamic entropy fits very well. While this was demonstrated explicitly for the electrically charged four-dimensional black holes of [10], it is clear that it also holds for
the higher dimensional black holes of [13].

The extremal black hole solutions considered in [10, 13] are electrically charged but magnetically uncharged. Dyonic black hole solutions were constructed in [14, 15] and considered from the point of view of state-counting in [16], where an expression for the density of string states was proposed. For large values of the charges, this expression is consistent with an entropy equal to a quarter of the area, which is of course the standard formula for non-extremal black holes. However, the black holes under consideration are extremal, and the experience of [6, 10, 11] indicates the entropy to be proportional to the mass rather than the area which goes like the square of the mass. It is therefore instructive to confront the arguments of [16] with other charged black holes which are better understood. Indeed, the clustering inequality used in [16] to motivate the proposed expression for the density of states can be easily seen to be fail for the known density of states for purely electrically charged black holes. The present investigation therefore seeks to develop a formula for the entropy or the density of states by avoiding the clustering argument and using known facts about extremal black holes instead.

2. In four dimensions the massless bosonic fields of heterotic string obtained by toroidal compactification lead to an effective action with an unbroken $U(1)^{28}$ gauge symmetry [10]:

$$S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} [R - \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi + \frac{1}{8} \text{Tr}(\partial^\mu \mathcal{M}\mathcal{L}\partial_\mu \mathcal{M}\mathcal{L})$$

$$- e^{-\Phi} F_{(a)\mu\nu}(\mathcal{L}\mathcal{M}\mathcal{L})_{ab} F^{(b)\mu\nu} - \frac{1}{12} e^{-2\Phi} H_{\mu\nu\rho} H^{\mu\nu\rho}] . \quad (1)$$

Here,

$$\mathcal{L} = \begin{pmatrix} -I_{22} & I_6 \\ I_6 & I_6 \end{pmatrix}, \quad (2)$$

with $I$ representing an identity matrix, $\mathcal{M}$ is a symmetric 28 dimensional matrix of scalar fields satisfying

$$\mathcal{M}\mathcal{L}\mathcal{M} = \mathcal{L}, \quad (3)$$

and there are 28 gauge field tensors

$$F_{(a)\mu\nu} = \partial_\mu A_{(a)\nu} - \partial_\nu A_{(a)\mu}, \quad a = 1, ..., 28 \quad (4)$$
as well as a third rank tensor $H$

$$H_{\mu \nu \rho} = \partial_\mu B_{\nu \rho} + 2A_{(a)\mu} \mathcal{L}_{ab} F_{(b)\nu \rho} + \text{cyclic permutations of } \mu, \nu, \rho \quad (5)$$
corresponding to an antisymmetric tensor field $B$. The theory possesses dyonic black hole solutions. We shall consider the extremal dyonic solution [14] considered in [16]. The asymptotic forms of the fields are as follows:

$$\langle g_{\mu \nu} \rangle = \eta_{\mu \nu}, \quad \langle e^{-\Phi} \rangle = 1, \quad \langle B_{\mu \nu} \rangle = 0, \quad \langle A_{(a)\mu} \rangle = 0. \quad (6)$$

In these circumstances, the $SL(2, R)$ (S-duality) symmetry is broken down to $SO(2)$. The magnetic and electric charges are 28-component vectors $\vec{P}$ and $\vec{Q}$. It is convenient to introduce the $O(6, 22)$ (T-duality) invariant magnitudes

$$Q_{R,L} = \left[ \frac{1}{2} \vec{Q}^T (\mathcal{L} \mathcal{M}_\infty \mathcal{L} \pm \mathcal{L}) \vec{Q} \right]^{\frac{1}{2}} \quad (7)$$

and similarly $P_{R,L}$. The ADM mass of the black hole is given by the T- and S- duality invariant form

$$M = \left[ 2P_R^2 + 2Q_R^2 + 4 \sqrt{P_R^2 Q_R^2 - \frac{1}{2} \vec{P}^T (\mathcal{L} \mathcal{M}_\infty \mathcal{L} + \mathcal{L}) \vec{Q}} \right]^{\frac{1}{2}} \quad (8)$$

The black hole is extremal and Bogomol’nyi-saturated.

A specially simple form of the charge vectors corresponds to the metric

$$ds^2 = -\frac{r^2}{R^2} dt^2 + \frac{R^2}{r^2} dr^2 + R^2 d\Omega^2, \quad (9)$$
where [14]

$$R = [(r + P_1)(r + P_2)(r + Q_1)(r + Q_2)]^{\frac{1}{2}} \quad (10)$$

with

$$P_{R,L} \propto P_1 \pm P_2, \quad (11)$$

etc.

The surface gravity (at the horizon $r = 0$), if calculated from the explicit expression (9) for the metric, is zero for nonvanishing $P_1P_2Q_1Q_2$. This way
of defining the temperature ceases to make sense when the value obtained is zero (or infinite). An alternative way of defining temperature is through the conical singularity that tends to arise on making the time imaginary. In cases like this, when the surface gravity vanishes, there is no conical singularity, though there may very well be other kinds of singularity [7], and the temperature is arbitrary [5].

3. For nonextremal black holes, the laws of black hole physics suggest that there is an entropy proportional to the area of the horizon. When the scale is fixed by comparing the temperature thus suggested with that given by the semiclassical calculations of [2], the entropy turns out to be a quarter of the area. If one is interested in an extremal black hole, one may be tempted to regard it as a special limiting case of a sequence of nonextremal black holes and thus infer that the same formula should hold for the entropy. However, it was pointed out in the context of Reissner-Nordstrom black holes [5] that the extremal and nonextremal cases of the euclidean version are topologically different, so that continuity need not hold. It was also argued that the temperature in this case is arbitrary. Subsequently it was shown [6, 12, 11] that if the derivation of an expression for the thermodynamic entropy along the lines of [17] is attempted afresh for these extremal cases, with due attention paid to the fact that the mass and charges are no longer independent as in the usual nonextremal cases, one obtains a result proportional to the mass of the black hole with an undetermined scale. Now the arguments of [17, 6, 11] will be adapted to the dyonic stringy black holes.

The first law of thermodynamics takes the form

\[ \tilde{T} dS = dM - \Phi_Q \cdot d\tilde{Q} - \Phi_P \cdot d\tilde{P}, \]  

(12)

where \( \Phi_Q \) represents the chemical potential corresponding to the charge \( \tilde{Q} \), etc. and the temperature has been written as \( \tilde{T} \) to indicate the possibility of its being different from the naively vanishing temperature. It is not difficult to derive expressions for the chemical potential in nonextremal cases, but we cannot use them here for two reasons: first, extremal black holes are not continuously connected to nonextremal black holes [5], and secondly, the standard expressions are calculated by differentiating the mass with respect to charges at constant area in the anticipation that constant area and constant entropy are synonymous, whereas in the case of extremal black holes this relation is not necessarily valid.
Consider a process in which the mass of the black hole and all its charges are scaled by the same factor \((1 + d\epsilon)\). The relation (8) will continue to be satisfied. The change in entropy is given by (12) to be

\[
\tilde{T}dS = d\epsilon(M - \tilde{\Phi}_Q \cdot \tilde{Q} - \tilde{\Phi}_P \cdot \tilde{P}).
\] (13)

Now the grand canonical thermodynamic potential

\[
W = M - \tilde{T}S - \tilde{\Phi}_Q \cdot \tilde{Q} - \tilde{\Phi}_P \cdot \tilde{P}
\] (14)

is related to the partition function by

\[
\exp\left(-\frac{W}{\tilde{T}}\right) = Z.
\] (15)

Moreover, in the leading semiclassical approximation, \(Z\) can be taken to be the exponential of the negative classical action, which in this case of arbitrary temperature has to be proportional to the reciprocal of the (arbitrary) temperature [5]. Hence \(W\) has to be treated as a constant in this approximation [6]. Comparing (13) with (14), we find

\[
\tilde{T}dS = d\epsilon(W + \tilde{T}S).
\] (16)

Since \(\tilde{T}\) is arbitrary and \(S\) cannot depend on it, it is clear that \(W\) must vanish, and moreover,

\[
dS = Sd\epsilon,
\] (17)

i.e., \(S\) is a homogeneous function of the charges of degree 1.

The entropy can be expected to depend only on combinations of the charges which are both T- and S-duality invariant. If it is further assumed to be independent of the moduli \(\mathcal{M}_\infty\) [16], the only combinations that can be involved are given by

\[
\mathcal{N} = \tilde{P}^T \mathcal{L} \tilde{P} + \tilde{Q}^T \mathcal{L} \tilde{Q} = P_R^2 - P_L^2 + Q_R^2 - Q_L^2
\] (18)

and

\[
\mathcal{A}^2 = \tilde{P}^T \mathcal{L} \tilde{P} \cdot \tilde{Q}^T \mathcal{L} \tilde{Q} - (\tilde{P}^T \mathcal{L} \tilde{Q})^2.
\] (19)
\( \mathcal{N} \) is of degree 2 and is the generalization to the dyonic case of the combination of the charges occurring in the expression for the square of the entropy of electrical black holes. Thus the square root of \( \mathcal{N} \) is a very natural guess for the answer. The other object \( \mathcal{A} \) is essentially the area of the horizon, but again it is of degree 2, so a square root will have to be taken. Of course, combinations of these two objects must also be thought of. But whenever the area enters, quarter powers of the charges are involved if negative powers are absent (note that in the special configuration given above, \( \mathcal{A} \propto \sqrt{P_1 P_2 Q_1 Q_2} \)). So it is reasonable to suppose that only \( \mathcal{N} \) appears in the expression for the entropy. This fixes the expression upto a constant. If the further guess is made that the constant is the same as in the case of purely electrically charged black holes, one is led to

\[
S = \text{const} \times \sqrt{P_R^2 - P_L^2 + Q_R^2 - Q_L^2} \sim 4\pi \sqrt{P_R^2 - P_L^2 + Q_R^2 - Q_L^2}.
\]

Like the expression proposed in [16], this reduces to the one found by counting string states in [10] when there is no magnetic charge, but unlike their proposal, it does not go over to the area when both electric and magnetic charges are large. We believe that something like this should be valid for these extremal black holes, for which there is no a priori reason for the area formula to hold.

To sum up, we have discussed the entropy of extremal dyonic black holes in the spirit of [16] but with two major differences: we have avoided using their clustering argument but have taken the extremality into account while adapting the procedure of [17] to this case. This results in a different proposal for the entropy than the one in [16].
References