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ON THE ULTRALOCAL
AND NON-ULTRALOCAL POISSON BRACKETS

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Abstract

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We consider the KdV field as an example of non-ultralocal mechanical system (i.e. its fundamental Poisson brackets contain the derivative of δ -function) and show that by appropriate gauge transformation the Poisson brackets of its monodromy matrix may be transformed into Heisenberg magnetic ones.

Аннотация

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Рассматривается поле КдФ как пример неультролокальной системы (не фундаментальные скобки Пуассона содержат производные β -функций). Показано, что подходящими калибровочным преобразованием над матрицей монодромии ее скобки могут быть трансформированы в скобки магнетика Гейзенберга.

The quantum inverse scattering method (QISM) works very effectively for a wide class of mechanical systems which have so called ultralocal commutation relation. The latter means that the commutator of fundamental variables is proportional to δ -function alone, without its derivatives. There were numerous attempts to extend QISM to the non-ultralocal case among which paper [1] seems most promising but the problem is still far from complete understanding.

At the same time the classical inverse scattering method (CISM) in the non-ultralocal case is as effective, as that in the ultralocal one. It worth mentioning that the first example that was considered in the frameworks of CISM was the KdV field, whose fundamental Poisson brackets (PB) are just δ' -function.

In a general case the CISM may be formulated with the help of L -operator [2]:

$$L = -\frac{d}{d\sigma} + l(\sigma, x), \quad (1)$$

where l belongs to Lie algebra g of Lie group G

$$l = \sum_a t^a K_a(\sigma, x) \quad (2)$$

and $K_a(\sigma, x)$ are functions of a set of dynamical variables $u_k(\sigma)$, which we shall take periodic in σ

$$u_k(\sigma + 2\pi) = u_k(\sigma), \quad (3)$$

the variable x in (1) and (2) is spectral parameter, $x \in C$. The spectral theory of L -operator could be formulated making use of its monodromy matrix

$$T_\sigma^{\sigma+2\pi}(x) = P \exp \int_\sigma^{\sigma+2\pi} d\sigma l(\sigma, x). \quad (4)$$

$T_\sigma^{\sigma+2\pi}(x)$ belongs to G and translates each solution of equation

$$L\psi = 0 \quad (5)$$

to the period 2π . In the ultralocal case, the PBs of fundamental variables are is proportional to δ -function

$$\{u_k(\sigma)u_{k'}(\sigma')\} \sim \Delta(\sigma - \sigma'), \quad (6)$$

where $\Delta(\sigma)$ is δ -function on a circle

$$\Delta(\sigma) = \sum_n e^{in\sigma}/2\pi. \quad (7)$$

The PB (6) induces the PB of monodromies

$$\{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}\} = Z(\sigma_1, \sigma_2, x, y). \quad (8)$$

The r.h.s. in (8) is a smooth function of σ_1, σ_2 and

$$\begin{aligned} Z(\sigma_1, \sigma_2, x, y)|_{\sigma_1 \rightarrow \sigma_2 + \epsilon} &= Z(\sigma_1, \sigma_2, x, y)|_{\sigma_1 \rightarrow \sigma_2 - \epsilon} \\ &= [r(x, y), T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_1}^{\sigma_1+2\pi}(y)]. \end{aligned} \quad (9)$$

This equation is r -matrix form of classical commutation relation and it states that the algebra of monodromies with coinciding initial points induced by PB(6) is closed provided $r(x, y)$ satisfies classical Yang-Baxter equation [2].

To illustrate this ultralocal case let us consider the Heisenberg magnetic.

$$L = -\frac{d}{d\sigma} + \frac{ik}{2}S_i(\sigma)\tau_i, \quad (10)$$

where τ_i — Pauly matrices. The PBs of fundamental variables are:

$$\{s_i(\sigma), s_j(\sigma')\} = -\epsilon_{ijk}s_k(\sigma)\Delta(\sigma - \sigma'). \quad (11)$$

To calculate the r.h.s. of (8) in this case we need a new object — the monodromy matrix with impurity:

$$Q_i(\sigma_2, \sigma_1, x) = \int_{\sigma_1}^{\sigma_1+2\pi} d\sigma \Delta(\sigma - \sigma_2) T_{\sigma}^{\sigma_1+2\pi} \tau_i T_{\sigma_1}^{\sigma}(x). \quad (12)$$

Note that inspite of the presence of δ -function in the integrand we cannot omit integration without spoiling periodicity of Q_i . Being a periodic function with respect to both σ_1 and σ_2 , Q_i is not a smooth function for $\sigma_1 \rightarrow \sigma_2$:

$$Q_i(\sigma_2, \sigma_1 x) = \begin{cases} \tau_i T^{\sigma_1+2\pi}(x) & \sigma_2 \rightarrow \sigma_1 - \epsilon \\ T_{\sigma_1}^{\sigma_1+2\pi}(x) \tau_i & \sigma_2 \rightarrow \sigma_2 + \epsilon \end{cases} \quad (13)$$

Making use of (12) one can get convinced that left and right limits $\sigma_1 \rightarrow \sigma_2$ coincide yielding a famous result:

$$\{T_{\sigma}^{\sigma+2\pi}(x) \otimes T_{\sigma}^{\sigma+2\pi}(y)\} = - \left[\frac{xy}{4(x-y)} \tau_i \otimes \tau_i, T_{\sigma}^{\sigma+2\pi}(x) \otimes T_{\sigma}^{\sigma+2\pi}(y) \right]. \quad (14)$$

In a non-ultralocal case the situation is quite different: the r.h.s. in eq.(8) is not a smooth function any longer, the left and right limits $\sigma_1 \rightarrow \sigma_2$ are different. This circumstance makes the algebra of monodromies unclosed and we have to complete it with an infinite set of monodromy with an arbitrary number of impurities. The simplest example here is provided by relativistic string [3]. The L -operator here coincides with the Heisenberg magnetic

$$L = -\frac{d}{d\sigma} + \frac{ix}{2}a_i(\sigma)\tau_i, \quad (15)$$

where functions $a_i(\sigma)$ describe the internal degrees of freedom of a string, they are periodic with vanishing zero mode

$$\int_0^{2\pi} a_i(\sigma) d\sigma = 0. \quad (16)$$

The fundamental PBs in a string case are

$$\{a_i(\sigma), a_j(\sigma')\} = -2\delta_{ij}\Delta'(\sigma - \sigma'). \quad (17)$$

Calculation of the PB of monodromies gives us the following result:

$$\begin{aligned} \{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}(y)\} &= -\frac{xy}{2(x-y)}(y[\tau_i, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes Q_i(\sigma_1, \sigma_2, y) \\ &+ xQ_i(\sigma_2, \sigma_1, x) \otimes [\tau_i, T_{\sigma_2}^{\sigma_2+2\pi}(y)]) \end{aligned} \quad (18)$$

Now, taking the limit $\sigma_1 \rightarrow \sigma_2$ from the right and from the left we obtain different results:

$$\begin{aligned} \{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}(y)\}_{\sigma_1 \rightarrow \sigma_2 + \epsilon} &= -\frac{xy}{2(x-y)}(y[\tau_i, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes T_{\sigma_1}^{\sigma_1+2\pi}(y)\tau_i \\ &+ x\tau_i T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes [\tau_i, T_{\sigma_1}^{\sigma_1+2\pi}(y)]), \\ \{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}(y)\}_{\sigma_1 \rightarrow \sigma_2 - \epsilon} &= -\frac{xy}{2(x-y)}(y[\tau_i, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes \tau_i T_{\sigma_1}^{\sigma_1+2\pi}(y) \\ &+ xT_{\sigma_1}^{\sigma_1+2\pi}(x)\tau_i \otimes [\tau_i, T_{\sigma_1}^{\sigma_1+2\pi}(y)]). \end{aligned} \quad (19)$$

One could find in literature an erroneous statement: some authors, making different regularization while calculating, obtain prescriptions for the P. B of monodromies with coinciding initial points, but this cure produces a new disease — violation of the Jacoby identity. Of course, the PB (12) does not violate the Jacoby identity if we keep three initial points different

$$\{\{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}(y)\} \otimes T_{\sigma_3}^{\sigma_3+2\pi}(z)\} + (cycl) = 0. \quad (20)$$

Equation (18) demonstrates what was said above about algebra of monodromies in the r.h.s.: there appears, besides simple monodromy, also the monodromy with one impurity. If we consider the PBs of this new object with itself and with simple monodromy we shall obtain in the r.h.s. monodromy with one, two and so on impurities.

From the point of view of CISM all these complications are unessential and the theories for PB (11) and (17) differ slightly in every aspect. Both theories have common involutory integrals, a functional form of finite-zone solutions. There is an unessential difference in its action-angle variables which, of course, feel the difference in the fundamental PB. The question arises however, in the quantum case where, up to now, only the single initial points commutation relations are the origin Bethe Ansatz construction of eigenstates of Hamiltonian. Therefore if we wish to use all these very useful achievements we have to learn how to get rid of ambiguity in commutation relation starting from the case. As we shall see below there exists a way to improve the situation with the PB of monodromy at least in the one-component field theory with the non-ultralocal PB. In the KdV field theory CISM may be provided by first order L -operator [2]

$$L = -\frac{d}{d\sigma} + \frac{ix}{2}\tau_3 + \frac{1}{2}\tau_+ + \frac{1}{2}\tau_- u(\sigma), \quad (21)$$

where

$$\tau_{\pm} = \tau_i \pm i\tau_2$$

and $u(\sigma)$ is real periodic function with PB

$$\{u(\sigma), u(\sigma')\} = \Delta'(\sigma - \sigma'). \quad (22)$$

More familiar, the second order Schrödinger L -operator arises if we exclude the second component of Pauli spinor from the system

$$L\psi = 0.$$

In the same notations as above, the PB of monodromy of (21) has the following form:

$$\begin{aligned} \{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}(y)\} = & \frac{1}{8} \left([\tau_-, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes Q_-(\sigma_1, \sigma_2, y) - \right. \\ & \left. - Q_-(\sigma_2, \sigma_1, x) \otimes [\tau_-, T_{\sigma_2}^{\sigma_2+2\pi}(y)] \right) + \text{Regular Part}, \end{aligned} \quad (23)$$

where $Q_{\pm} = Q_1 \pm Q_2$ and

$$\begin{aligned} \text{Regular Part} = & -\frac{1}{x^2 - y^2} \left\{ [\tau_-, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes (Q_+(\sigma_1, \sigma_2, y) - \right. \\ & - i(x-y)Q_3(\sigma_1, \sigma_2, y) + \frac{(x-y)^2}{4}Q_-(\sigma_1, \sigma_2, y)) + \\ & + [\tau_+, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes Q_-(\sigma_1, \sigma_2, y) \\ & + [\tau_3, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes (2Q_3(\sigma_1, \sigma_2, y) + i(x-y)Q_-(\sigma_1, \sigma_2, y)) \Big\} - \\ & - \frac{1}{x^2 - y^2} \left\{ (Q_+(\sigma_2, \sigma_1, x) + i(x-y)Q_3(\sigma_2, \sigma_1, x) + \right. \\ & + \frac{(x-y)^2}{4}Q_-(\sigma_2, \sigma_1, x)) \otimes [\tau_-, T_{\sigma_1}^{\sigma_1+2\pi}(x)] + \\ & + Q_+(\sigma_2, \sigma_1, x) \otimes [\tau_-, T_{\sigma_2}^{\sigma_2+2\pi}(x)] + \\ & + (2Q_3(\sigma_2, \sigma_1, x) - i(x-y)Q_3(\sigma_2, \sigma_1, x)) \otimes [\tau_3, T_{\sigma_2}^{\sigma_2+2\pi}(y)] \Big\}. \end{aligned} \quad (24)$$

As its name says Regular Part is regular when $\sigma_1 \rightarrow \sigma_2$, while the first term in parenthesis in (23) produces a jump in this point

$$\begin{aligned} & \{T_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes T_{\sigma_2}^{\sigma_2+2\pi}(y)\}_{\sigma_1 \rightarrow \sigma_2+\epsilon} - \{\}_{\sigma_1 \rightarrow \sigma_2-\epsilon} = \\ & = -\frac{1}{4}[\tau_-, T_{\sigma_1}^{\sigma_1+2\pi}(x)] \otimes [\tau_-, T_{\sigma_1}^{\sigma_1+2\pi}(y)]. \end{aligned} \quad (25)$$

To cure the monodromy matrix, let us introduce additional dynamical variable q into our system, which will be canonically conjugated to annihilator of PB (22). The latter is

$$p = \int_0^{2\pi} d\sigma u(\sigma). \quad (26)$$

It is clear, that

$$\{p, u(\sigma)\} = 0 \quad (27)$$

and

$$u(\sigma) = p/2\pi + \sum_{n \neq 0} u_n e^{in\sigma}. \quad (28)$$

Together with a new variable, $u(\sigma)$ will form function $\varphi(\sigma)$:

$$\varphi(\sigma) = q + \frac{P}{2\pi}\sigma + \sum_{n \neq 0} \frac{u_n}{in} e^{in\sigma}. \quad (29)$$

If we set

$$\{q, p\} = 1, \quad (30)$$

then

$$\{\varphi(\sigma), \varphi(\sigma')\} = -\Theta(\sigma - \sigma'), \quad (31)$$

where

$$\Theta(\sigma) = \frac{\sigma}{2\pi} + \sum \frac{e^{in\sigma}}{in \cdot 2\pi} \quad (32)$$

is a “step function” on a circle with the following properties:

$$\begin{aligned} \Theta(\sigma)_{\sigma \rightarrow \pm 0} &= \pm 1/2, \\ \Theta(\sigma + 2\pi) &= \Theta(\sigma) + 1, \\ \Theta'(\sigma) &= \Delta(\sigma). \end{aligned} \quad (33)$$

It is clear that

$$u(\sigma) = \varphi'(\sigma).$$

The new variable allows us to construct the gauge transformation

$$g(\sigma, x) = \exp\left[-\frac{p}{4\pi}\sigma + \varphi(\sigma) - \frac{ix}{2}\right]\tau_-/2. \quad (34)$$

Note that

$$g(\sigma + 2\pi)g^{-1}(\sigma, x) = e^{\tau - P/4}. \quad (35)$$

Let us define gauge transformed monodromy matrix $t_\sigma^{\sigma+2\pi}(x)$

$$t_\sigma^{\sigma+2\pi}(x) = g^{-1}(\sigma + 2gx)T_\sigma^{\sigma+2\pi}g(\sigma, x). \quad (36)$$

It is not difficult to calculate the PB of these new monodromies and find that it is regular when $\sigma_1 \rightarrow \sigma_2$:

$$\begin{aligned} \{t_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes t_{\sigma_2}^{\sigma_2+2\pi}(y)\}_{\sigma_1 \rightarrow \sigma_2 + \epsilon} &= \{t_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes t_{\sigma_2}^{\sigma_2+2\pi}(y)\}_{\sigma_1 \rightarrow \sigma_2 - \epsilon} = \\ &= -\frac{2}{x^2 - y^2}[\tau_i, \otimes \tau_i, t_{\sigma_1}^{\sigma_1+2\pi}(x) \otimes \tau_{\sigma_1}^{\sigma_1+2\pi}(y)]. \end{aligned} \quad (37)$$

The appearance of classical r -matrix of the Heisenberg magnetic in r.h.s. of (37) guaranties the fulfilment of the Jacoby identity and therefore closeness of the algebra of monodromies with a fixed initial point. It is easy to find also the gauge transformed L -operator:

$$\begin{aligned} g^{-1}(\sigma)Lg(\sigma) &= -\frac{d}{d\sigma} + \left\{ \tau_3\left(\varphi(\sigma) - \frac{P}{4\pi}\sigma\right) + \frac{1}{2}\tau_+ \right. \\ &\quad \left. + \frac{1}{2}\tau_- \left[\frac{P}{4\pi} - \left(\varphi(\sigma) - \frac{P}{4\pi}\right)^2 - \frac{x^2}{4} \right] \right\}. \end{aligned} \quad (38)$$

This example shows that in this non-ultralocal case we can find a set of monodromies which form a closed algebra with respect to the PB. Remarkably that this result probably survives quantization: in [4] it has been stated that properly ordered quantum monodromy exhibits the same commutation relation as in the ultralocal case, though the authors did not check the coincidence of the left and right limits of initial points.

Several comments have to be made in conclusion. In [5] the author considered the same problem not for periodic boundary conditions but vanishing potentials on a infinite line. The statement is that monodromy (scattering matrix in this case) forms the closed algebra even without gauge transformation (36), (38). The transition from the periodical case to the infinite line is very delicate when the dependence of initial points is nontrivial and we cannot compare our results. The other question is the universality of the present construction. We failed to find an appropriate gauge transformation in the string case — (8), (19). However situation here is the following: the distinction in left and right limits in r.h.s. of (18) starts from the term of order x^3y^3 of spectral parameters. We found the gauge transformation which removes discontinuity up to order x^4y^4 . Principally one could go along this line and remove it up to the next order, but it seems that it is a wrong way, which leads to trivialization of monodromy matrix by extracting half of variables out of it.

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