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FOR COMPOSITE FERMIONS
AT $1/3$ FILLING FACTOR

Aurora Pérez Martínez
Alejandro Cabo
and
Valia Guerra

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Aurora Pérez Martínez
Departamento de Física del CINVESTAV, Apartado Postal 14-740, Mexico 07000, DF Mexico.

Alejandro Cabo
International Centre for Theoretical Physics, Trieste, Italy

and

Valia Guerra
Departamento de Matemática, Instituto de Cibernética, Matemática y Física, Calle E No. 309 Esq. a 15, Vedado, La Habana 4, Cuba.

ABSTRACT

The collective mode spectrum of the composite fermion state at 1/3 filling factor is evaluated. At zero momentum, the result coincides with the cyclotron energy at the external magnetic field value, and not at the effective field, in spite of the fact that only the former enter in the equations, thus, the Kohn theorem is satisfied. Unexpectedly, in place of a magneto-roton minimum, the collective mode gets a threshold indicating the instability of the mean field composite fermion state under the formation of crystalline structures. However, the question about if this outcome only appears within the mean field approximation should be further considered.

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1Permanent address: Grupo de Física Teórica, Instituto de Cibernética, Matemática y Física, Calle E No. 309 Esq. a 15, Vedado, La Habana, Cuba.

1 Introduction

In previous papers [1]-[2], we have introduced a generating functional approach to the composite fermion model of FQHE of Jain ([3],[4]) through considering the statistical interaction with a slowly varying parameter taken in the fermion limit [4]. In [1] the Dyson equation were solved in the Hartree-Fock approximation. The paper [2] was devoted to construct a perturbative approach based on the generating functionals and mean statistical field values.

In this work we continue the analysis of the implications of the general approach in [2]. Concretely the Bethe-Salpeter equation and its contribution to the linear response of the FQHE model is considered in order to discuss the elementary excitation spectrum predicted by the model. Technically, the discussion in this paper is very close the one in the work [5] in which the fractional statistics gas was investigated. Here, the FQHE problem is examined the discussion includes the Coulomb interaction and the presence of the external magnetic field. The intra Landau level collective mode spectrum was already investigated in [6] following an approach proposed by Feynman. A nonvanishing energy gap associated to magneto-roton excitations were obtained.

Nowadays, the general point of view is that the gap for the $\nu = 1/3$ state should be near $0.1 e^2/r_o$, a value which is compatible with recent experimental results [7]. Therefore, taking into account the also widespread interest created by the composite fermion description of the FQHE, in this work it is intended to determine the properties for the collective mode spectrum in this approach. The method employed was the numerical solution of the Bethe-Salpeter equation for the four-points Green's function taken in the first approximation for the interaction kernel. Then, as the electromagnetic response kernel is linearly given in terms of the four-point function at coinciding points, it follows [5] that the collective mode spectrum can also be obtained as a byproduct.

Section 2 is devoted to review the previous introduced general background. The scheme of calculation is presented in Section 3. In Section 4 the results for the collective mode spectrum are given.

2 Review

In [2] a functional way of description for the composite fermion model was proposed. The treatment was based in the Green's function generating functional

$$ Z[J, J_0, \eta^{\ast}, \eta, j_0] = \int D\phi D\phi^{\ast} D\psi D\psi^{\ast} \exp \left\{ -\int \psi^{\ast} \eta + \eta^{\ast} \psi + j_0 \phi + j_0 \phi^{\ast} + \phi^2 dz_1 + \phi^{\ast 2} dz_1 \right\} \exp \left\{ \int \left( J_0 A_i + j_0 A_0 \right) dz_4 \right\}, \tag{1} $$

where the arguments of $Z$ are sources for all the fields to be specified below and

$$ \hat{Z}[\psi, \psi^{\ast}, A_0, a_0, \phi] = \exp(S), \tag{2} $$

in which the action in terms of the composite fermion, statistical and electromagnetic fields has the form
\[ S = \frac{1}{\hbar c} \int d^2 x d^2 x' \psi^*(x) \left( -ieA_k \frac{\partial}{\partial x^k} - \frac{1}{2m}(p + e/cA^2) \right)^2 - ieA_k + p \psi(x) \]

\[ - \int \frac{e^2}{2m^2c^2} q_0(x) \left( \frac{e^2}{2m} \right)^2 \varepsilon \psi^*(x) \psi(x) \right) d^2 x d^2 x' \]

\[ + \frac{1}{\lambda^2} \int \left( \frac{e}{\hbar} \frac{\partial}{\partial x^k} \right) \left( \frac{e}{\hbar} \frac{\partial}{\partial x'^k} \right) V_1(\psi', \psi) \]

\[ - \frac{1}{16\pi} \iiint \left( \partial_{x^k} A_k - \partial_{x'^k} A_k \right) d^2 x d^2 x' \]

the interaction vertex \( V_1 \) is defined by

\[ V_1 = -\frac{1}{\lambda^2} \iiint \left[ \psi^*(x) \psi^*(x') U(x - x') \psi(x') \psi(x) \right]_{x=x'} d^2 x d^2 x' \]

\[ + \frac{1}{\lambda^2} \iiint \left[ \psi^*(x) \psi(x') U(x - x') \psi(x') \psi(x) \right]_{x=x'} d^2 x d^2 x' \]

\[ - \frac{e^2}{2m^2c^2} \iiint d^2 x d^2 x' \left[ \psi^*(x) \psi^*(x') A^4(x - x') \psi(x') \psi(x) \right]_{x=x'} \]

Above, \( \kappa_0 \) represents the compensating charge density of the jellium. \( \psi^* \) and \( \psi \) are the composite fermion fields. \( a_\nu \) is the statistical field, \( A_\nu^\prime = \phi^\prime_\nu + A_\nu \) is an auxiliary variable and \( A_\nu \) is the vector potential of the electromagnetic field in the gauge \( A^\prime = 1/2 \theta \times \vec{x} \). \( A_\nu \) is an external classical field. The statistical field is related to the fermion density through the usual formula

\[ a_\nu(x) = \int A_\nu(x - x') \psi^*(x') \psi(x') d^2 x', \]

\[ a_\nu(x) = 0, \]

where, the centered at \( x' \) solenoidal vector potential \( A_\nu \) is given by

\[ A_\nu(x - x') = \frac{\mu_0 c}{4\pi} \frac{e}{c} \frac{(x - x')}{||(x - x')||^2}, \]

\[ \epsilon^{21} = -\epsilon^{12} = 1, \epsilon^{22} = 0, \]

the fractional statistical parameter \( \theta \) defines the solenoid flux in [6], which will selected here as equal to two flux quanta and flowing in the direction opposed to the sense in which the magnetic field points. This selection defines the 1/3 filling factor state \([1].\]

In references [1]-[2] the Dyson equation in the Hartree-Fock approximation was solved and various general properties of the exact propagator and dielectric tensor obtained. One interesting fact following was that the mean field solutions are exactly characterized by a constant effective field which correspond to the external magnetic field plus an addition created by the statistical field. The selfenergy spectrum was evaluated and will be used here as a necessary piece in discussing the BS equation.

3 Scheme of Calculation

In this section the technique for the evaluation of the collective mode spectrum is considered. The general formulations developed in [8] were closely followed. From them, it becomes clear that the collective mode spectrum should appear as included in the spectrum of the four-point Green function. This occurs because, at least in the approximation to be considered, the electromagnetic linear response kernel is also linearly related with the mentioned Green function through magnitudes which are not singular in the frequency. The approach followed consisted in reducing the Bethe-Salpeter equation to a matrix equation by using the magneto-exciton two-particle wavefunctions [8] and then solve for the frequency numerically by selecting a finite number of functions in the basis.

The integral Bethe-Salpeter equation for the four-point Green function, after introducing the compact notation of representing spacetime arguments by integer numbers and space vectors by the same numbers with arrows, can be written as

\[ F(1, 1', 2, 2') = F(1, 1', 2, 2') \]

\[ + \int \int d^4 s d^4 t F(1, 1', 2, 2') W(3, 4, 5, 6, 7, 8, 9) F(5, 6, 2', 2), \]

where \( F \) is the interaction free four-point function and the interaction kernel in the first approximation is taken as the following functional derivative

\[ W = \frac{\delta W_{HF}(3, 4)}{\delta G(5, 6)} \]

in which \( W_{HF}(1, 2) \) is the Hartree-Fock inverse propagator [1], [9].

As mentioned above, it is possible to simplify the integral equation (7) introducing magneto-exciton wave functions [5], [6] which have the form

\[ \psi_{mn} = \frac{(-1)^m}{(2\pi)^{1/2}} \frac{1}{2 \pi^{1/2}} \left[ \begin{array}{c} \psi_{m} \\ \psi_{n} \end{array} \right], \]

\[ exp(-((z_1^2 + z_2^2 + z_3^2)/4)). \]

\[ z = x + y. \]

These two-particle states are constructed from the one-body wave functions for electrons and holes in the mth and nth Landau levels. They are characterized by a center of mass momentum which can be written in complex form as \( z_0 = \bar{q} + \bar{q}_0 \). The main simplification introduced by these states is that the \( z_0 \) is conserved and all the matrix elements are diagonal in this quantum number.

Then, the integral equation (7) can be written as a linear matrix equation in the form

\[ F_{mn, m', n'}(\alpha, \omega) = \sum_{m', n'} F_{mn}(\alpha, \omega) W_{mn} \]

\[ + \sum_{m', n'} F_{mn}(\alpha, \omega) \sum_{m', n'} F_{mn}(\alpha, \omega). \]

The matrix elements of the free four point function \( F_4 \) and the interaction kernel \( W \) are given by the expressions
\[ \left\langle \frac{m'}{m} \left| \mathcal{F} \right| \frac{n'}{n} \right\rangle = \iint \iint \Psi_{\alpha_\nu}^* (1, \beta, \beta) \mathcal{F}_\alpha (1, \beta, \beta) \Psi_{\alpha_\nu} (1, \beta, \beta) d\beta d\beta d\beta, \] (9)

\[ \left\langle \frac{m'}{m} \left| \mathcal{W} \right| \frac{n'}{n} \right\rangle = \iint \iint \Psi_{\alpha_\nu}^* (1, \beta, \beta) \mathcal{W} (1, \beta, \beta) \Psi_{\alpha_\nu} (1, \beta, \beta) d\beta d\beta d\beta, \] (10)
in which the dependence on the \( \alpha \) quantum number has been omitted and a vertical representation of the bracket indices is also used.

The interaction kernel becomes the sum of twenty contributions arising from the anyonic like interactions and two from the Coulomb one in the form

\[ \mathcal{W} = \sum_{i=1}^{20} \mathcal{W}_{\nu} + \sum_{i=1}^{2} \mathcal{W}_{\nu}. \]

The explicit formulae for these terms and the one for \( \mathcal{F} \) are given in Appendix A. Obtained The Feynman diagram representation of these contributions can be found in [5].

Let us discuss now the parameters characterizing our specific problem. The statistical parameter will be selected as satisfying \( \theta = -2\pi \). It corresponds with the composite fermions description in which an even number of flux quanta are attached to each electron [1]-[2]. In addition, due to the fact that we are interested in analyzing the 1/3 filling factor case, only one Landau will be filled, then \( k = 1 \) will be fixed [1]. This selection is related with a negative mean statistical field which cancels 2/3 of the external magnetic field [1]. All the lengths and energies here can be expressed in units of the magnetic length in the effective magnetic field remaining after the external field is partially compensated \( L \) and the cyclotron energy \( \hbar \omega_c \) in the same field, respectively.

The collective modes of the system, correspond with the lowest frequencies leading to zero eigenvalues of the inverse kernel

\[ \mathcal{F}^{-1} = \left( \mathcal{F}_\nu \right)^{-1} - \mathcal{W}, \] (11)
in which addition are also singularities of the dielectric response tensor component with both indices being temporal [5].

Due to the fact that \( \mathcal{F}_\nu \) is qualitatively different for particle-hole and hole-particle channels it becomes useful to divide the matrix equation (10) in four sub-blocks. Concretely, the block representation is obtained by restricting, as conceived in [5], the indices \( m \) and \( n \) to empty Landau levels and \( m' \) and \( n' \) to filled ones and define the block matrices \( \mathcal{E} \) and \( d \) through

\[ \left\langle \frac{m'}{m} \left| \mathcal{W} \right| \frac{n'}{n} \right\rangle \equiv \left\langle \frac{m'}{m} \left| \mathcal{E} \right| \frac{n'}{n} \right\rangle, \]

and

\[ \left\langle \frac{m'}{m} \left| \mathcal{W} \right| \frac{n}{n'} \right\rangle \equiv \left\langle \frac{m'}{m} \left| d \right| \frac{n}{n'} \right\rangle, \]
in which the properties of the magneto-exciton wave function also implies

\[ \left\langle \frac{m'}{m} \left| \mathcal{W} \right| \frac{n}{n'} \right\rangle \equiv \left\langle \frac{m'}{m} \left| \mathcal{E} \right| \frac{n}{n'} \right\rangle^*, \]

The explicit expressions for the matrix elements of the matrices \( \mathcal{E} \) and \( d \) are given in Appendix B. The twenty anyonic terms associated to each of these blocks \( \mathcal{E} \) and \( d \) were received just by using the almost identical expressions considered in [5].

Now, introducing the matrix \( \Delta \) as

\[ \left\langle \frac{m'}{m} \left| \Delta \right| \frac{n}{n'} \right\rangle \equiv \delta_{mn} \delta_{n'n'} (\epsilon_m - \epsilon_n) \] (12)
in order to compact the blocks coming from the matrix representation of \( \mathcal{F}_\nu \), the homogeneous Bethe-Salpeter equation (10) can be reduced to the matrix form [5].

\[ \left[ \begin{array}{cc} \hbar \omega - \Delta \epsilon - \mathcal{E} + i\eta & -d \\ -d^* & -\hbar \omega - \Delta \epsilon - \mathcal{E} + i\eta \end{array} \right]^{-1} \] (13)

This representation was then used to determine numerically the collective mode dispersion relation. For this purpose a finite number of basis functions corresponding to the Landau levels index up to a maximum value were retained in constructing the BS matrix (15). The results are described in next section.

4 Results and discussion

The roots of the determinants of the matrix (15) were found numerically after considering only the lowest Landau level indices for the magneto-exciton basis function up to a maximum value \( n_c \). This number was also varied in order to check the convergence of the results. The lowest energy collective mode dispersion relation is shown in Fig 1 as a function of center of mass momentum taken along the \( z \) axis. As noticed before, the energies are given in units of the cyclotron frequency of the effective magnetic field \( \omega_c \) and the momentum in units of \( q_0 \). The curve corresponds to a number \( n_c = 15 \) of Landau levels.

It can be observed that the collective mode energy decreases as the center of mass momenta grows up to a threshold value near \( q_0 = 2.165 \). At this point unstable modes having imaginary part of the frequency appear. These kind of modes remain up to near the \( q_0 = 3.3 \) momentum value. Then, the unstability disappear and the real energy begin to grow with the momentum of center of mass. It can be noticed that unstability region begins for wavevector values greater than \( q_0 = 1 \).

The above described behaviour is not significantly affected by the presence of the Coulomb interaction. The growing and spatially periodic oscillations have a wavelength of the order of the magnetic length in the effective magnetic field.

An important property which follows from Fig 1 is that the Kohn theorem is exactly satisfied at zero momentum. That is, the collective mode energy became equal to three times \( \hbar \omega_c \), or what is the same, equal to the cyclotron energy in the external magnetic field \( \omega_c \). This outcome is an independing checking of the calculation in the low momentum limit. Another verification performed consisted in taking the parameter \( \theta = 1 \) by
also excluding the Coulomb interaction. After this, it was possible to reproduce the collective mode dispersion for a bose fluid obtained in Ref. [5]. Finally, previous results for the inter Landau level collective excitation of Kallin and Halperin were also repeated by considering the coulomb interaction and assuming zero flux for the statistical solenoidal interaction, that is taking $\text{E}_c = 0$.

It is known that the perturbative calculation of the energy within the composite fermion approach has difficulties. One of them, for example, is that after disconnecting the coulomb interaction the calculated values can hardly interpreted as reproducing the properties of the noninteracting electron system. Therefore, it cannot be discarded that the detected instability could disappear after including correlations in the same way that those correlations are expected to repair the above mentioned trouble with energy. However, under the acceptance that the results remain valid in the exact theory, the instability could be imagined to be related with the largely discussed question about the role of crystal fluctuations in the FQHE ground state. Further analysis related with these possibilities will be considered elsewhere.

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Appendix A

a. The selfenergies and their statistical and coulomb contributions for each Landau level of index $n$

$$\epsilon_n = \epsilon_R(n) + \epsilon_c(n)$$

$$\epsilon_R(n) = -\frac{\epsilon^2}{4} + \frac{\epsilon^2}{2} \left( \frac{1}{2} + \frac{1}{2n(1+n)} - \sum_{k=1}^n \frac{1}{k} \right)$$

$$\epsilon_c(n) = \frac{\epsilon^2}{4} \left( -\frac{1}{2} + \frac{1}{2n} \right) - \frac{\epsilon^2}{2} \left( \frac{1}{2} + \frac{1}{2n(1+n)} - \sum_{k=1}^n \frac{1}{k} \right),$$

where $n$ is the index of the Landau level in the effective field and the constant $\epsilon_c$ is given by

$$\epsilon_c = \frac{\epsilon^2}{2}\left( \frac{\hbar e B}{mc} \right)$$

b. The matrix elements of $F_c$

$$\langle m' | F_c | n' \rangle = \delta_{m,n} \delta_{m',n'} \left[ \begin{array}{cl} (\hbar \omega - (\epsilon_n - \epsilon_c) + i\eta)^{-1} & n \text{ empty}, n' \text{ occupied} \\ (-\hbar \omega - (\epsilon_n - \epsilon_c) + i\eta)^{-1} & n' \text{ empty}, n \text{ occupied} \\ 0 & \text{otherwise} \end{array} \right]$$

c. The twenty two contributions to the matrix elements of interaction kernel $W$

In the formulae below, the $\Psi_A$ and $\Psi_B$ represents any two states of the magnetoexciton basis, $\Phi$ is the vector potential for the effective magnetic field and $A_{12}$ means the solenoid vector (6) with spatial arguments 1 and 2. As in [3] dimensionless quantities $c, m, c$ and $\hbar$ are set equal to 1, which produces energies and lengths in units of $\hbar/e$ and $\hbar e B/c$.

$$W_{14}^A = \theta \lim_{\lambda \to 1} \int d1 \int d2 \Psi_B(1,2)A_{12}(P_1 + \Phi)\Psi_A(3,1),$$

$$W_{24}^A = \theta \lim_{\lambda \to 1} \int d1 \int d2 \Psi_B(2,2)A_{12}(P_1 + \Phi)\Psi_A(1,3),$$

$$W_{34}^A = -\theta \int d1 \int d2 \Psi_B(2,1)|A_{12}(P_1 + \Phi)\Psi_A(2,1),$$

$$W_{44}^A = -\theta \int d1 \int d2 \Psi_A(1,2)|A_{12}(P_1 + \Phi)\Psi_B(1,2),$$
\[
W_0^4 = \theta^2 \int d1 \int d2 |A_{12}|^2 \Psi_A^*(2,2) \Psi_B(1,1),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 |A_{12}|^2 \Psi^*(1,2) \Psi_B(1,2),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,1) \Psi^*(2,3) \Psi_B(2,3),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,1) \Psi^*(2,3) \Psi_B(2,2),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,1) \Psi^*(2,2) \Psi_B(1,3),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,2) \Psi^*(3,3) \Psi_B(2,1),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(2,1) \Psi^*(2,1) \Psi_B(3,3),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(3,2) \Psi^*(1,1) \Psi_B(2,3),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(2,3) \Psi^*(2,3) \Psi_B(1,1),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,1) \Psi^*(3,3) \Psi_B(2,2),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,3) \Psi^*(2,3) \Psi_B(2,1),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(3,1) \Psi^*(2,1) \Psi_B(2,3),
\]
\[
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\]
\[
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\[
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\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(2,1) \Psi^*(3,1) \Psi_B(2,1),
\]
\[
W_0^4 = -\theta^2 \int d1 \int d2 \int dA_{12} A_{12} \Pi(1,1) \Psi^*(2,2) \Psi_B(1,1).
\]

where the projection operator in the first Lanczos level \( \Pi \) is defined as
\[
\Pi(1,2) = \frac{1}{2\pi i \hbar} \exp[-(|z_1|^2 + |z_2|^2)/4 + z_1^* z_2/2].
\]

As all the vectors appearing are spatial ones the arrow over them has been omitted for avoid more cumbersome writing.

**Appendix B**

a. Definitions of various auxiliary quantities

\[
c(m,n) = (-1)^{m+n} \frac{s_m s_n}{(2m+n+1)!},
\]

\[
d(m,n) = (-1)^{m+n} \frac{s_m s_n}{(2m+n+1)!},
\]

\[
co(m,n) = \frac{m!}{(n-m)!n!},
\]

\[
b = \frac{1}{2} |z_n|^2.
\]

\[
f_n = e^{-1} \sum_{k=0}^{n} \frac{k^p}{k!}.
\]

\[
g_n = \sum_{k=0}^{n} \frac{k!}{k+1} f_k.
\]

\[
h_n = \sum_{k=0}^{n} \frac{k!}{k+1} [1 - f_k].
\]

\[
E_b = \frac{1}{2} \sum_{p=1}^{\infty} \frac{(-b)^p}{p!}.
\]

\[
X = e^{-b} \sum_{p=1}^{\infty} \frac{(b)^p}{p!}.
\]

\[
Y_{m,n} = m! n! e^{-b} \sum_{p=1}^{\infty} \frac{(p-1)!}{(p+m)!(p+n)!} b^p.
\]

b. Divergent integrals

\[
l_1 = e^{-b} \lim_{q \to 0} \int_0^\infty dr \frac{r}{r^2 + \eta^2} j_0(q r),
\]

\[
l_2 = e^{-b} \lim_{q \to 0} \int_0^\infty dr \frac{r}{r^2 + \eta^2} e^{-r^2/2},
\]
c. The $E$ block matrix components for $m, n > 0$

\[
E(m, n) = \sum_{r=1}^{22} E_r(m, n),
\]

\[
E_1(m, n) = -\frac{\theta^c(m, n)}{2} e^{-b} \left( \frac{m}{b} - 1 \right),
\]

\[
E_2(m, n) = -\frac{\theta^c(m, n)}{2} e^{-b} \left( \frac{n}{b} - 1 \right),
\]

\[
E_3(m, n) = -\frac{c(m, n)}{2} e^{-b},
\]

\[
E_4(m, n)|_{n=2m} = -\frac{\theta}{2} \left[ \delta_{mn} + c(m, n) \left( \frac{m!}{b^m} f_m e^{-b} + \frac{n!}{b^n} (1 - f_{n-1}) \right) \right],
\]

\[
E_5(m, n) = \theta^c c(m, n) I_1,
\]

\[
E_6(m, n)|_{n=2m} = -\frac{\theta}{2} \left( 2 I_2 + X + g_{m-1} - h_{n-1} \right),
\]

\[
E_7(m, n)|_{n=2m} = \theta^2 \left[ E_8 - E_9 + \frac{e^{-b} - b}{2} \frac{1}{2} \sum_{p=0}^m \frac{1}{k-p+1} \left( \frac{c(m, n)}{2} (1 - \delta_{mn}) \right) \right] \left( \frac{m! f_{m-1}}{b^m} + \frac{n!}{b^n} (1 - f_{n-1}) \right),
\]

\[
E_8(m, n) = \frac{\theta^c c(m, n)}{4} e^{-b} \left( \frac{m}{b} - 1 \right),
\]

\[
E_9(m, n) = \frac{\theta^c c(m, n)}{4} e^{-b} \left( \frac{n}{b} - 1 \right),
\]

\[
E_{10}(m, n) = -\frac{\theta^c c(m, n)}{4} e^{-b} \left( \frac{1}{b} + \frac{1}{n+1} \right),
\]

\[
E_{11}(m, n) = -\frac{\theta^c c(m, n)}{4} e^{-b} \left( \frac{1}{b} + \frac{1}{m+1} \right),
\]

\[
E_{12}(m, n) = -\frac{\theta^2}{2} c(m, n) \left( I + I_2 \right) + \frac{e^{-b} - b}{2} \sum_{k=1}^{m} \frac{1}{k},
\]

\[
E_{13}(m, n) = -\frac{\theta^2}{2} c(m, n) \left[ (I + I_2) + \frac{e^{-b} - b}{2} \sum_{k=1}^{m} \frac{1}{k} \right],
\]

\[
E_{14}(m, n) = \frac{\theta^c c(m, n)}{2} e^{-b} \left( \frac{1}{b} \right),
\]

\[
E_{15}(m, n) = \frac{\theta^c c(m, n)}{4} e^{-b},
\]

\[
E_{16}(m, n) = \frac{\theta^c c(m, n)}{4} e^{-b},
\]

\[
E_{17}(m, n)|_{n=2m} = \frac{\theta^2 c(m, n)}{4} \left( \frac{1}{b} \right),
\]

\[
E_{18}(m, n)|_{n=2m} = \frac{\theta^2 c(m, n)}{4} \left( \frac{1}{b} \right),
\]

\[
E_{19}(m, n)|_{n=2m} = \frac{\theta^2 c(m, n)}{4} \left( \frac{1}{b} \right),
\]

\[
E_{20}(m, n)|_{n=2m} = \frac{\theta^2 c(m, n)}{4} \left( \frac{1}{b} \right),
\]

\[
E_{21}(m, n) = \frac{\theta^c c(m, n)}{4} e^{-b} \left( \frac{1}{b} \right),
\]

\[
E_{22}(m, n) = \frac{\theta^c c(m, n)}{4} e^{-b} \left( \frac{1}{b} \right),
\]

\[
E_{23}(m, n) = \frac{\theta^2}{2} c(m, n) \left( I + I_2 \right) + \frac{e^{-b} - b}{2} \sum_{k=1}^{m} \frac{1}{k},
\]

\[
d(m, n) = \sum_{r=1}^{22} d_r(m, n),
\]
\begin{align*}
d_1(m, n) &= -\theta^2 \frac{d(m, n)}{e^{t}\left(\frac{m}{b} - 1\right)}, \\
d_2(m, n) &= -\theta^2 \frac{d(m, n)}{e^{t}\left(\frac{n}{b} - 1\right)}, \\
d_3(m, n) &= -\frac{d(m, n)}{2} \left( e^{-t} - \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_4(m, n) &= -\frac{d(m, n)}{2} \left( e^{-t} - \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_5(m, n) &= \theta^2 d(m, n) i, \\
d_6(m, n) &= \frac{d(m, n)}{2} (2l_0 + X - h_{m+n-1}), \\
d_7(m, n) &= -\frac{d(m, n)}{2} \left( \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_8(m, n) &= -\frac{d(m, n)}{4} e^{-t} \left( \frac{1}{m+1} + \frac{1}{b} \right), \\
d_9(m, n) &= \frac{d(m, n)}{4} e^{-t} \left( \frac{n}{b} - 1 \right), \\
d_{10}(m, n) &= -\frac{d(m, n)}{4} e^{-t} \left( \frac{1}{b} + \frac{1}{n+1} \right), \\
d_{11}(m, n) &= \frac{d(m, n)}{4} e^{-t} \left( \frac{m}{b} - 1 \right), \\
d_{12}(m, n) &= -\frac{d(m, n)}{2} \left( (l_1 - l_2) + \frac{1}{2} e^{-t} \sum_{k=1}^{n} \frac{1}{k} \right), \\
d_{13}(m, n) &= \frac{d(m, n)}{2} \left( (l_1 - l_2) + \frac{1}{2} e^{-t} \sum_{k=1}^{m} \frac{1}{k} \right), \\
d_{14}(m, n) &= \frac{d(m, n)}{2} e^{t} \left( \frac{1}{m+1} e^{-t} + \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{15}(m, n) &= \frac{d(m, n)}{4} \left( e^{-t} - \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{16}(m, n) &= \frac{d(m, n)}{4} \left( e^{-t} + \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{17}(m, n) &= \frac{d(m, n)}{4} \left( e^{-t} - \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{18}(m, n) &= \frac{d(m, n)}{4} \left( e^{-t} - \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{19}(m, n) &= \frac{d(m, n)}{4} \left( e^{-t} + \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{20}(m, n) &= \frac{d(m, n)}{4} \left( e^{-t} + \frac{(m + n - 1)!}{b^{m+n}} [1 - f_{m+n-1}] \right), \\
d_{21}(m, n) &= -u_c d(m, n) e^{-t} \sum_{i=0}^{m+n} co(m, i)(1/4i!2iz_0) \Gamma(\frac{i+1}{2}) \\
&\quad \cdot \frac{\frac{1}{F_1(i+1/2, 1, 1, 1, 1; x^2/2)}{\sqrt{2(1/2)\Gamma(\frac{i+1}{2})}}}{}, \\
d_{22}(m, n) &= u_c d(m, n) e^{-t} |iz_0| \\
\end{align*}
References


Figure Caption

Fig. 1. The collective mode energy in units of $\hbar \omega /1$ as a function of the center of mass momentum. Note that the Kohn theorem is satisfied at zero momentum and that the system develops a threshold after which unstable modes appear in a band of momentum values.