Some remarks on finite-gap solutions of the Ernst equation

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It is explicitly shown that the class of algebro-geometrical (finite-gap) solutions of the Ernst equation constructed several years ago in [1] contains the solutions recently constructed by R. Meinel and G. Neugebauer [2] as a subset.

1. Algebro-geometrical solutions of Ernst equation

The Ernst equations which arises from certain dimensional reduction of 4D Einstein’s equations has the following form:

$$(\mathcal{E} + \mathcal{E})\Delta \mathcal{E} = 2(\mathcal{E}_x^2 + \mathcal{E}_\rho^2)$$

where $\mathcal{E}(x, \rho)$ is a complex-valued Ernst potential and

$$\Delta = \partial_x^2 + \frac{1}{\rho} \partial_\rho + \partial_\rho^2$$

is a cylindrical Laplacian operator. For $\mathcal{E} \in \mathbb{R}$ Ernst equation reduces to the classical Eulers-Darboux equation

$$\Delta \log \mathcal{E} = 0$$

corresponding to static space-times. Denote $x + i \rho$ by $\xi$ and consider the hyperelliptic algebraic curve $\mathcal{L}$ of genus $g$ defined by

$$w^2 = (\lambda - \xi)(\lambda - \bar{\xi}) \prod_{j=1}^{g}(\lambda - E_j)(\lambda - F_j)$$

with $\xi = x + i \rho$ symmetric with respect to antiholomorphic involution $\lambda \rightarrow \bar{\lambda}$ that entails for some $m \leq g$

$$E_j, F_j \in \mathbb{R}, \quad j = 1, \ldots, g$$

Introduce on $\mathcal{L}$ the canonical basis of cycles $(a_j, b_j) \quad j = 1, \ldots, g$. Each cycle $a_j$ is chosen to surround the branch cut $[E_j, F_j]$; cycle $b_j$ starts on one bank of the branch cut $[\xi, \bar{\xi}]$, goes on the other sheet through branch cut $[E_j, F_j]$ and comes back. The dual basis of holomorphic differentials $dU_j, \ j = 1, \ldots, g$ is normalized by

$$\int_{a_j} dU_k = \delta_{jk}$$

Define $g \times g$ matrix of b-periods $\mathbf{B}_{jk} = \oint_{b_j} dU_k$ and related $g$-dimensional theta-function $\Theta(z|\mathbf{B})$.

Differentials $dU_j$ are linear combinations of non-normalized holomorphic differentials

$$dU_j^0 = \frac{\lambda^{j-1} d\lambda}{w} \quad j = 1, \ldots, g$$

General algebro-geometrical solution of the Ernst equation may be written in many different forms (see [1, 5]). Here it is convenient to use the original form of [1]:

$$\mathcal{E} = \frac{\Theta(U|\mathcal{L})^2 + B_{\Omega} - K)}{\Theta(U|\mathcal{L})^2 + B_{\Omega} - K)} \times \exp\{\Omega|_{\infty}^{\infty}\}$$

where the new objects are defined as follows: $K$ is a vector of the Riemann constants of $\mathcal{L}$; $D$ is a set (divisor) of $g$ $(\xi, \bar{\xi})$-independent points $D_1, \ldots, D_g$ on $\mathcal{L}$;

$$\left(U|_{D}^{\infty} \right)_k \equiv \int_{P_0}^{\infty} dU_k - \sum_{j=1}^{g} \int_{P_0}^{D_j} dU_k$$

with an arbitrary base point $P_0$ (entering also vector $K$).

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It remains to define $\Omega(\infty^1) - \Omega(\infty^2)$ and vector $B_\Omega$. Let $d\Omega(P)$ be an arbitrary locally holomorphic 1-form on $L$ with $(\xi, \bar{\xi})$-independent singularities and related singular parts satisfying the normalization conditions

$$\int_{a_j} d\Omega = 0 \quad j = 1, \ldots, g$$

Define its vector of $b$-periods

$$(B_\Omega)_j = \int_{b_j} d\Omega$$

and require the reality conditions

$$\Omega(\infty^2) - \Omega(\infty^1) \in \mathbb{R} \quad \text{Re}(B_\Omega)_k = \pm \frac{1}{4}$$

Now solution (7) is completely defined. If one takes $g = 0$ then combination of theta-functions in (7) disappears and we get

$$\mathcal{E} = \exp\left\{ \Omega_0(\infty^2) - \Omega_0(\infty^1) \right\} \in \mathbb{R}$$

i.e. static solution, which serves as a static background of solution (7). It is easy to show that by an appropriate choice of differential $d\Omega_0$ on the Riemann surface $\mathcal{L}_0$ given by

$$w^2 = (\lambda - \xi)(\lambda - \bar{\xi})$$

one can get arbitrary static solution. Namely, take an arbitrary solution $\mathcal{E}_0 \in \mathbb{R}$ (for definiteness, asymptotically flat i.e. $\mathcal{E}_0(\xi = \infty) = 1$) satisfying (2) and define 1-form $d\Omega_0$ by

$$d\Omega_0(\lambda, \xi, \bar{\xi}) = \frac{d\lambda}{4} \int_{\xi} d\xi' \frac{\partial}{\partial \lambda} \left( \frac{\lambda - \xi'}{\lambda - \bar{\xi}'} \right) \frac{\partial \log \mathcal{E}_0(\xi', \bar{\xi}')}{\partial \xi'}$$

and analogous equation with respect to $\xi$ (which are compatible as a corollary of (2)); we have

$$d\Omega_0(\lambda) = d\Omega_0(\bar{\lambda})$$

This is a simple example of "direct scattering procedure" (and analog of Fourier transform): the positions and structure of singularities of $d\Omega_0$ carry the whole information about solution $\mathcal{E}_0$.

The 1-form $d\Omega$ on $L$ which enters (7) inherits all singularities of $d\Omega_0$ on $\mathcal{L}_0$ and is assumed to have additional simple poles at the branch points $E_j$ with the residues $1/2$ and at the branch points $F_j$ with the residues $-1/2$, $j = 1, \ldots, g$.

Therefore, for fixed genus $g$ the solution (7) is defined by the following set of data: an arbitrary background solution $\mathcal{E}_0$ of the Ehlers-Darboux equation (2) and $(\xi, \bar{\xi})$-independent points \( \{E_j, F_j, D_j\} \).

2. Reduction to Meinel-Neugebauer construction

To obtain the solutions constructed in [2, 3] as a special case of (7) one have to take $m = g$ i.e. for all $j = 1, \ldots, g$ one assume

$$F_j = \bar{E}_j$$

Then to rewrite solutions (7) in the form of [2] introduce on $L$ meromorphic 1-form $dW$ having the 1st order poles at $\lambda = \infty^1$ and $\lambda = \infty^2$ with the residues $-1$ and $+1$ respectively normalized by

$$\int_{a_j} dW = 0 \quad j = 1, \ldots, g$$

The following simple identity:

$$\exp\{W(D) - W(D)\} \equiv \frac{\Theta(U(\infty^1) - U(D) - K)}{\Theta(U(\infty^1) - U(D) - K)} \times \frac{\Theta(U(\infty^2) - U(D) - K)}{\Theta(U(\infty^2) - U(D) - K)}$$

is valid for arbitrary two sets of $g$ points $D$ and $\bar{D}$ and may be verified by simple comparison of the pole structure of both sides with respect to every $D_j$ and every $\bar{D}_j$.

Thus solution (7) may be rewritten as follows:

$$\mathcal{E} = \exp\left\{ W|_{D} + \Omega|^{\infty^2}_D \right\}$$

where divisor $D$ consists of the points $\bar{D}_1, \ldots, \bar{D}_g$ defined by the following system of equations:

$$U(D) - U(D) = -B_\Omega$$

(15)
The vector in the l.h.s. is understood as
\[(U(\hat{D}) - U(D))_k = \sum_{j=1}^{g} \int_{D_j} dU_k\]

The problem of determining points of \(\hat{D}\) from (15) is called the Jacobi inversion problem.

Equations (15) may be rewritten in terms of non-normalized basis of holomorphic differentials given by (6) as follows:
\[\sum_{k=1}^{g} \int_{D_k} \lambda_{j-1}^i d\lambda \int_{\partial \hat{L}} \lambda_{j-1}^i d\lambda = \int_{\partial \hat{L}} \Omega \lambda_{j-1}^i d\lambda \]  
(16)

for \(j = 1, \ldots, g\) where \(\partial \hat{L}\) is the boundary of 4g-sided fundamental polygon \(L\) of surface \(\hat{L}\) which is obtained if we cut \(L\) along all basic cycles;
\[\Omega(P) = \int_{P_0}^P d\Omega \quad P \in L\]

with arbitrary base point \(P_0 \in L\); choice of \(P_0\) does not influence the r.h.s. of (16). Expression (16) may be easily derived from (15) using the general formula valid for any two 1-forms \(W_{1,2}\) on \(L\) [4]:
\[\int_{\partial \hat{L}} W_1 dW_2 = \sum_{j=1}^{g} \left\{ A_{W_1}^j B_{W_2}^j - B_{W_1}^j A_{W_2}^j \right\} \]  
(17)

where \(A_{W_1}^j, B_{W_2}^j\) are \(a\) and \(b\) periods of the forms \(dW_{1,2}\) (to derive (16) one should take \(dW_1 = d\Omega, dW_2 = dU_j\)).

Introducing differential
\[dW_0^{\lambda} \equiv \frac{\lambda^0 d\lambda}{w}\]

(which coincides with \(dW\) up to some combination of holomorphic differentials (6) which provide vanishing of all \(a\)-periods of \(dW\)), and applying (17) to \(d\Omega\) and \(dW_0\), we rewrite (14) as follows:
\[\mathcal{E} = \exp \left\{ W_0^{D} + \int_{\partial \hat{L}} \Omega dW_0 + \int_{\infty}^\infty d\Omega \right\} \]  
(18)

Formulas (16) and (18) after identification
\[u_j = \int_{\partial \hat{L}} \Omega \frac{\lambda^j d\lambda}{w} \quad j = 0, \ldots, j - 1\]  
(19)

may be rewritten in the following way:
\[\sum_{k=1}^{g} \int_{D_k} \lambda_{j-1}^i d\lambda \int_{\partial \hat{L}} \lambda_{j-1}^i d\lambda = u_j \quad j = 0, \ldots, j - 1\]  
(21)

and
\[\mathcal{E} = \exp \left\{ \sum_{j=1}^{g} \int_{D_j} \frac{\lambda^j d\lambda}{w} + u_g \right\} \]  
(22)

which precisely coincide with expressions of [2]. Functions \(u_j, j = 1, \ldots, g\) satisfy the Laplace equation
\[\Delta u_j = 0\]

and the recurrent equations
\[u_j \xi = \frac{1}{2} u_{j-1} + \xi u_{j-1} \xi\]  
(23)

as a corollary of the relations
\[\Delta \frac{1}{w} = 0\]

and the residue theorem applied to the contour integral over \(\partial \hat{L}\).

The static background of solution (22) is given by an arbitrary solution of the Laplace equation \(u_g\) (one could take any other function \(u_j\), since it would almost uniquely determine the others according to (23)), which may, therefore, be alternatively expressed as
\[u_g = \int_{\infty}^\infty d\Omega_0\]
in terms of the differential \(d\Omega_0\) (11).

Let us show how to choose the parameters of the present construction to get the “dust disc” solution [3] posed at \((z = 0, \rho = \rho_0)\). We shall present the arguments that this solution has naked singularities and is therefore unphysical.

One should take \(g = 2\), choose some complex \(E_1\) (related to parameter \(\mu\) of [3]) and put \(F_1 =\)
$E_1; E_2 = -F_1; F_2 = E_2$. Points $D_1$ and $D_2$ are chosen to coincide with points $E_1$ and $E_2$, respectively. Finally, the 1-form $d\Omega$ should be taken in the form

$$d\Omega(\lambda) = \int_{i\rho_0}^{i\rho_0} f(\gamma)d\Omega(\gamma)(\lambda)d\gamma + d\Omega$$

where the integral is taken along the imaginary axis; $d\Omega(\gamma)(\lambda)$ is meromorphic 1-form on $\mathcal{L}$ with vanishing $\alpha$-periods and unique pole of the second order at $\lambda = \gamma$ with leading coefficient equal to 1; $f(\gamma)$ may be an arbitrary measure satisfying $f(\gamma) = f(\gamma)$ (for example, $f \in \mathbb{R}$). $d\Omega$ is a meromorphic 1-form on $\mathcal{L}$ having simple poles with the residues $1/2$ at $E_{1,2}$ and $-1/2$ at $F_{1,2}$. Related static background will be given by [1]

$$\log \mathcal{E}_0 = \int_{-i\rho_0}^{i\rho_0} \frac{f(\gamma)d\gamma}{\{(\gamma - \xi)(\gamma - \xi^*)\}^{1/2}}$$

Specifying $f(\gamma)$ in some special way (see [3]) one arrives to the “dust disc” solution of [3].

Here one should mention that analysis of [5] shows that the invariants of the Weyl tensor on the rings $\xi = E_1$ and $\xi = E_2$ are singular, and the space-time is ramified in their neighbourhood (this is checked in [5] for $g = 1$ case, but clearly this fact is pure local: presence or absence of the other branch cuts does not influence the qualitative space-time structure, say, near $\xi = E_1$). Thus solution discussed in [3] has naked singular “branch rings” posed at $x + i\rho = E_1$ and $x + i\rho = E_2$ outside of the “dust disc” itself.

3. Summary

We have shown that the solutions of the Ernst equation obtained recently in [2] (and, in particular, some partial solution of this class exploited in [3] to describe rigidly rotating dust disc) constitute a subclass of the algebro-geometric (finite-gap) solutions found before in [1].

However, in spite of the solutions derived in [2, 3] are not new, the physical interpretation of special solution of this class proposed in [3] would be very interesting if it would really describe the dust disc. Unfortunately, this solution will have at least two naked ring singularities outside of the “dust disc”; in the neighbourhood of these rings the space-time will have non-trivial topological structure which makes its physical interpretation doubtful.

Nevertheless, we hope that some of the finite-gap solutions will find reasonable physical application (see also [5] for discussion, where, in particular, we describe a solution with toroidal ergosphere).

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