Abstract

The aim of this letter is to indicate the differences between the Rovelli-Smolin quantum volume operator and other quantum volume operators existing in the literature. The formulas for the operators are written in a unifying notation of the graph projective framework. It is clarified whose results apply to which operators and why.
It should be emphasized at the very beginning that the letter has been motivated by very nice calculations made recently by Loll [1] in connection with the lattice quantization of the volume and by the work of Pietri and Rovelli [2] who studied the Rovelli-Smolin volume operator [3]. In the full continuum theory, there is still one more candidate for the quantum volume operator proposed in [4, 5, 6] which will be referred to as the ‘projective limit framework’ volume operator. The misunderstanding which we indicate here is that in [1, 7] the lattice operator tends to be considered as the restriction of the Rovelli-Smolin as well as the projective limit framework volume operators. In fact, the first lattice volume, that of [1], corresponds to neither of them. The corrected lattice volume operator of [7], on the other hand, has been modified to agree with the graph projective framework volume operator. However this is still different then the Rovelli-Smolin operator. We don’t discuss here the origins of the differences between the two continuum theory operators or compare their statuses. This is the subject of the coming paper [8]. Therein, the Rovelli-Smolin operator has been written in terms of the graph projective framework which makes the comparison possible. Below we present the derived formula and show the difference between the Rovelli-Smolin and the graph projective framework operators. Finally, we explain why some of the arguments of Loll are still true for the both remaining operators and what is the origin of the relations between some eigen values derived by Loll and those established by Pietri and Rovelli.

All the operators in question come from the same classic formula for the volume of a 3-surface $\Sigma$ of initial data expressed by the Ashtekar variables,

$$\text{vol} = \int_{\Sigma} d^3x \sqrt{\frac{1}{3!} |\epsilon^{ijk}\epsilon_{abc}E^a_i(x)E^b_j(x)E^c_k(x)|},$$

(1)

where $E^a_i$ is the $su(2)$ valued vector-density momentum canonically conjugate to the Ashtekar connection $A^i_a$, $a, b, c$ being the space and $i, j, k$ the internal indices. The space of the connections is denoted by $\mathcal{A}$.

The Hilbert space in which we will define the operators is constructed from the gauge invariant cylindrical functions on $\mathcal{A}$ [9, 10, 11, 12, 4]. Recall that a function $\Psi$ defined on $\mathcal{A}$ is called *cylindrical* if there exists a finite family $\gamma = \{e_1, ..., e_n\}$ of finite curves and a complex valued function $\psi \in$
\( C^0(SU(2)^n) \) such that

\[
\Psi(A) = \psi(U_{e_1}(A), ..., U_{e_n}(A)), \quad U_p = P\exp\int_p^- A; \tag{2}
\]

that is, where given a curve \( p \) in \( \Sigma \), \( U_p \) is the parallel transport matrix with respect to \( A \). If we admit only piecewise analytic curves then for every cylindrical function we can choose \( \gamma = \{ e_1, ..., e_n \} \) to be an embedded graph, meaning that every curve (edge) \( e_i \) is an analytic embedding of an interval and two distinct elements of \( \gamma \) can share, if any, only one or the both end points (called the vertices of \( \gamma \)). The Hilbert space is the space \( L^2(\mathcal{A}) \) of cylindrical functions integrable with the square with respect to the following integral

\[
\int_{\mathcal{A}} \Psi = \int_{SU(2)^n} \psi(g_1, ..., g_n) dg_1 ... dg_n \tag{3}
\]

\( dg \) being the probability Haar measure on \( SU(2) \).

The volume operator involves third derivatives, so for its initial domain (later on extended by the essential self-adjointness) we will take the space \( \text{Cyl}^{(3)} \) of the cylindrical functions given by the \( C^3 \) functions \( \psi \).

Before writing the explicit formulas we need a 'basis' of first order differential operators acting on the cylindrical functions. To a pair, an analytic curve \( p \) and its marked endpoint \( v \), and to a an element \( \tau_i \) of a fixed basis in \( su(2) \) we assign an operator \( X_{vpi} \) acting on cylindrical functions as follows. Given a cylindrical function, represent it by (2) using a graph such that one of its edges, say \( e_I \), is a segment of \( p \) containing the point \( v \). Then, respectively

\[
X_{vpi}(A) := \begin{cases} 
(U_{e_I} \tau_i)^A_B \frac{\partial}{\partial U_{e_I} B} \psi(U_{e_1}(A), ..., U_{e_n}(A)), \\
(\tau_i U_{e_I})^A_B \frac{\partial}{\partial U_{e_I} B} \psi(U_{e_1}(A), ..., U_{e_n}(A)),
\end{cases} \tag{4}
\]

when \( e_I \) is outgoing and when \( e_I \) is incoming, at the vertex \( v \) (the result is graph independent).

The 'graph projective framework' volume operator of \([4, 5, 6]\) acts on a cylindrical function (2) in the following way (below, we skip an overal constant factor which is the same for all the operators)

\[
\hat{\text{vol}} \Psi = \sum_v \sqrt{|\hat{q}_v|} \Psi \tag{5}
\]
where the sum ranges over all the vertices in the graph $\gamma$ used to represent $\Psi$ by (2) and the operator $\hat{q}_v$ assigned to a point $v$ of $\Sigma$ is defined by

$$\hat{q}_v \Psi = \frac{1}{8 \cdot 3!} \sum_{(e_I, e_J, e_K)} i\epsilon(e_I, e_J, e_K)\epsilon_{ijk} X_{ve_i} X_{ve_j} X_{ve_k}, \quad (6)$$

the sum ranging over all the ordered triples of edges of $\gamma$ at $v$ and where $\epsilon(e_I, e_J, e_K)$ depends only on the orientation of the vectors $(\dot{e}_I, \dot{e}_J, \dot{e}_K)$ at $v$,

$$\epsilon(e_I, e_J, e_K) = \begin{cases} 
1, & \text{when the orientation is positive}, \\
-1, & \text{when negative}, \\
0, & \text{when the vectors are linearly dependent}. 
\end{cases} \quad (7)$$

On the other hand, the Rovelli-Smolin volume regularization [3] written in terms of the graph projective framework turns out to give the following result [8]

$$\hat{\text{vol}}^{\text{RS}} \Psi = \sum_v \sqrt{\hat{q}_v^{\text{RS}}} \Psi \quad (8)$$

where

$$\hat{q}_v^{\text{RS}} \Psi = \frac{1}{8 \cdot 3!} \sum_{(e_I, e_J, e_K)} |\epsilon_{ijk} X_{ve_i} X_{ve_j} X_{ve_k}| \quad (9)$$

using the same sum convention as above. (Surprisingly, the factor 8 in (6) appears on the quantum level whereas in (8) it comes from the classical expression used to approach the determinant.)

To understand the difference between $\hat{\text{vol}}$ and $\hat{\text{vol}}^{\text{RS}}$, notice that the volume operator $\hat{\text{vol}}$ is sensitive on diffeomorphic characteristics of intersections in a given graph. For a planar graph for instance each $\hat{q}_v$ is identically zero. On the other hand, the Rovelli-Smolin operator $\hat{q}_v^{\text{RS}}$ extracted by our decomposition, regards each triple of edges at an intersection equally, irrespectively whether they are tangent or not. Thus, whereas the ‘graph projective framework’ volume $\hat{\text{vol}}$ is diffeomorphism of $\Sigma$ invariant, the Rovelli-Smolin volume $\hat{\text{vol}}^{\text{RS}}$ is preserved by all the homeomorphisms! To be more precise: given a cylindrical function $\Psi$ of (2), if a diffeomorphism (homeomorphism) $\varphi$ of $\Sigma$ happens to carry a graph $\gamma$ representing $\Psi$ into a graph which is again piecewise analytic, then the induced action of $\varphi$ on $\Psi$ commutes with the action of $\hat{\text{vol}}$ (respectively, with $\hat{\text{vol}}^{\text{RS}}$).
Let us turn now to the Loll graph operators. Now, a graph $\gamma$ is a fixed cubic lattice. The first operator published in [1] is given by the Eqs (5, 6) however with the sum in (6) ranging only over the outgoing edges (with respect to an orientation of the lattice) and without the factor $\frac{1}{8}$. Finally, the modified lattice volume operator of [7] coincides with the ‘graph projective framework’ volume operator (5,6) (modulo some mistakes: taken literally as it stands in [7], that definition would give products of operators $X_{ve_i}X_{v'e'_i}$ at different points $v \neq v'$ which wouldn’t be gauge invariant; but a simple correction corresponding to the previous verbal description removes this problem.) Let us denote these operators by $\hat{\text{vol}}^{L_1}$ and $\hat{\text{vol}}^{L_2}$.

Studying the operator $\hat{\text{vol}}^{L_1}$, Loll proved that it annihilates all the cylindrical functions on a lattice which at each vertex involve not more then three intersecting edges. Remarkably, this result continues to be true for any of the remaining volume operators [5, 7, 2]. Moreover, recently the eigen values of $\hat{\text{vol}}^{L_2}$ for four valent graphs embedded in a lattice (applicable to the operator $\hat{\text{vol}}$ automatically) obtained in [7] are being compared and confirmed by the team studying the original Rovelli-Smolin volume operator $\hat{\text{vol}}^{RS}$. Let us take advantage of our unified notation to see how it could happen.

In 3-valent case, at a vertex $v$ there are at most three edges. Thus the formulas (5,6) and (8,9) just coincide. Moreover, every term $X_{ve_i}X_{ve_j}X_{ve_k}\epsilon_{ijk}$ kills a gauge invariant cylindrical function individually [1]. We show this by an alternative calculation [5]. Fix a vertex $v$ of a graph $\gamma$ and denote the edges at $v$ by $e_1, e_2, e_3$ say. Restricted to the gauge invariant cylindrical functions, the differential operators satisfy the following constraint, (we drop $v$)

$$X_{e_1i} + X_{e_2i} + X_{e_3i} = 0 \quad (10)$$

From that we evaluate

$$\epsilon_{ijk}X_{e_1i}X_{e_2j}X_{e_3k} = -\epsilon_{ijk}X_{e_1i}X_{e_2j}(X_{e_1k} + X_{e_2k}) = 0. \quad (11)$$

Consider now a four valent case. Let $e_1, ..., e_4$ be the edges of $\gamma$ incident at a fixed vertex $v$. Then, the Gauss constraint reads

$$X_{e_1i} + X_{e_2i} + X_{e_3i} + X_{e_4i} = 0. \quad (12)$$

From that we derive

$$\epsilon_{ijk}X_{e_1i}X_{e_2j}X_{e_3k} = -\epsilon_{ijk}X_{e_1i}X_{e_2j}X_{e_4k} \quad (13)$$
etc.. That is, at a vertex, all the triple products $X_{ve_i}X_{ve_j}X_{ve_k} \epsilon_{ijk}$ are equal modulo a sign. Hence, for a four valent case, at every vertex

$$\hat{q}_v = \kappa(v)\hat{q}_{RS}^v,$$  

the constant factor being given by the diffeomorphic versus topological characteristics of the intersection at $v$. This explains, why the eigen values of [7] can be in a direct relation with the results of of [2] even in the 4-valent case as long as a single vertex is considered.

Since the Rovelli-Smolin operator $vol^{RS}$ is presented here for the first time in the above form let us briefly explain how in the graph projective framework one can see its self adjointness (as well as the self adjointness of the operator $vol$). For every fixed graph $\gamma$ the operators $iX_{ve_i}X_{ve_j}X_{ve_k} \epsilon_{ijk}$ are essentially-self adjoint differential operators acting in the domain $\mathcal{C}^3(SU(2)^n)$ in $L^2(SU(2)^n)$ (which justifies taking the absolute value). From this one proves, that in the terminology of [4] the operator $vol^{RS}$ defines a consistent family of essentially self-adjoint operators. Therefore, as proven therein, it is essentially self adjoint in $Cyl^{(3)}$.

To show the discreteness of the spectrum of $vol^{RS}$ (proven in [2]) and of the operator $vol$, it is enough to prove the discreteness of the operators reduced to any graph and the corresponding $L^2(SU(2)^n)$ [8]. For a fixed graph $\gamma$ we can use just the spin-network functions [13, 14] (we are using the notation of [15]). The operators $X_{ve_i}X_{ve_j}X_{ve_k} \epsilon_{ijk}$ acting on the spin-network functions $T_{\gamma,\vec{\pi},\vec{c}}$ preserve the labeling $\vec{\pi}$ of the edges of $\gamma$ by irreducible representations and only affect in a pointwise way the intertwining operators assigned to the vertices by $\vec{c}$ [3, 1, 5]. Thus they restrict to finite dimensional real and antisymmetric matrices which completes the proof.

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