USING SALPETER'S PROPAGATOR FOR SOLVING THE BETHE-SALPETER EQUATION

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SUMMARY. We transform the Bethe-Salpeter equation for two fermions into an equivalent Salpeter equation, with a 3D potential directly computable from a series of reducible and irreducible Feynman graphs. We present a perturbation calculation of the energy spectrum, taking Salpeter's equation as the unperturbed equation. We compare this approach with the 4D approach of Bodwin and Yennie, and, briefly, with other 3D reduction methods.

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1. Introduction.

One of the most popular tools for computing bound states from quantum field theory is the Bethe-Salpeter equation (BSE) [1,2]. The principal difficulty of this equation comes from its 4-dimensional character, due to the presence of the "relative time-energy" degree of freedom. The opposite sign in the metrics implies that this supplementary degree of freedom can not directly be treated like the three usual "relative position-momentum" degrees of freedom.

A radical, often used and somewhat arbitrary way of solving the relative time problem consists in making an "instantaneous approximation". This kills immediately the relative time degree of freedom, and transforms the Bethe-Salpeter equation into the 3D equation known as Salpeter's equation [3]. For more refined calculations, one performs usually a perturbation calculation starting with a carefully chosen 3D equation [3-20]. Although all choices of this 3D equation should theoretically lead to the same results if all orders could be computed, important differences may appear when approximations (like the truncation of series) are made. In order to get a sufficiently simple unperturbed 3D equation (like an effective Schrödinger-Coulomb or Dirac-Coulomb equation), one is obliged to approximate both the free propagator and the BS kernel (the "4D potential"). One remains however free to choose which approximation will be performed first, and/or to choose some less effective 3D equation (like Salpeter's [3], Breit's [21] or Sazdjian's [14] equation) as an intermediate step. These equations incorporate more physics (like the particle-antiparticle symmetry) but must themselves be solved by perturbations.

One must thus choose to approximate either the BS kernel or the free propagator first. If one approaches the BS kernel by some instantaneous approximation, the resulting unperturbed equation can be immediately transformed into a Salpeter equation. The perturbation calculation is however performed in the 4D space [12,17], and we shall therefore call this approach the 4D approach. If one approaches the free propagator by some expression containing a δ fixing the relative energy, one gets a 3D equation, in which the remainder of the propagator appears in a series of higher-order contributions (to the 3D potential), which we shall call the reduction series. This series is to be combined with the series giving the BS kernel itself. This 3D equation, with the resulting 3D potential, is in principle equivalent to the original Bethe-Salpeter equation. The 3D

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potential is of course too complicated to be used in an exact calculation, and will therefore be approximated by a part of the first term of the reduction series, the remainder of the first term and the other terms being treated by the ordinary 3D perturbation methods. We shall call this approach the 3D approach. It has been more largely used [3-11,13-16,18-20]. It is indeed more versatile, as we are not limited to Salpeter's propagator: we can also choose (for example) Breit's or Sazdjian's propagator, or effective Schrödinger or Dirac propagators, if we want to start with a directly solvable unperturbed equation. Instead of a single 3D equation, in which the relative time-energy degree of freedom is completely eliminated, it is also possible to write a pair of coupled Dirac equations, in the spirit of the "relativistic quantum mechanics" (RQM) [22-27], related to the "constraint relativistic classical mechanics". The relative time-energy degree of freedom is not completely eliminated, but behaves as in the free equations.

In an earlier work [19], we proposed such a 3D reduction method, inspired by Sazdjian [14] and the constraint relativistic mechanics theory [22-27], for the more general two-fermion plus potential problem. The free propagator was approximated by its integral with respect to the relative energy, times a δ fixing the relative energy to the value given by the no-mutual interaction Dirac equations. Our 3D equation was a Salpeter equation (generalized in order to take the external potential into account), with a non-local and energy-dependent potential, directly computable from a series of reducible and irreducible Feynman graphs. A generalization of the well-known projecting operator of Salpeter's equation appeared naturally, eliminating the "continuum dissolution" problem.

This paper is devoted to the 3D approach with Salpeter's propagator, combining the advantages of a straightforward relation with RQM and the possibility of easily introducing an external potential, without continuum dissolution. It has also the property, not shared by all 3D reductions, that the 3D potential reduces to the first term of the reduction series when the BS kernel is instantaneous. This feature makes possible a term by term comparison with a 4D perturbation calculation around the same unperturbed problem. To be definite, the common unperturbed problem chosen here is Salpeter's equation for two fermions with a Coulomb potential, and the 4D perturbation calculation is taken from the review articles of Bodwin and Yennie [12] and Murota [17] on the muonium ($e^-\mu^+$) and positronium (e^-e^+) problems. The comparison is performed up to the $(K_R)^2$ contributions to the energy (K_R = rest of the BS kernel). Our investigations confirm that both methods give the same results (it was a wise precaution to check it). It is however important to see how the terms of one expansion are transformed and combined into terms of the other one. In practice, indeed, supplementary approximations will be made (limited expansions in powers of α or v/c , approximations on the sum on the intermediate states) and these approximations could be the origin of nonnegligible differences between the results of both methods.

1.1 Organization of the paper.—In section 2, we present the BS machinery: two-fermion propagator and amplitude, expansion of the propagator in term of amplitudes, inhomogeneous and homogeneous BSE, normalization of the BS amplitudes. In section 3, we present our 3D reduction method of the BSE. In section 4, we specialize to an instantaneous interaction, compute the norm of the wave functions (generalizable to a scalar product) and the 3D propagator. In section 5, we present a perturbation calculation around the Salpeter-Coulomb equation. In section 6, we present the 4D approach of Bodwin and Yennie [12] and show the equivalence of their expression of the energy spectrum with our's (up to $(K_R)^2$). Section 7 is devoted to a brief comparison of the two approaches with each other and with some other 3D reduction methods, and to conclusions.

1.2 Notations.— We denote by x_i the position of the particle i , by p_i its conjugate momentum and by m_i its mass. The total (or external, or CM) and relative variables will be

$$X = \frac{1}{2}(x_1 + x_2), \quad P = p_1 + p_2, \quad (1.1)$$

$$x = x_1 - x_2, \quad p = \frac{1}{2}(p_1 - p_2). \quad (1.2)$$

The Dirac hamiltonian of the free fermion i is

$$h_i = \bar{\alpha}_i \cdot \bar{p}_i + \beta_i m_i, \quad (1.3)$$

and we shall define

$$S = h_1 + h_2, \quad s = \frac{1}{2}(h_1 - h_2), \quad (1.4)$$

and the projection operators on the positive or negative energy free states:

$$\Lambda_i^\pm = \frac{1}{2} \left(1 \pm \frac{h_i}{\sqrt{h_i^2}} \right), \quad \Lambda^{\alpha\beta} = \Lambda_1^\alpha \Lambda_2^\beta. \quad (1.5)$$

The free propagator of the fermion i will be:

$$G_{0i} = \frac{\gamma_i \cdot p_i + m_i}{p_i^2 - m_i^2 + i\epsilon} = \frac{1}{p_{i0} - h_i + i\epsilon h_i}, \quad (1.6)$$

where the infinitesimal part can be obtained by making $p_{i0} \rightarrow p_{i0} + i\epsilon p_{i0}$ (in the second expression, $i\epsilon p_{i0}$ has been replaced by $i\epsilon h_i$, which has the same sign near the pole when diagonalized).

In order to preserve the readability, we left a certain amount of ambiguity in the notations, the meaning of some symbols being precised by the context. For example, p_0 is in general the relative energy operator. It is a number in the momentum representation and it is $i \frac{\partial}{\partial t}$ in the position representation. We write φ for a quantum mechanics wave function, $\varphi(\vec{x})$ for its position representation and $\varphi(\vec{p})$ (the Fourier transform of the latter) for its momentum representation, without using tilda's, as we use sometimes a position representation for the relative variables combined with a momentum representation for the total variables.

We work in a spinor-spinor representation. The spinor-conjugate spinor representation should be useful with a particle-antiparticle system ($e^+ e^-$, $q \bar{q}$), in order to compute the charge conjugation properties and the annihilation graph more easily. Everything can of course be done in both representations.

We work in an arbitrary Lorentz frame. In such a frame, it would be easy to add external potentials $V_i(\vec{x}_i, \gamma_i)$ (referring to a "laboratory frame") to the hamiltonians h_i , as we did in a previous work [19]. In this case P_0 is conserved but not $\vec{P}$, and the BSE is not covariant. In the absence of such potentials, the BSE is covariant and $\vec{P}$ is also conserved, becoming a simple parameter (in sections (3-7) we shall work in a fixed $\vec{P}$ subspace without indexing all operators and wave functions). The choice $\vec{P} = \vec{0}$ (center of mass frame) is of course a must for the actual calculations. Furthermore, our various formulas (deduced from a covariant BSE but not a priori covariant themselves) can then be easily covariantized by working in the center of mass frame and by making all quantities covariant with the help of the 4-vector P .

2. Bethe-Salpeter equation.

This section contains a short presentation of the Bethe-Salpeter equation, in which various conventions (signs, normalizations...) used in this work are introduced.

The propagator of a two-fermion system can be written, in the position representation

$$G(x'_1, x'_2; x_1, x_2) = \langle 0 | T (\Psi_1(x'_1) \Psi_2(x'_2) \bar{\Psi}_1(x_1) \bar{\Psi}_2(x_2)) | 0 \rangle = \frac{1}{(2\pi)^4} \int d^4 P \exp[-i P \cdot (X' - X)] G_P(x', x) \quad (2.1)$$

and obeys the inhomogeneous BSE

$$G = G_0 + G_0 K G, \quad (2.2)$$

where the BS kernel K is the sum of an infinity of terms represented by the irreducible Feynman graphs, while the free propagator G_0 can be written as

$$G_0 = (p_{10} - h_1 + i\epsilon h_1)^{-1} (p_{20} - h_2 + i\epsilon h_2)^{-1} \beta_1 \beta_2. \quad (2.3)$$

The BS amplitudes and conjugated amplitudes for the fermion and the antifermion are

$$\chi_{P,\alpha}(x'_1, x'_2) = \langle 0 | T(\Psi_1(x'_1) \Psi_2(x'_2)) | P, \alpha \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2E}} \exp[-iP \cdot X'] \chi_{P,\alpha}(x'), \quad (2.4)$$

$$\bar{\chi}_{P,\alpha}(x_1, x_2) = \langle P, \alpha | T(\bar{\Psi}_1(x_1) \bar{\Psi}_2(x_2)) | 0 \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2E}} \exp[iP \cdot X] \bar{\chi}_{P,\alpha}(x), \quad (2.5)$$

$$\eta_{-P,\alpha}(x'_1, x'_2) = \langle P, \alpha | T(\Psi_1(x'_1) \Psi_2(x'_2)) | 0 \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2E}} \exp[iP \cdot X'] \eta_{-P,\alpha}(x'), \quad (2.6)$$

$$\bar{\eta}_{-P,\alpha}(x_1, x_2) = \langle 0 | T(\bar{\Psi}_1(x_1) \bar{\Psi}_2(x_2)) | P, \alpha \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2E}} \exp[-iP \cdot X] \bar{\eta}_{-P,\alpha}(x). \quad (2.7)$$

with

$$E = (\bar{P}^2 + M^2)^{1/2}. \quad (2.8)$$

The amplitudes and the conjugated amplitudes are related by

$$\bar{\chi}_{P,\alpha}(x) = -\chi_{P,\alpha}^+(x) \beta_1 \beta_2 \Big|_{\theta(x_0) \rightleftharpoons \theta(-x_0)}, \quad \bar{\eta}_{P,\alpha}(x) = -\eta_{P,\alpha}^+(x) \beta_1 \beta_2 \Big|_{\theta(x_0) \rightleftharpoons \theta(-x_0)}. \quad (2.9)$$

This permutation of the θ 's means that the $x_0 > 0$ and the $x_0 < 0$ parts must be replaced by the analytic continuation of each other.

In the propagator, we can isolate the poles of a bound state and an antibound state of mass M [28]:

$$G_P(x', x) = i \sum_{\alpha} \left\{ \frac{1}{P_0 - E + i\epsilon} \chi_{\bar{P},\alpha}(x') \bar{\chi}_{\bar{P},\alpha}(x) - \frac{1}{P_0 + E - i\epsilon} \eta_{\bar{P},\alpha}(x') \bar{\eta}_{\bar{P},\alpha}(x) \right\} + \dots \quad (2.10)$$

with

$$\chi_{\bar{P},\alpha}(x') = \chi_{E,\bar{P},\alpha}(x'), \quad \bar{\chi}_{\bar{P},\alpha}(x) = \bar{\chi}_{E,\bar{P},\alpha}(x), \quad \eta_{\bar{P},\alpha}(x') = \eta_{-E,\bar{P},\alpha}(x'), \quad \bar{\eta}_{\bar{P},\alpha}(x) = \bar{\eta}_{-E,\bar{P},\alpha}(x). \quad (2.11)$$

If we put expression (2.10) into the inhomogeneous BSE (2.2), factorizing the $\delta^4(P' - P)$, and keeping the residu of the pole at $P_0 = E$, we get the homogeneous BS equation for the bound states amplitudes:

$$\chi = G_0 K \chi, \quad \bar{\chi} = \bar{\chi} K G_0, \quad (2.12)$$

where the amplitude χ is a double spinor function of (x_1, x_2) , (P, x) or (P, p) ... according to the chosen representation. For the scattering states, an independent term χ_0 would be added. These equations are true only at $P_0 = E$.

If we multiply the inhomogeneous BSE (2.2) by GG_0^{-1} at left, use the expansion (2.10) of G and keep the residue of the pole at $P_0 = E$ (or $P_0 = -E$), we get an orthonormalization condition for BS amplitudes sharing a common total energy momentum $(E, \vec{P})$:

$$i \int d^4x d^4y' \bar{\chi}_{\vec{p},\alpha}(x) \left[\frac{\partial}{\partial P_0} G_P^{-1}(x, y') \right]_{P_0=E} \chi_{\vec{p},\beta}(y') = \delta_{\alpha\beta} \quad (2.13)$$

with $G^{-1} = G_0^{-1} - K$. For different energies, we can generalize (2.13) to [10,14,29]

$$i \int d^4x d^4y' \bar{\chi}_{\vec{p},\alpha}(x) \frac{G_{E,\vec{p}}^{-1}(x, y') - G_{E',\vec{p}}^{-1}(x, y')}{E_\alpha - E_\beta} \chi_{\vec{p},\beta}(y') = \delta_{\alpha\beta}. \quad (2.14)$$

3. 3D reduction.

As we want to obtain a pair of coupled Dirac equations, we shall first write the free propagator G_0 in terms of the operators $(P_0 - S)$, $(p_0 - s)$, which are respectively the sum and the half-difference of the operators $(p_{10} - h_1)$, $(p_{20} - h_2)$ used in the free Dirac equations. We shall then approach the propagator G_0 by an operator G_δ proportional to $\delta(p_0 - s)$. A straightforward choice is

$$G_0 \approx G_\delta = \delta(p_0 - s) \int dp_0 G_0(p_0) = \frac{-2i\pi\tau}{P_0 - S} \delta(p_0 - s) \beta_1 \beta_2 \quad (3.1)$$

with

$$\tau = \frac{1}{2}(\tau_1 + \tau_2) = \Lambda^{++} - \Lambda^{--}, \quad \tau_i = \frac{h_i}{\sqrt{h_i^2}} = \frac{h_i}{\sqrt{\vec{p}_i^2 + m_i^2}} = \text{sign}(h_i). \quad (3.2)$$

The denominator of G_δ must also contain an infinitesimal imaginary part, obtained by replacing P_0 by $P_0 + i\epsilon P_0$, as usual (this imaginary part can be forgotten in the bound state region). Writing G_0 as $G_\delta + G_R$ in equations (2.12), we get

$$\chi = G_0 K \chi = (G_\delta + G_R) K \chi = \Psi + G_R K \chi, \quad (3.3)$$

with

$$\Psi = G_\delta K \chi \quad (= G_\delta G_0^{-1} \chi). \quad (3.4)$$

Solving (formally) the inhomogeneous equation (3.3) and putting the result into (3.4), we get

$$\Psi = G_\delta K (1 - G_R K)^{-1} \Psi = G_\delta K_T \Psi \quad (3.5)$$

where

$$K_T = K (1 - G_R K)^{-1} = K + K G_R K + \dots = (1 - K G_R)^{-1} K \quad (3.6)$$

obeys

$$K_T = K + K G_R K_T = K + K_T G_R K. \quad (3.7)$$

The higher-order terms of the series (3.6) re-introduce the reducible Feynman graphs into the kernel, but with G_0 replaced by G_R . We can write (3.5) as the sum and the half-difference of a pair of coupled Dirac equations by multiplying it by $P_0 - S$ and $p_0 - s$ respectively:

$$(P_0 - S)\Psi = -2i\pi\tau\delta(p_0 - s)\beta_1\beta_2 K_T \Psi, \quad (p_0 - s)\Psi = 0. \quad (3.8)$$

In the BS→RQM transformation, we have “trivialized” the relative time: the relative time dependence of Ψ is that of a wave function of free fermions. In the RQM→QM (quantum mechanics) transformation, we shall completely eliminate the relative time with the transformation

$$\Psi = \exp[-isx_0]\psi, \quad \psi = \Psi(x_0 = 0). \quad (3.9)$$

Eqs. (3.8) become then

$$(P_0 - S)\psi = \tau V\psi, \quad p_0\psi = 0, \quad (3.10)$$

with

$$V = -i \int dt' dt \exp[ist'] \beta_1 \beta_2 K_T(t', t) \exp[-ist]. \quad (3.11)$$

The second equation (3.10) can be forgotten, provided we also forget the relative time-energy degree of freedom (x_0, p_0) , which had been already “trivialized” by the BS → RQM transformation.

Let us also write (3.11) in the (p'_0, p_0) representation:

$$V = -2i\pi \int dp'_0 dp_0 \delta(p'_0 - s) \beta_1 \beta_2 K_T(p'_0, p_0) \delta(p_0 - s). \quad (3.12)$$

It is interesting to compute G_0 , G_δ and $G_R = G_0 - G_\delta$ in the (t', t) representation. For G_R we have

$$G_R(t', t) = G_R(t' - t) = \frac{1}{2\pi} \int dp_0 \exp[-ip_0(t' - t)] G_R(p_0), \quad (3.13)$$

and similarly for G_0 and G_δ . Performing this integral by residues for the different signs of $(t' - t)$, h_1 , h_2 and gathering the results, we get

$$\begin{aligned} G_R(t' - t) = & -\frac{i}{H_0} \exp[-is(t' - t)] \left\{ \left(\exp\left[i \frac{H_0}{2} \tau |t' - t|\right] - 1 \right) \tau \right. \\ & \left. + \left(\exp\left[i \frac{H_0}{2} (t' - t)\right] - \exp\left[-i \frac{H_0}{2} (t' - t)\right] \right) (\theta(t' - t) \Lambda^{+-} - \theta(t - t') \Lambda^{-+}) \right\} \beta_1 \beta_2 \end{aligned} \quad (3.14)$$

where $H_0 = P_0 - S$. The contribution of the -1 term corresponds to $-G_\delta(t' - t)$:

$$G_\delta(t' - t) = \frac{-i\tau}{H_0} \exp[-is(t' - t)] \beta_1 \beta_2 \quad (3.15)$$

the rest of (3.14) being $G_0(t' - t)$. There is no pole at $H_0 = 0$ in G_R , as the numerator vanishes. We see also that G_R is zero when $t' - t = 0$. This implies that the reduction series (3.6) giving K_T is limited to its first term K in the case of an instantaneous interaction, an interesting feature not shared by all 3D reductions.

The operator τ is $+1, -1, 0$ when applied on eigenstates of h_1, h_2 with positive, negative or mixed eigenvalues respectively. Its presence in equation (3.10) allows for the existence of solutions entirely in the $\tau^2 = 1$ subspace. This fact eliminates several difficulties, like the “continuum dissolution” problem when an external potential is present [19,30-32] (in our formalism, an external potential can be introduced by simply adding a $V_i(\vec{x}_i, \gamma_i)$ term to the definition (1.3) of h_i [19]).

4. Instantaneous interaction.

In sections 4-7, we shall work in a constant $\vec{P}$ subspace, removing a phase (as in (2.1)) from the wave functions and BS amplitudes, and a $\delta(P' - P)$ from the operators, without writing $\vec{P}$ indexes everywhere.

We call “instantaneous” a BS kernel $K = K_c$ which takes the form

$$K_c(t', t) = \delta(t') \bar{K}_c \delta(t) \quad (4.1)$$

in the (t', t) representation. As $G_R(0, 0) = 0$, we have also

$$K_T = K_c + K_c G_R K_c + \dots = K_c \quad (4.2)$$

so that the reduction series is limited to the first term. There is no convergence problem anymore, and the 3D reduction of the BS equation is straightforward. The QM potential V becomes

$$V = V_c = -i\beta_1 \beta_2 \bar{K}_c \quad (4.3)$$

and the QM equation (3.10) becomes Salpeter's equation [3]

$$(P_0 - S) \varphi = \tau V_c \varphi. \quad (4.4)$$

with $\varphi = \psi_c$. The BS amplitude χ is given in terms of the RQM wave function φ by

$$\chi = \chi_c = (1 - G_R K_c)^{-1} \Psi_c = (1 + G_R K_c) \Psi_c \quad (4.5)$$

$$\begin{aligned} \chi_c(t) &= \{ \exp[-ist] + G_R(t) \bar{K}_c \} \varphi \\ &= \exp[-ist] \left\{ \exp \left[i \frac{H_0}{2} \tau |t| \right] + \frac{2}{H_0} \sin \left(\frac{H_0}{2} t \right) [\theta(t) \Lambda^{+-} - \theta(-t) \Lambda^{-+}] \beta_1 \beta_2 \bar{K}_c \right\} \varphi \end{aligned} \quad (4.6)$$

replacing $G_R(t)$ by its expression (3.14) and using the QM equation (4.4).

If K_c does not depend on P_0 , the BS orthonormalization (2.14), calculated in terms of the QM wave functions φ , sharing or not sharing the same energy, but sharing a common spatial momentum $\vec{P}$ is

$$(\varphi_i, \varphi_j) = \int d^3x \varphi_i^+(\vec{x}) (\tau \varphi_j)(\vec{x}) = (\varphi_i^+ \tau \varphi_j) = \pm \delta_{ij}, \quad (4.7)$$

where the sign (-) is for the negative energy states (this scalar product could also be derived directly from (4.4) [12,29]). In the last, more compact expression, the spatial integration has been included in the contraction with the final wave function. If we want to consider (4.7) as a 6-dimensional equation, we must multiply the wave functions by $\exp(i \vec{P} \cdot \vec{X})$, and complete the scalar product by an integration on $\vec{X}$, which gives a $\delta(\vec{P}' - \vec{P})$. The hamiltonian $S + \tau V$, which is not symmetric for the usual scalar product $(\varphi_i^+ \varphi_j)$, is symmetric for the scalar product (4.7) (in the $\tau^2 = 1$ subspace) and the wave functions corresponding to different values of the energy are orthogonal.

If we can find an hermitian operator Γ , which anticommutes with the h_i , commutes with V_c and squares to 1 (such as $\beta_1 \beta_2 \gamma_{51} \gamma_{52}$ when V_c is a Coulomb potential), we can immediately show that $\Gamma \varphi$ is also a solution of Salpeter's equation with energy $-P_0$ and norm -1 (this is an easy way of computing negative energy solutions, this is not the charge conjugation operation).

The homogeneous BSE (2.12) leads directly to another expression of χ_c in terms of φ :

$$\chi_c = G_0 K_c \varphi, \quad \bar{\chi}_c = -\bar{\varphi} K_c G_0 \Big|_{i\epsilon \rightleftharpoons -i\epsilon} \quad (4.8)$$

with $\bar{\varphi} = \varphi^+ \beta_1 \beta_2$ (we recover the usual definition of the “barred” spinor). This expression of χ_c is of course equivalent to (4.6). The QM wave function φ obeys a 3D Bethe-Salpeter equation, which can be obtained by writing (4.8) at $t = 0$:

$$\varphi = g_0 V_c \varphi \quad (4.9)$$

where g_0 is Salpeter's free propagator

$$g_0 = iG_S(0,0)\beta_1\beta_2 = iG_0(0,0)\beta_1\beta_2 = \frac{\tau}{H_0}. \quad (4.10)$$

The BS equation for the G_c propagator

$$G_c = G_0 + G_0 K_c G_c = G_0 + G_c K_c G_0 = G_0 + G_0 K_c G_0 + G_0 K_c G_c K_c G_0 \quad (4.11)$$

can also be reduced to its 3D form by taking it at $t'=t=0$. Writing

$$g_c = iG_c(0,0)\beta_1\beta_2, \quad (4.12)$$

we get

$$g_c = g_0 + g_0 V_c g_c = g_0 + g_c V_c g_0 = g_0 + g_0 V_c g_0 + g_0 V_c g_c V_c g_0. \quad (4.13)$$

From (2.10), we can deduce an expansion of g_c , explicitating the contributions of the positive and negative energy two-body bound and scattering states (the sum becomes an integral for the scattering states):

$$g_c = \sum_i \left[\frac{1}{P_0 - E_i + i\epsilon} (\varphi_i \varphi_i^+)_+ - \frac{1}{P_0 + E_i - i\epsilon} (\varphi_i \varphi_i^+)_- \right] + \dots \quad (4.14)$$

Multiplying (4.13) by $(P_0 - S)$, we find that $(P_0 - S - \tau V_c)g_c = \tau$. If we assume that the φ_i are a basis of the $\tau^2 = 1$ subspace, we can write the completeness relation

$$\sum_i [(\varphi_i \varphi_i^+)_+ - (\varphi_i \varphi_i^+)_-] = \tau \quad (4.15)$$

which corresponds to the scalar product (4.7). Writing then

$$g_c = \frac{1}{P_0 - (S + \tau V_c) + i\epsilon P_0} \tau = \frac{1}{P_0 - (S + \tau V_c) + i\epsilon P_0} \sum_i [(\varphi_i \varphi_i^+)_+ - (\varphi_i \varphi_i^+)_-] \quad (4.16)$$

we get the explicitated (no "+"...) terms of the expansion (4.14).

A final remark: we must not deduce from (4.15) that the two parts of the completeness expansion are equal to Λ^{++} and $-\Lambda^{--}$ separately, as the φ_i are not eigenstates of τ , but only of τ^2 .

Salpeter's equation (4.4) will be used as the zero-order equation for the perturbation calculation of the exact energy spectrum. Unfortunately, this equation itself is still too complicated to be solved exactly. In Refs. [12,17], approximated solutions are computed by iterations. We give elsewhere [29] a calculation of Salpeter's energy spectrum, wave functions and full propagator: Salpeter's equation is transformed into an effective Schrödinger or Dirac equation with a modified potential, and the difference between this modified potential and the original potential V_c is treated as a perturbation. If we take for V_c the Coulomb potential

$$V_c = \frac{-e^2}{|\vec{x}|} \quad (4.17)$$

we get a Schrödinger-Coulomb or a Schrödinger-Dirac equation. The energy spectra and the solutions of these equations are analytically known [33], and for the first one there exists also a compact expression for the full propagator [34-36].

5. Perturbation calculation around Salpeter's equation.

We start with the 3D equation (3.10), writing V as $V_c + V_{RT}$:

$$(P_0 - S)\psi = \tau(V_c + V_{RT})\psi \quad (5.1)$$

to be solved by perturbations around Salpeter's equation

$$(P_0 - S)\varphi = \tau V_c \varphi \quad (5.2)$$

with V_c computed from an instantaneous P_0 -independent BS kernel K_c :

$$V_c = -i\beta_1\beta_2\overline{K}_c. \quad (5.3)$$

In order to match the preceding section, we take for V_c the Coulomb potential (4.17), which is the most important contribution of the one-photon exchange graph in the Coulomb gauge [19]. The perturbation potential V_{RT} is given by

$$V_{RT} = -i \int dt' dt \exp[ist'] \beta_1 \beta_2 K_{RT}(t', t) \exp[-ist] \quad (5.4)$$

where K_{RT} is given by $K_R = K - K_c$ (the rest of the one-photon exchange graph and the other irreducible graphs) plus the iterations of $K_c + K_R$:

$$K_{RT} = K_R + (K_c + K_R) \sum_{n=1}^{\infty} \{G_R(K_c + K_R)\}^n. \quad (5.5)$$

This is an expansion in powers of α . Let us now hold all powers of K_c and the first and second powers of K_R . Keeping in mind that $K_c G_R K_c = 0$, we get 12 terms:

$$\begin{aligned} K_{RT} = & K_R + K_R G_R K_c + K_c G_R K_R + K_R G_R K_R \\ & + K_c G_R K_R G_R K_c + K_c G_R K_R G_R K_R + K_R G_R K_R G_R K_c + K_R G_R K_c G_R K_R \\ & + K_c G_R K_R G_R K_c G_R K_R + K_c G_R K_R G_R K_R G_R K_c + K_R G_R K_c G_R K_R G_R K_c \\ & + K_c G_R K_R G_R K_c G_R K_R G_R K_c. \end{aligned} \quad (5.6)$$

Four terms (nrs. 1,2,3,5) are of first order in K_R and the other ones are of second order.

Writing (5.1) as

$$(P_0 - S - \tau V_c) \psi = \tau V_{RT} \psi \quad (5.7)$$

and inverting the operator of the left-hand side (which can be done when P_0 is not an eigenvalue of the Salpeter-Coulomb hamiltonian), we get

$$\psi = g_c V_{RT} \psi. \quad (5.8)$$

We shall search for a solution ψ_i by perturbations around a solution φ_i of the unperturbed equation. Using the expansion (4.14) of g_c :

$$g_c = g_{ci} + \overline{g}_{ci}, \quad g_{ci} = \frac{1}{P_0 - E_i} \varphi_i \varphi_i^+ \quad (5.9)$$

we have

$$\psi_i = \varphi_i \frac{(\varphi_i^+ V_{RT} \psi_i)}{P_0 - E_i} + \overline{g}_{ci} V_{RT} \psi_i \quad (5.10)$$

in which

$$(\varphi_i^+ V_{RT} \psi_i) = (\varphi_i^+ [\tau \tau V - V_c \tau \tau] \psi_i) = (\varphi_i^+ [\tau(P_0 - S) - (E_i - S)\tau] \psi_i) = (P_0 - E_i)(\varphi_i^+ \tau \psi_i). \quad (5.11)$$

If we normalize ψ_i to

$$(\varphi_i^+ \tau \psi_i) = 1 \quad (5.12)$$

we get

$$P_0 - E_i = (\varphi_i^+ V_{RT} \psi_i), \quad \psi_i = \varphi_i + \bar{g}_{ci} V_{RT} \psi_i \quad (5.13)$$

and, after iterating:

$$P_0 - E_i = (\varphi_i^+ T_i \varphi_i), \quad T_i = V_{RT} + V_{RT} \bar{g}_{ci} V_{RT} + \dots \quad (5.14)$$

We must not forget that T_i depends on P_0 via $\bar{g}_{ci}$ and via V_{RT} . Equation (5.14) is thus a numerical equation in P_0 , that we can solve at each order by expanding $T_i(P_0)$ around $P_0 = E_i$. The second-order result is

$$P_0 - E_i = \left\{ (\varphi_i^+ V_{RT} \varphi_i) + (\varphi_i^+ V_{RT} \bar{g}_{ci} V_{RT} \varphi_i) + (\varphi_i^+ V_{RT} \varphi_i) \frac{\partial}{\partial P_0} (\varphi_i^+ V_{RT} \varphi_i) \right\}_{P_0=E_i} \quad (5.15)$$

with

$$(\varphi_i^+ V_{RT} \bar{g}_{ci} V_{RT} \varphi_i) = \sum_{j \neq i} (\varphi_i^+ V_{RT} \varphi_j) \frac{\epsilon_j}{P_0 - \epsilon_j E_j} (\varphi_j^+ V_{RT} \varphi_i), \quad (5.16)$$

denoting by ϵ_j is the sign of the energy of the solution φ_j .

For an expansion up to $(K_R)^2$, we must keep 12 terms of K_{RT} in V_{RT} , and only four terms in V_{RT}^2 . For an expansion up to α^2 , we should keep the first line of (5.6) in the term in V_{RT} , and only K_R in the term in V_{RT}^2 . This refers only to a naïve vertex counting. The bound state wave functions depend also on α , and this dependence brings supplementary powers of α into the matrix elements, in a non-trivial way.

Instead of performing the sum on the intermediate states, we could try to use a compact form of $\bar{g}_{ci}$. As a first approximation, we could use a known compact form of the Schrödinger-Coulomb propagator [34-36], and remove the pole. In Ref.[29], we show how to improve this approximation. We could also try to use eqs.(4.13) for g_c . As it would be difficult to remove the pole at $P_0 = E_i$ from g_c , since the bound state poles appear only after an infinite number of iterations, we shall transform eqs. (4.13) into equations directly written for $\bar{g}_{ci}$ by using (5.9) and the following identities derived from the BS equations for the wave functions:

$$g_0 V_c \varphi_i = \varphi_i - \frac{P_0 - E_i}{P_0 - S} \varphi_i, \quad \varphi_i^+ V_c g_0 = \varphi_i^+ - \varphi_i^+ \frac{P_0 - E_i}{P_0 - S}. \quad (5.17)$$

The resulting equations are

$$\bar{g}_{ci} = [g_0 - \frac{1}{H_0} \varphi_i \varphi_i^+] + g_0 V_c \bar{g}_{ci} \quad (5.18)$$

$$\bar{g}_{ci} = [g_0 - \varphi_i \varphi_i^+ \frac{1}{H_0}] + \bar{g}_{ci} V_c g_0 \quad (5.19)$$

$$\bar{g}_{ci} = [g_0 - \frac{1}{H_0} \varphi_i \varphi_i^+ - \varphi_i \varphi_i^+ \frac{1}{H_0} + (P_0 - E_i) \frac{1}{H_0} \varphi_i \varphi_i^+ \frac{1}{H_0}] + g_0 V_c g_0 + g_0 V_c \bar{g}_{ci} V_c g_0. \quad (5.20)$$

6. Perturbation calculation in the 4-dimensional space.

This perturbation calculation has been performed by Bodwin and Yennie [12], directly in the 4D space. The second-order energy shift is given by an expression similar to (5.15):

$$P_0 - E_i = i \left\{ (\bar{\chi}_{ci} K_R \chi_{ci}) + (\bar{\chi}_{ci} K_R \bar{G}_{ci} K_R \chi_{ci}) + i (\bar{\chi}_{ci} K_R \chi_{ci}) \frac{\partial}{\partial P_0} (\bar{\chi}_{ci} K_R \chi_{ci}) \right\}_{P_0=E_i}. \quad (6.1)$$

where $\overline{G}_{ci}$ is G_c with the pole at $P_0 = E_i$ removed. The contraction with the final wave function, in (6.1) and in (6.9) below, includes now a 4D integration. It seems difficult to evaluate $\overline{G}_{ci}$ directly. However, if we consider the following iteration of G_c :

$$G_c = G_0 + G_0 K_c G_0 + G_0 K_c G_c K_c G_0 \quad (6.2)$$

we see that $G_c(t', t)$ can be replaced by $G_c(0, 0) = -i g_c \beta_1 \beta_2$ in the right-hand side, if K_c is instantaneous. Removing the pole of G_c at $P_0 = E_i$ and taking the $P_0 \rightarrow E_i$ limit, we get

$$\overline{G}_{ci} \Big|_{P_0=E_i} = \left[G_0 + G_0 K_c G_0 + G_0 K_c \overline{G}_{ci} K_c G_0 + \frac{\partial}{\partial P_0} [G_0 K_c \chi_{ci} \overline{K}_{ci} K_c G_0] \right]_{P_0=E_i}. \quad (6.3)$$

The last term will be combined with the third part of (6.1). Using the BS equations (2.12) and writing the 4D amplitude χ_{ci} in terms of the QM wave function φ_i (4.8), we get finally

$$\begin{aligned} P_0 - E_i = & \left[-i(\overline{\varphi}_i K_c G_0 K_R G_0 K_c \varphi_i) \right. \\ & - i(\overline{\varphi}_i K_c G_0 K_R [G_0 + G_0 K_c G_0 + G_0 K_c \overline{G}_{ci} K_c G_0] K_R G_0 K_c \varphi_i) \\ & \left. - (\overline{\varphi}_i K_c G_0 K_R G_0 K_c \varphi_i) \frac{\partial}{\partial P_0} (\overline{\varphi}_i K_c G_0 K_R G_0 K_c \varphi_i) \right]_{P_0=E_i} \end{aligned} \quad (6.4)$$

In the third part, originally, only K_R had to be derived. The derivations of G_0 are provided by the terms transferred from the second part.

In the works of Bodwin and Yennie [12] and Murota [17], the choice for K_c is an instantaneous part of the one-photon exchange graph which leads to a Coulomb potential. In K_R we find the rest of the (instantaneous or not) contributions of the one-photon exchange graph [19] and the contributions of all other irreducible graphs. The unperturbed equation is then Salpeter's equation with a Coulomb potential, which has been used in our 3D perturbation calculation of the preceding section.

We expect that the energy shift (6.4), obtained in the 4D formalism, is equal to the energy shift (5.15), obtained in the 3D formalism. It is however a good precaution to check it. Furthermore, it will be interesting to see how the terms of one expression transform into the terms of the other one.

The expression (6.4) can be transformed into the expression (5.15) by making the replacements:

$$G_0(t' - t) = -i g_0 \exp[-is(t' - t)] \beta_1 \beta_2 + G_R(t' - t) \quad (6.5)$$

$$\overline{G}_{ci}(0, 0) = -i \overline{g}_{ci} \beta_1 \beta_2 \quad (6.6)$$

and eliminating g_0 . Most g_0 can be eliminated immediately by using the wave equations

$$g_0 V_c \varphi_i = \varphi_i, \quad \varphi_i^+ V_c g_0 = \varphi_i^+ \quad (P_0 = E_i), \quad (6.7)$$

but we remain with the derivations of g_0 in the third line of (6.4) and with the central expression between the two K_R in the second line. In this expression, we can use the BS equations and iterated equations (5.18-5.20) for $\overline{g}_{ci}$ to eliminate the terms in $\overline{g}_{ci} V_c g_0$, $g_0 V_c \overline{g}_{ci}$ and $g_0 V_c \overline{g}_{ci} V_c g_0$. We remain then with the contribution in $\overline{g}_{ci}$ to the 3D energy shift (5.15), a $G_R + G_R K_c G_R$ term to be transferred to the first line, and terms in $\varphi_i \varphi_i^+$ which will cancel the derivations of g_0 in the third line.

This completes the chain of transformations (6.1) $\rightarrow$ (6.4) $\rightarrow$ (5.15), which is far from being trivial. Although (6.1) and (5.15) seem to be similar, we can not identify the three parts separately, as several terms have been exchanged between them.

7. Comparison between different reductions and conclusions.

7.1 Equivalence of the 4D and 3D approaches.— We have verified above the equivalence of both approaches, up to the second order in K_R , and up to iterations of $\bar{g}_{ci}$ using eqs. (5.18-5.20). As these equations are satisfied only by the exact $\bar{g}_{ci}$, any approximation or truncation of $\bar{g}_{ci}$ could break the equivalence. This is not a fundamental difference between both methods, however, as we could also use such iterations in the 3D approach as well, for pure computational reasons.

The comparison has been made by keeping the two first orders in K_R and all orders in K_c in the 3D expansion. For a given order in K_R , the 3D expansion contains more terms, but they are simpler to compute. The double expansion (3D reduction, followed by the perturbation expansion) of the 3D way is balanced by the complication of the computation of $\bar{G}_{ci}$ in the 4D way. In the 3D way, however, we are tempted to make an expansion of K_{RT} into powers of α instead of K_R (for an expansion up to α^2 , we would keep only the first line of (5.6)). The modified propagator G_R , which has no physical cut, is less liable to modify the relative importance of the various contributions than G_0 . One must of course be very careful when estimating the order of the different contributions in a bound state problem, as the coupling constant is already present in the unperturbed wave functions.

An important feature of the 3D or relativistic quantum theory framework is its key position between quantum field theory, ordinary quantum mechanics and relativistic constraint classical mechanics.

7.2. Salpeter's propagator with a scalar constraint.— There exists of course an infinity of possible decompositions of G_0 into a G_δ containing the most important singularities of G_0 and a rest. In ref.[20], we examine several possibilities. We can first replace the constraint $\delta(p_0-s)$ by the constraint $\delta(p_0-\mu)$, with

$$\mu = \frac{h_1^2 - h_2^2}{2P_0} = \frac{m_1^2 + \vec{p}_1^2 - m_2^2 - \vec{p}_2^2}{2P_0} = \frac{Ss}{P_0} \quad (= \frac{m_1^2 - m_2^2}{2P_0} \text{ in the CM frame}), \quad (7.1)$$

in the definition of G_δ (the operator $(p_0-\mu)$ is given by the half-difference of the iterated (second-order) free equations). The matricial operator s is now replaced by the scalar μ . All calculations above can easily be adapted. The second-order energy shift is again (5.15). The 3D potential is now defined with exponentials $\exp[i\mu\tau]$, $\exp[-i\mu\tau]$ in (3.11), and this change is balanced by a different definition of G_R , which appears in the expansion (3.6) of K_T .

7.2. Other propagators.— A lot of possible G_δ , combining the scalar constraint $\delta(p_0-\mu)$ and a propagator with no τ operator, were examined in ref.[20]. Among these propagators were Breit's propagator (just Salpeter's propagator without the operator τ), Sazdjian's propagator (based on the second-order equations), and also effective Schrödinger and Dirac propagators. With these effective propagators, the solutions of the unperturbed problem (in the case of a Coulomb potential) are analytically known. In contrast, with Salpeter's, Breit's or Sazdjian's propagator, the solutions of the unperturbed problem itself must be computed by perturbations, so that we must finally perform a two-step perturbation calculation. The advantage of these propagators lies in the fact that they preserve more of the fundamental symmetries (like the particle-antiparticle symmetry) than do the effective propagators.

The calculations above can again be easily adapted to all the non-Salpeter propagators. We find however that $G_\delta(0,0)$ is not equal to $G_0(0,0)$ anymore, so that $K_c G_R K_c$ is no more automatically zero, and that the expansion (5.6) should now contain an infinity of terms. Salpeter's propagator (with the scalar of matricial constraint), is thus the most suitable one for a perturbation calculation around an instantaneous approximation. Another advantage of the τ operator in Salpeter's propagator lies in the fact that it allows for the existence of solutions entirely in the $\tau^2 = 1$ subspace. Without this operator, the wave function would have $\tau^2 = 0$ components, which could become a source of problems, like the "continuum dissolution" problem when an external potential is added [19-20,30-32]. This operator τ has however the drawback of complicating the one-body limit of the 3D equation: at the $m_2 \rightarrow \infty$ limit, we must get fermion 1 in the Coulomb potential of the heavy particle 2. This physical fact is translated more or less directly into the equations obtained with the various 3D reductions. While the matricial constraint leads to complicated one-body limits [20], the unperturbed equation obtained with a scalar constraint and Breit's propagator leads already to a Dirac

equation with a Coulomb potential, and all higher-order contributions cancel mutually [6,37-39,20]. With Salpeter's propagator, the 3D wave function can be obtained by applying τ on the Breit wave function, and its one-body limit by applying the projector Λ_1^+ on the Dirac-Coulomb wave function. This limit is thus easy to compute, but it obeys a non-compact equation obtained by solving the Dirac-Coulomb equation with respect to the Λ_1^+ part of the wave function [20]. Salpeter's propagator seems thus to be not so well adapted to the computation of the proton recoil effects in an hydrogen atom, for example.

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