THIRD-ORDER RESONANCE SLOW EXTRACTION FROM
ALTERNATING GRADIENT SYNCHROTRONS

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1. INTRODUCTION

The equation for the radial betatron oscillations around the equilibrium orbit in the presence of a sextupole perturbation, in a normalized system of units (Appendix I) is:

\[ \frac{d^2x}{dt^2} + Q^2x = kx^3 \cos n\phi, \]  \hspace{1cm} (1.1)

where \( x \) is the radial displacement from the equilibrium orbit, \( Q \) is the frequency of the free betatron oscillations, and \( \phi = \int (ds/Q\beta) \). The right-hand side represents a continuously distributed sextupole perturbation of strength \( k \) and frequency \( n \). Substituting for \( x \) the general expression of the free betatron oscillations, on the right-hand side of Eq. (1.1) the frequencies \( 2Q + n, 2Q - n \), and \( n \) are found \(^1\). A resonance occurs if one of these frequencies is equal to the frequency \( Q \) of the free betatron oscillations. The sextupole perturbation excites the third-order resonance if the \( Q \)-value is:

\[ \bar{Q} = \frac{n}{3}, \]

i.e. if the wavelength of the sextupole perturbation is \( 1/3 \) of the betatron wavelength. In this case the motion of the particles is unstable everywhere in the phase plane.

If the \( Q \)-value differs from the resonant value \( \bar{Q} \), namely

\[ \Delta Q = Q - \bar{Q} < \Delta x, \]  \hspace{1cm} (1.2)

small amplitude oscillations are stable; in the phase plane a stable region surrounds the equilibrium orbit. The frequency of the stable non-linear oscillations of very small amplitude is \( Q \). The limit amplitude for stable oscillations is the amplitude of the non-linear oscillation having frequency \( \bar{Q} \) \(^1\). Introducing the new variable:
\[ u = \frac{k}{Q^2 - \bar{Q}^2} x \quad (1.3) \]

Eq. (1.1) becomes:

\[ \frac{d^2u}{ds^2} + \bar{Q}^2 u = - (Q^2 - \bar{Q}^2) u + (Q^2 - \bar{Q}^2) u^2 \cos n\phi. \quad (1.4) \]

The non-linear oscillation having frequency \( \bar{Q} \) can be expressed as a Fourier series, where because of Eq. (1.2) the first harmonic is predominant. If the expression of the first harmonic is substituted for \( u \) in Eq. (1.4) and the terms having frequency \( \bar{Q} \) are retained, the equation which is obtained gives the phase and the amplitude of the first harmonic\(^1\). Since oscillations having frequency \( \bar{Q} \) make the left-hand side of Eq. (1.4) zero, all these terms have the common factor \( Q^2 - \bar{Q}^2 \), which then cancels out. The amplitude of the first harmonic is therefore independent from \( k \) and \( Q^2 - \bar{Q}^2 \). This, combined with Eq. (1.3), shows that in the phase plane \( (x, x') \) the limit amplitude for stable oscillations is proportional to \( (Q^2 - \bar{Q}^2)/k \), i.e. to \( \Delta Q/k \), in the range defined by Eq. (1.2). This result remains valid if the sextupole perturbation is given by sextupole lenses, excited in such a way that the periodicity of the distribution of the sextupole lenses around the ring contains the \( n^{th} \) harmonic.

The third-order resonance can be used as follows for the extraction of particles\(^2\). By means of quadrupole fields the \( Q \)-value of the machine is tuned near to the resonant value \( \bar{Q} \); \( \Delta Q \) must be such that the stable region is large enough to hold the initial beam. Moving the \( Q \)-value to the resonant value, the stable area is progressively reduced and the particles are spilled out from the stable region. Spilt particles perform resonant oscillations, the amplitude of which rapidly increases and the phase of which is approximately independent from the amplitude, until they enter an extraction septum magnet. A fraction of the particles is lost on the septum.

This report gives a theory of third-order resonance extraction. The perturbation is assumed to be given by sextupole lenses; the thin lens approximation is made. The machine is assumed to have a linear behaviour. The theory is valid in the range defined by Eq. (1.2).

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Appendix II describes a computer program which is used for computations on resonant extraction from alternating gradient synchrotrons.

2. TUNED MACHINE

The tuning quadrupole fields and the perturbing sextupole fields, when they have a dipole component on the equilibrium orbit different from zero, cause a distortion of the equilibrium orbit. The tuning quadrupole fields and the quadrupole components of the perturbing sextupole fields on the distorted equilibrium orbit, cause a perturbation of the linear properties of the betatron motion. If the dipole and the quadrupole components of the tuning quadrupole fields and of the perturbing sextupole fields on the distorted equilibrium orbit are considered as belonging to the machine, the machine which is obtained is called "tuned". The equilibrium orbit of the tuned machine is the distorted equilibrium orbit, and the betatron motion of particles is perturbed only by sextupole fields. The machine in itself is called "non-tuned".

In this chapter the third-order resonance extraction is studied in a system of units "N" normalized with respect to the tuned machine and having the distorted equilibrium orbit as reference orbit for the coordinate system. This system of units has been used in writing Eq. (1.4).

The properties of the radial betatron phase plane are studied using the following assumptions:

i) The sextupole perturbation is given by sextupole lenses placed at azimuthal positions corresponding to the extreme of the cosine wave on the right-hand side of Eq. (1.1); sextupole lenses of the same polarity are spaced by multiples of $q/n$ and sextupole lenses of opposite polarity by odd multiples of $q/2n$.

ii) The sextupole lenses are placed at regular intervals around the machine and have the same strength and the same polarity.

It is convenient to refer to periods of the sequence of sextupoles (called periods here) rather than to revolutions around the machine. The advance of the betatron phase function, $\Psi = \int (ds/\beta)$, per period is denoted
by \( \mu \). The resonance value of \( \mu \) is \( \bar{\mu} = (2\pi \times \text{integer})/3 \). The properties of the radial phase plane are computed at the longitudinal mid-points of the sextupole lenses.

2.1 Fixed points

A point in the phase plane is a fixed point \(^3\) of order \( m \) if it represents a trajectory which repeats itself every \( m \) periods. This trajectory gives \( m \) fixed points of order \( m \).

The origin of the phase plane is, as it has been defined, a first-order fixed point \( X_0 \). The presence of a triplet of third-order fixed points is next investigated.

On account of assumptions i) and ii) the sextupole distribution is aximuthally symmetrical with respect to the longitudinal mid-points of the sextupole lenses (S-points) and to points midway between the sextupole lenses (mid-S points). Hence in the phase plane at these points the triplet of third-order fixed points is symmetrical with respect to the \( x' = 0 \) axis, and the third-order fixed points lie at the vertices of an isosceles triangle, which for \( \mu \to \bar{\mu} \) tends to be an equilateral triangle centred on the origin of the phase plane; one of the fixed points lies on the \( x' = 0 \) axis. If the point \( X_0 \) is disregarded, because of the symmetry of the situation, the condition for a point on the \( x' = 0 \) axis of the phase plane at an S-point to be a third-order fixed point is that this point is transformed after \( \frac{\pi}{2} \) periods to a point on the \( x' = 0 \) axis of the phase plane at mid-S. This condition is expressed by:

\[
\frac{\pi}{2} \Delta \mu + \Sigma \Delta \Phi_S = 0, \tag{2.1}
\]

where \( \Sigma \Delta \Phi_S \) is the sum of the phase shifts given by the sextupoles in the \( \frac{\pi}{2} \) periods. The physical meaning of Eq. (2.1) is that the phase shifts, given by the sextupoles to the trajectory which gives the triplet of third-order fixed points, compensate exactly for the difference of \( \mu \) from the resonance value \( \bar{\mu} \) and therefore bring the phase advance in three periods to an integer number of turns.
The third-order fixed points are computed as a first approximation solution of Eq. (2.1) for $\Delta \mu = \mu - \bar{\mu} \ll 1$; this means that the kicks given by the sextupoles are considered small compared to the amplitude of oscillation. A trajectory starting at the abscissa $x$ on the $x' = 0$ axis of the phase plane at an S-point, receives from the downstream half-sextupole the kick $K_S x^2/2$ and consequently the phase advance $-K_S x/2$, where $K_S$ is the sextupole strength. From the sextupole, which is one period downstream, the trajectory receives the kick $K_S (x \cos \bar{\mu})^2$, and the phase advance $-K_S x \cos^3 \bar{\mu}$. Therefore:

$$\sum \Delta \psi = -\frac{K_S x}{2} - K_S x \cos^3 \bar{\mu} = -\frac{1}{2} K_S x.$$

By introducing this expression into Eq. (2.1) and by solving for $x$, one obtains the abscissa $x_i$ of the third-order fixed point $X_i$, lying on the $x' = 0$ axis. The coordinates of the successive third-order fixed points of the triplet $X_2, X_3$ are found by using the transfer matrix $M_\mu$ of the piece of tuned machine, under resonant conditions, between two successive sextupoles:

$$X_i - X_o = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{4\Delta \mu}{K_S}$$  \hspace{1cm} (2.2.a)

$$X_2 - X_o = M_\mu (X_1 - X_o) = \begin{pmatrix} \cos \bar{\mu} \\ -\sin \bar{\mu} \end{pmatrix} \frac{4\Delta \mu}{K_S} = \begin{pmatrix} -1/2 \\ \pm \sqrt{3}/2 \end{pmatrix} \frac{4\Delta \mu}{K_S}$$  \hspace{1cm} (2.2.b)

$$X_3 - X_o = M_\mu^{-1} (X_1 - X_o) = \begin{pmatrix} \cos \bar{\mu} \\ \sin \bar{\mu} \end{pmatrix} \frac{4\Delta \mu}{K_S} = \begin{pmatrix} -1/2 \\ \pm \sqrt{3}/2 \end{pmatrix} \frac{4\Delta \mu}{K_S}.$$  \hspace{1cm} (2.2.c)

The alternative signs refer to $\bar{\mu}/2\pi = \text{integer} \pm 1/3$. Eqs. (2.2) give the coordinates of the third-order fixed points at the S-points.
2.2 Local properties of the fixed points

The linearized motion in the neighbourhood of the fixed points is considered next. The local properties of each fixed point are deduced from the transfer matrix for a number of periods equal to the order of the fixed point.

The first-order fixed point $X_0$ corresponds to the stable closed orbit. In the phase plane particles move around $X_0$ in circles and with phase advance per period $\mu$.

The local properties of the third-order fixed points will be computed for the fixed point $X_1$. Those of the others can be deduced by a rotation of $\pm \bar{\mu}$ in the phase plane. The three-period matrix for the linearized motion in the neighbourhood of the fixed point $X_1$, has the expression:

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2K_3x_1 & 1 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2K_3x_2 & 1 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K_5x_1 & 1 \end{pmatrix}$$

where $c = \cos \mu$, $s = \sin \mu$.

Taking into account Eq. (2.2) and neglecting terms of higher order in $\Delta \mu$, one obtains:

$$U = \begin{pmatrix} 1 & 0 \\ 4\Delta \mu & 1 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4\Delta \mu & 1 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4\Delta \mu & 1 \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4\Delta \mu & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos 3\mu + (\cos 3\mu - c^3) (4\Delta \mu)^2; \sin 3\mu + (\cos 3\mu - c) 4\Delta \mu \\ -\sin 3\mu + (\cos 3\mu - c) 4\Delta \mu; \cos 3\mu + (\cos 3\mu - c^3) (4\Delta \mu)^2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + \frac{27}{2} \Delta \mu^2 \quad 9\Delta \mu \\ 3\Delta \mu \quad 1 + \frac{27}{2} \Delta \mu^2 \end{pmatrix}.$$

The half trace of $U$ is:

$$\frac{\text{Tr} U}{2} = 1 + \frac{27}{2} \Delta \mu^2 > 1.$$  (2.3)
The eigenvalues are:

\[ \lambda_1, 2 = 1 \pm \sqrt{27} \Delta \mu. \quad (2.4) \]

and the slopes of the eigenvectors are:

\[ \frac{x'}{x} = \pm \frac{1}{\sqrt{3}}. \quad (2.5) \]

The third-order fixed points are therefore unstable fixed points. The eigenvectors define the direction of phase plane separatrices\(^2,3\) of order three. The separatrices divide the phase plane into regions inaccessible from one another in a three-period step. A trajectory starting on a separatrix, after three periods returns on the same separatrix.

2.3 Phase-plane diagram

In Fig. 1 the theoretical phase-plane diagram is shown for two sets of parameters, and is compared with the computed one. If the polarity of the sextupoles is inverted, the direction of the axes must be reversed, according to the scaling law described in Ref. 3). The denomination of the separatrices and the direction of the three-period steps which particles take on them are given in the figure. The figure shows the coordinates 1,2,3,... at the S-points in successive periods, of a trajectory on the outward-going separatrices: from period to period this trajectory goes cyclically from one of these separatrices to the next, and the cycle is repeated every three periods with increased amplitude of oscillation. A similar cycle is performed by trajectories on the closed separatrices and by the trajectory which gives the third-order fixed points.

In the region which surrounds the stable fixed point and which is limited by the closed separatrices, the motion of the particles is stable. The theoretical stable region has the shape of an equilateral triangle the area of which is:

\[ A = 12 \sqrt{3} \left( \frac{\Delta \mu}{K_S} \right)^2. \quad (2.6) \]

The limit amplitude for stable non-linear oscillations is the amplitude of oscillation of the trajectory which gives the third-order fixed points.
If the assumption ii) of Chapter 2 is not valid one can expect a distortion of the phase-plane diagram because of the different contributions of the sextupoles to the phase shift $\Sigma \Delta \Psi_s$. Since for $\Delta \mu \ll 1$ the kicks given by the sextupoles are small compared to the amplitude of oscillation, this distortion is small. The formulae which have been derived in Sections 2.1, 2.2, and 2.3 remain valid, provided that the substitutions of $2\pi \Delta Q$ for $\Delta \mu$ and of the total sextupole strength for $K_S$ are made. This total sextupole strength must be computed by giving to the strengths of the sextupoles excited as described in assumption i) of Chapter 2 the same sign as the strength of the sextupole where the phase plane is considered. The phase-plane diagram at a sextupole is determined by the ratio between $\Delta Q$ and the total sextupole strength.

2.4 Resonant conditions

In the resonant conditions ($\Delta \mu = 0$) the stable area is zero. Particle motion on the outward-going separatrices and on the inward-going separatrices is studied considering the kicks given by the sextupoles to be small with respect to the amplitude of oscillation; second order terms and higher are neglected. Under this assumption a trajectory starting at $X = (x, x')$ in the phase plane at an S-point, arrives after three periods at a point displaced by:

$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \cos 3\bar{\mu} \sin 3\bar{\mu} & 0 \\ -\sin 3\bar{\mu} \cos 3\bar{\mu} & 1 \end{pmatrix} K_S \frac{x^2}{2} +$$

$$+ \begin{pmatrix} \cos 2\bar{\mu} \sin 2\bar{\mu} & 0 \\ -\sin 2\bar{\mu} \cos 2\bar{\mu} & 1 \end{pmatrix} K_S (x \cos \bar{\mu} + x' \sin \bar{\mu})^2 +$$

$$+ \begin{pmatrix} \cos \bar{\mu} & 0 \\ -\sin \bar{\mu} & 1 \end{pmatrix} K_S (x \cos 2\bar{\mu} + x' \sin 2\bar{\mu})^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} K_S \frac{x^2}{2}.$$  

Substituting for $\bar{\mu}$ one obtains:

$$\Delta X = \frac{3}{4} \kappa \frac{x^2 - x'^2}{(x^2 - x'^2)^{1/2}}. \quad (2.7)$$
The condition for a particle to move on a separatrix is:

\[
\frac{\Delta x}{\Delta x'} = \frac{x}{x'}.
\]

Introducing into this the expression for \(\Delta x/\Delta x'\) given by Eq. (2.7) one obtains the equation:

\[x (x' - 3x'^3) = 0.\]

The separatrices are therefore defined by:

\[x = 0; \quad \frac{x'}{x} = \pm \frac{1}{\sqrt{3}}.\] (2.8)

For a point \(X = \begin{pmatrix} 0 \\ x' \end{pmatrix}\) the expression (2.7) becomes:

\[
\Delta X = -\frac{3}{4} K_S x'^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\] (2.9)

from which the direction and the magnitude of the three-period steps along the \(x = 0\) separatrix can be deduced. A rotation of \(\pm \mu\) gives the direction and the magnitude of the steps along the other separatrices.

The increase of the amplitude of oscillation \(r = \sqrt{x'^2 + x'^2}\) after three periods is:

\[
\Delta r = \frac{3}{4} K_S r^2.\] (2.10)

In Fig. 2 is schematized the phase-plane diagram and the mechanism of the three-period steps which particles take from A to B along the separatrices. A trajectory which at an S-point is at the point A on a separatrix, during the three following periods receives from the sextupoles the kicks 1, 2, 3 and at the S-point three periods downstream is at the point B on the same separatrix. The three-period step from A to B is found by linear composition of the images \(I_1, I_2, I_3\) of the kicks received by the trajectory.
2.5 Extraction of the particles

The expression (2.6) of the stable area shows that the particles can be spilled out from the stable region by moving the $Q$-value to the resonant value $\bar{Q}$. To obtain this $Q$-shift one can use tuning quadrupole fields or, if the $Q$-value depends on the radial position of the stable closed orbit (giving the first-order fixed point) one can move the stable closed orbit $x_0$ towards the radial position where $Q$ has the resonant value $\bar{Q}$.

If the spill-out is slow, in the adiabatic approximation spilt particles move on the outward-going separatrices. A three-revolution step taken by a trajectory along an outward-going separatrix, defines the width of a beam which contains all the spilt particles. This "spilt" beam moves from an outward going separatrix to the next and every three revolutions returns on the same separatrix, with increased amplitude of oscillation and increased beam width. The linear particle density of spilt particles on the outward-going separatrix therefore decreases with increasing amplitude of oscillation.

The spilt beam must be extracted when its outer trajectory has the maximum oscillation amplitude $r_m$ admitted by the machine. The septum of the extraction magnet "E", which deflects the particles away from the machine, is placed at the radial position corresponding to the inner trajectory. At the azimuthal position of E, one of the outward-going separatrices must have a sufficiently small slope that all the particles in the extracted beam have a larger displacement than the particles circulating in the machine. If at the first sextupole upstream E the extracted beam is not on the outward-going separatrix which is approximately vertical, and consequently receives from this sextupole a relatively large kick, the largest amplitude of oscillation occurs only between this sextupole and E.

Before it enters E, the extracted beam can be separated from the circulating particles by means of a "thin" septum magnet "T". The "thick" septum of E is inserted in the gap produced by T between the circulating particles and the extracted beam. Particle loss occurs on the thin septum. The particle loss on T per unit septum thickness is:

$$P_T = \frac{\rho(r_T)}{\cos \phi_T} \quad ,$$

(2.11)
where $\psi_T$ is the phase, measured in the clockwise direction from the x-axis of the phase plane at $T$, of the outward-going separatrix where the extracted beam is at $T$; the linear density of split particles on the same separatrix at the oscillation amplitude $r_T$ corresponding to the septum of $T$, is denoted by $\rho(r_T)$ and is a decreasing function of $r_T$. In order to determine the optimum azimuthal position of $T$ and $E$, a factor of merit $f$ can be defined as follows:

$$f = \frac{\Delta x_E/\Delta x_T'}{P_T}, \quad (2.12)$$

where $\Delta x_E$ is the displacement produced at $E$ by a kick $\Delta x_T'$ given by $T$. If between $T$ and $E$ no kick is given by the sextupoles to the extracted beam, $r_T'\lambda$ has the largest possible value and the factor of merit is:

$$f \sim \cos (\psi_E - \psi_{ET}) \sin \psi_{ET} \sim \sin (2\psi_{ET} - \psi_E) + \sin \psi_E \quad (2.13)$$

where $\psi_{ET}$ is the phase advance from $T$ to $E$, and $\psi_E = \psi_T + \psi_{ET}$. If

$$\psi_{ET} = \frac{\psi_E}{2} + (2h + 1) \frac{\pi}{4}; \quad h = \text{integer}$$

the factor of merit has the extreme value:

$$f \sim (-1)^h \sin \psi_E.$$

The maximum value of $|f|$ is proportional to $1 + |\sin \psi_E|$. The upper limit for $|\sin \psi_E|$ is imposed by the condition that at $E$ the extracted beam must be displaced more than the circulating particles.

If a single septum magnet is used to separate the extracted beam from the circulating particles and to deflect it away from the machine, and if this septum magnet is denoted by $T$, the particle loss on it per unit septum thickness is still expressed by Eq. (2.1).

If the sextupole strength is increased, a larger growth rate of the resonant oscillations is obtained. Consequently the linear particle density on the outward-going separatrices is decreased, but the septums
must be shifted to inner radial positions, where the linear particle
density on the outward-going separatrixes is larger; there is a value
of the sextupole strength which gives a minimum particle loss.

2.6 Vertical optics

The vertical motion is perturbed in two ways:

i) The tuning quadrupole fields and the quadrupole components of the
   perturbing sextupole fields on the distorted equilibrium orbit
   cause a static perturbation.

ii) The quadrupole components of the sextupole fields acting on the
    split beam become stronger when the radial amplitude of oscilla-
    tion of the split beam increases, and cause a dynamic perturbation
    of its optical properties.

In the system of units $N_1$, only the dynamic perturbation is given.
The strength of the perturbing quadrupole components of the sextupole
fields is proportional to the radial displacement of the split beam from
the stable closed orbit. While recurring periodically, these quadrupole
components are increasing in strength from period to period of the
perturbation.

A perturbation in general causes a beat oscillation of the beam
height with respect to the unperturbed beam height. The beat factor $G$
is defined as the ratio of the major to the minor axis of the perturbed
emittance ellipses, in a system of units normalized with respect to the
unperturbed machine. The beat factor is azimuthally constant in each
piece of unperturbed machine. The expression $\sqrt{G-1}$ gives the amplitude
of the beat oscillation, compared to the unperturbed beam height.

An expression for the beat factor due to the dynamic perturbation
of the split beam is derived, in the system of units $N_1$, using the
following assumptions:
i) the sextupoles are placed as in assumption i) of Chapter 2 and at regular intervals around the machine circumference (consequently they are excited with the same polarity or with alternate polarities). The normalized absolute value of the perturbing quadrupole fields has in all the sextupoles the same linear dependence on the radial displacement from the stable closed orbit;

ii) the increase of the amplitude of the radial oscillation of the split beam during the period of the perturbation is negligible;

iii) the theoretical expression of the separatrices is valid;

iv) the normalized strengths of the perturbing quadrupole fields are small compared to one.

If at a sextupole the beam is on the vertical separatrix, it is not perturbed by any quadrupole field. On account of assumption i) at the following sextupole the beam is on one of the two other separatrices; by introducing the theoretical expression of the separatrices, one finds that the perturbing quadrupole field has normalized strength

$$|K| = \sqrt{3} \, K_{s} \beta_{\nu} / \beta_{h},$$

where $\beta_{\nu}, \beta_{h}$ are the unnormalized vertical and radial betatron amplitude functions at the location of the sextupole and $r$ is the amplitude of the radial oscillation of the beam. On account of assumptions i), ii) and iii) the quadrupole field given by the following sextupole has strength $-K$. The sequence of quadrupole fields of strength 0, $+K$, $-K$ covers a period of the perturbation. Under the adiabatic approximation the properties of the motion can be deduced from the transfer matrix for one period of the perturbation. At the entrance of the quadrupole field of strength $+K$ this matrix is expressed by:

$$V_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix} = \begin{pmatrix} \cos 3\mu_{\nu} + Ks(c^2 - s^2) - 2K^2s^2c & \sin 3\mu_{\nu} - 2Ks^2c \\ -\sin 3\mu_{\nu} - 2Ks^2c - K^2s(c^2 - s^2) & \cos 3\mu_{\nu} - Ks(c^2 - s^2) \end{pmatrix}$$
where $\mu_v$ is the vertical phase advance between two successive sextupoles and $c = \cos \mu_v$, $s = \sin \mu_v$. At the exit of the quadrupole field of strength $+K$, the matrix for one period of the perturbation is expressed by:

$$V_2 = \begin{pmatrix}
\cos 3\mu_v - 2Ks^2 & \sin 3\mu_v - 2Ks^2 c \\
-\sin 3\mu_v - 2Ks^2 c & \cos 3\mu_v + 2Ks^2 - 2K^2 s^2 c
\end{pmatrix}.$$ 

The half-trace of the matrices $V_1$ and $V_2$ is:

$$\frac{\text{Tr} V}{2} = \cos 3\mu_v - K^2 s^2 c = \cos 3\mu_v \left(1 + \frac{K^2 s^2}{3 - 4c^2}\right). \quad (2.14)$$

From Eq. (2.14) one can see that the perturbation introduces instability in the vicinity of:

$$\mu_v = \frac{n\pi}{3} \quad (n = \text{integer}, \frac{n}{3} \neq \text{integer}). \quad (2.15)$$

If $n/3$ is integer, the instability is not introduced because the perturbing quadrupole fields $+K$, $-K$ compensate exactly. Writing $\mu_v = \frac{n\pi}{3} + \epsilon$ and neglecting terms in $\epsilon$ and $K$ of the third order and higher, one has:

$$\frac{\text{Tr} V}{2} = \cos \frac{n\pi}{3} \left(1 - \frac{9\epsilon^2}{2}\right) - K^2 \sin^2 \frac{n\pi}{3} \cos \frac{n\pi}{3}. \quad (2.16)$$

By solving the equation $\left(\text{Tr} V\right)/2 = \cos n\pi$ for $\epsilon$, one obtains the extremes of the stopband, given by:

$$\epsilon = \pm K \frac{\sin \left(\frac{n\pi}{3}\right)}{3} \sqrt{-2 \frac{\cos(n\pi/3)}{\cos n\pi}} = \pm \frac{K}{2\sqrt{3}}. \quad (2.17)$$

For the stable motion the beat factor at the entrance of the quadrupole field of strength $+K$ is $G_1$, expressed, as a function of the elements of the one-period matrix $V_1$, by:

$$G_1^2 = \frac{|b - c| + \sqrt{(b + c)^2 + (a - d)^2}}{|b - c| - \sqrt{(b + c)^2 + (a - d)^2}} > 1.$$
By introducing the expressions of the matrix elements one obtains:

\[ G_1^2 = \frac{4c^2 - 1 + K^2 (c^2 - s^2)/2 + |K| \sqrt{1 + K^2/4}}{|4c^2 - 1 + K^2 (c^2 - s^2)/2| - |K| \sqrt{1 + K^2/4}}. \]

Similarly, at the exit of the quadrupole field of strength +K the beat factor is \( G_2 \), expressed by:

\[ G_2^2 = \frac{4c^2 - 1 + K^2 c^2 | + 2 |Kc| \sqrt{1 + K^2/4}}{|4c^2 - 1 + K^2 c^2 | - 2 |Kc| \sqrt{1 + K^2/4}}. \]

If \( K < 4c^2 - 1 \), i.e., if \( \mu_v \) is far enough from the stopbands defined by Eq. (2.15), the first order approximation expressions of \( G_1 \) and \( G_2 \) are

\[ G_1 = 1 + \frac{K}{4c^2 - 1} = 1 + \frac{\sin \mu_v}{\sin 3\mu_v}, \]

\[ G_2 = 1 + \frac{2cK}{4c^2 - 1} = 1 + \frac{\sin 2\mu_v}{\sin 3\mu_v}. \]

The functions \((G_1 - 1)/|K|\) and \((G_2 - 1)/|K|\) are plotted in Fig. 3.

3. NON-TUNED MACHINE

In this Chapter the results obtained in Chapter 2 are written in a system of units "N_2" normalized with respect to the non-tuned machine; the ideal equilibrium orbit is taken as reference curve for this coordinate system. If the system of units is not specified differently, the symbols contained in this Chapter denote quantities which are expressed in the system of units N_2.

The transformation from the system of units N_2 to the system of units N_4 is the inverse of a normalization with respect to the tuning. This transformation is determined (Appendix I) by the betatron functions \( \beta \), \( \alpha \) and by the position \( X_0 \) of the stable closed orbit of the tuned machine, in the system of units N_2.
Assuming $\beta_n = 1$ and denoting by $\beta_S$ and $\alpha_S$ the betatron functions at the location of a sextupole, where the phase-plane is considered, the coordinates of the third-order fixed points and the stable area are expressed by:

$$x_1 - x_0 = \frac{1}{\sqrt{\beta_S}} \begin{pmatrix} \beta_S & 0 \\ -\alpha_S & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{k \Delta \mu}{K_S \beta_S^{3/2}} = \begin{pmatrix} \beta_S \\ -\alpha_S \end{pmatrix} \frac{k \Delta \mu}{K_S \beta_S^{3/2}}$$

(3.1.a)

$$x_2 - x_0 = \frac{1}{\sqrt{\beta_S}} \begin{pmatrix} \beta_S & 0 \\ -\alpha_S & 1 \end{pmatrix} \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \frac{k \Delta \mu}{K_S \beta_S^{3/2}} = \begin{pmatrix} -\beta_s/2 \\ \alpha_s/2 \pm \sqrt{3}/2 \end{pmatrix} \frac{k \Delta \mu}{K_S \beta_S^{3/2}}$$

(3.1.b)

$$x_3 - x_0 = \frac{1}{\sqrt{\beta_S}} \begin{pmatrix} \beta_S & 0 \\ -\alpha_S & 1 \end{pmatrix} \begin{pmatrix} -1/2 \\ \pm \sqrt{3}/2 \end{pmatrix} \frac{k \Delta \mu}{K_S \beta_S^{3/2}} = \begin{pmatrix} -\beta_s/2 \\ \alpha_s/2 \pm \sqrt{3}/2 \end{pmatrix} \frac{k \Delta \mu}{K_S \beta_S^{3/2}}$$

(3.1.c)

$$A = 12 \sqrt{3} \left( \frac{\Delta \mu}{K_S \beta_S^{3/2}} \right)^2 = \frac{12 \sqrt{3}}{\beta_S^3} \left( \frac{\Delta \mu}{K_S} \right)^2.$$ 

(3.2)

Eqs. (3.1) and (3.2) are derived from Eqs. (2.2) and (2.6) and through Eqs. (A4), (A6) and (A7) of Appendix I.

The position of a third-order fixed point $x_1$ does not change during the spill-out process if the following condition is satisfied:

$$\frac{dX_0}{d\tau} = -\frac{d(x_1 - x_0)}{d\tau}$$

where $x_1 - x_0$ is expressed by Eq. (3.1). This is advantageous for obtaining a small radial emittance of the extracted beam.
3.1 Tuning by means of distributed quadrupole fields

If the tuning is made by using distributed quadrupole fields and the consequent variation of the amplitude betatron function of the machine is neglected, β = 1, α = 0. The above-mentioned transformation is a translation of the origin of the phase plane by $X_0$.

3.2 Tuning by means of a single quadrupole lens

The perturbation to the radial betatron functions caused by a tuning quadrupole lens $L$ of strength $K_L$ is considered first. The one-revolution transfer matrix at an azimuth $\Psi_0$ after $L$, in the system of units $N_2$ is:

$$M_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} \cos \Psi_0 & \sin \Psi_0 \\ -\sin \Psi_0 & \cos \Psi_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\mu_0 - \Psi_0) \sin(\mu_0 - \Psi_0) \\ -\sin(\mu_0 - \Psi_0) \cos(\mu_0 - \Psi_0) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \mu_0 + K_L \sin \Psi_0 \cos(\mu_0 - \Psi_0) & \sin \mu_0 + K_L \sin \Psi_0 \sin(\mu_0 - \Psi_0) \\ -\sin \mu_0 + K_L \cos \Psi_0 \cos(\mu_0 - \Psi_0) & \cos \mu_0 + K_L \cos \Psi_0 \sin(\mu_0 - \Psi_0) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \mu_0 \pm \frac{K_L}{2} (\sin \mu_0 + \sin(2\Psi_0 - \mu_0)) & \sin \mu_0 \pm \frac{K_L}{2} [-\cos \mu_0 + \cos(2\Psi_0 - \mu_0)] \\ -\sin \mu_0 \pm \frac{K_L}{2} (\cos \mu_0 + \cos(2\Psi_0 - \mu_0)) & \cos \mu_0 \pm \frac{K_L}{2} [\sin \mu_0 - \sin(2\Psi_0 - \mu_0)] \end{pmatrix}$$

where $\mu_0 = 2\pi Q_0$ is the phase advance per revolution. The half-trace of the matrix $M_0$ is:

$$h = \cos \mu_0 + \frac{K_L}{2} \sin \mu_0.$$  

The betatron functions $\beta, \alpha$ are obtained in the usual way from the matrix $M_0$, as follows:
\[
\beta = \frac{2 \sin \mu_0 - K_L \cos \mu_0 + K_L \cos (2 \psi_o - \mu_0)}{2 \sqrt{1-h^2} \ b_o/|b_o|} \quad (3.3)
\]

\[
\tilde{a} = \frac{K_L \sin (2 \psi_o - \mu_o)}{2 \sqrt{1-h^2} \ b_o/|b_o|}.
\]

Assuming that \( \Psi_o \) is the phase advance from the quadrupole to the sextupole, where the phase plane is considered, these formulae give the values of \( \beta_S \), \( a_S \) necessary for insertion in Eqs. (3.1) and (3.2).

A quadrupole lens which brings \( 3 \Phi_o \) to the nearest integer, i.e. which makes \( h = -1/2 \), is defined as "normal", the required strength of it being:

\[
\tilde{K}_L = \frac{1 + 2 \cos \mu_0}{\sin \mu_0} \quad (3.4)
\]

The phase advance per revolution \( \mu \) and the position \( X_C \) of the stable closed orbit, in the tuned machine, depend on the strength of the quadrupole and of the sextupoles and on the displacements of the equilibrium orbit from their centre line in the non-tuned machine \( x_L, x_S \). In the one-sextupole case\(^5\) the expressions of \( \cos \mu \) and of the position \( X_C \) of the stable closed orbit at the longitudinal mid-point of the sextupole, have been determined in Chapter 2 of Ref. 5), as follows:

\[
\cos \mu = 1 - \sqrt{\Delta}
\]

\[
X_C = \begin{pmatrix} X_o \\ X'_C \end{pmatrix} = \begin{pmatrix} \frac{1-h-\sqrt{\Delta}}{K_S b_o} \\ \frac{(d_0 - a_0) (x_C - x_S)}{K_L x_L} - \frac{K_L x_L [\sin \psi_o - \sin (\mu_0 - \psi_o)]}{2 b_o} \end{pmatrix} \quad (3.5)
\]

\[FS/5731\]

\[p/mn\]
\[ \Delta = (1-h)^2 - K_S \ b_0 \left( K_L x_L \left[ \sin \psi_o + \sin (\mu_o - \psi_o) \right] + 2x_S (1-h) \right). \]

The subscript \(_o\) denotes the unperturbed machine. The dependence of \(\mu\) and \(x_o\) in \(\mu_o, K_L, x_L, x_S\) can be linearized in the neighbourhood of the resonant conditions, defined by Eq. (3.4), and in the neighbourhood of \(x_L = x_S = 0\), as follows:

\[ \Delta \mu \approx \frac{\Delta \cos \mu}{\sin \mu} = \frac{\Delta h + K_S b_0 \left\{ (K_L x_L/3) \left[ \sin \Psi_o + \sin (\bar{\mu}_o - \bar{\psi}_o) \right] + x_S \right\}}{-\sin \bar{\mu}}. \]

\[ x_o \approx \frac{K_L x_L}{3} \left[ \sin \bar{\psi}_o + \sin (\bar{\mu}_o - \bar{\psi}_o) \right] + x_S \]

\[ \frac{2}{3} \frac{K_L}{x_L} \sin (\bar{\mu}_o - 2\bar{\psi}_o) \left[ \sin \bar{\psi}_o + \sin (\bar{\mu}_o - \bar{\psi}_o) \right] - \left[ \sin \bar{\psi}_o - \sin (\bar{\mu}_o - \bar{\mu}_o) \right] \]

where:

\[ \Delta h = h - \bar{h} = \left( -\sin \bar{\mu}_o + \frac{K_L}{2} \cos \bar{\mu}_o \right) \Delta \mu_o + \frac{\sin \bar{\mu}_o}{2} \Delta K_L. \]

The bar above the symbols denotes that the corresponding quantities are evaluated for the resonant conditions and for \(x_L = x_S = 0\).

3.3 Vertical optics

The dynamic perturbation evaluated in Section 2.6 must be superimposed on the static perturbation, which is conveniently studied in the system of units \(N_s\). Deviations of the stable closed orbit from the centre line of the sextupoles cause a static perturbation of the beam contained in the stable region of the radial phase plane. Since usually these deviations
are small compared to the radial oscillation amplitude of the extracted beam this static perturbation can be neglected compared to the dynamic perturbation of the extracted beam. This section covers the static perturbation caused by the tuning quadrupole fields.

If the tuning is obtained by means of quadrupole fields distributed all around the machine, usually this static perturbation is negligible.

If the tuning is obtained by means of a single tuning quadrupole $L$, having for the vertical motion normalized strength $K_{LV}$, the beat factor caused by $L$ is expressed by:

$$G_L^2 = \frac{2 + K_{LV} \tan \pi Q_{OV}}{2 - K_{LV} \cot \pi Q_{OV}},$$

where $Q_{OV}$ is the nominal $Q$-value for the vertical motion. If the right-hand side is smaller than one, its reciprocal must be taken, such that $G_L > 1$. The normalized quadrupole strength for the vertical motion $K_{LV}$ is related to the normalized strength for the radial motion $K_L$, by:

$$K_{LV} = - K_L \frac{\beta_v}{\beta_h},$$

where $\beta_h$ and $\beta_v$ are the unnormalized $\beta$-values at the location of $L$. 
NORMALIZED SYSTEMS OF UNITS

Denoting by $s$ the distance along the equilibrium orbit and by $x$ either the radial or the vertical component of the displacement from the equilibrium orbit, the general expression of the betatron oscillation in an alternating gradient synchrotron is:

$$ x(s) = a \sqrt{\beta(s)} \cos (\psi(s) + \delta) \quad \text{(A1)} $$

where $\beta(s)$ is the betatron amplitude function, $\psi(s) = \int (ds/\beta)$ is the betatron phase function and $a$ and $\delta$ are arbitrary constants. This expression is reduced to a harmonic oscillation by introducing "normalized" variables:

$$ \tilde{x} = \sqrt{\beta_n/\beta} x \quad \text{(A2)} $$

$$ \psi = \int (ds/\beta) \quad \text{(A3)} $$

where $\beta_n$ is a constant which has the same dimensions as $\beta$.

According to Eqs. (A2), (A3) at any particular azimuth the phase plane $(x, x')$ linearly transforms into the "normalized" phase plane $(\tilde{x}, \tilde{x}')$; the primes denote differentiation respectively by $s$ and by $\psi$. In matrix notation:

$$ \begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} = N \begin{pmatrix} x \\ x' \end{pmatrix} \quad \text{(A4)} $$

The matrix $N$ is expressed by:

$$ N = \sqrt{\frac{\beta_n}{\beta}} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \quad \text{(A5)} $$

where $\alpha = -(ds/ds)/2$. The determinant of $N$ is $\det N = \beta_n$. The inverse matrix of $N$ is:
\[ N^{-1} = \frac{1}{\sqrt{\beta_n \beta_n^*}} \begin{pmatrix} \beta & 0 \\ -\alpha & 1 \end{pmatrix} \]  
(A6)

An area \( A \) in the phase plane \((x, x')\) is related to the corresponding area \( \bar{A} \) in the normalised phase plane \((\bar{x}, \bar{x}')\) by

\[ \bar{A} = \beta_n A. \]  
(A7)

The matrix for the transformation of the phase plane \((x, x')\) from an azimuth \( s_1 \) to an azimuth \( s_2 \) is expressed as:

\[
\begin{pmatrix}
\frac{\beta_2}{\beta_1} (\cos \Psi + \alpha_1 \sin \Psi) & \sqrt{\beta_1 \beta_2} \sin \Psi \\
-\frac{(1 + \alpha_1 \alpha_2) \sin \Psi + (\alpha_2 - \alpha_1) \cos \Psi}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \Psi - \alpha_2 \sin \Psi)
\end{pmatrix} 
\]  
(A8)

when \( \Psi = \Psi(s_2) - \Psi(s_1), \beta_1 = \beta(s_1), \) etc. The matrix for the transformation of the normalised phase plane is:

\[
\begin{pmatrix}
\cos \Psi & \sin \Psi \\
-\sin \Psi & \cos \Psi
\end{pmatrix} 
\]  
(A9)

If a trajectory receives a kick \( \Delta x' \), in the normalised phase plane it receives the kick:

\[ \Delta \bar{x}' = \sqrt{\beta_n} \Delta x' \]  
(A10)

A thin lens which gives to particles displaced by \( \Delta x \) from its centre the kick:

\[ \Delta x' = K \Delta x^j, \]

where \( K \) is the lens strength and \( j \) an integer, gives to particles having normalised displacement \( \Delta \bar{x} \) from its centre, the normalised kick:
\[ \Delta \bar{x}' = \sqrt{\beta/\beta_n} \Delta x' = \sqrt{\beta/\beta_n} K \Delta x^j = \sqrt{\beta/\beta_n} (\beta/\beta_n)^{j/2} \chi (\Delta \bar{x})^j. \]

One can therefore define a "normalized" lens strength:

\[ \bar{K} = \sqrt{\beta/\beta_n} (\beta/\beta_n)^{j/2} K. \] (A11)

This shows that at any particular azimuth, Eqs. (A2), (A3) define a "normalized" system of units. \( \beta_n \) can be used as a scaling factor, i.e. numerical computations performed in a normalized system of units can be utilized for different sets of actual parameters by assigning different values to \( \beta_n \).

Perturbations to the betatron oscillations can be studied in a system of units normalized with respect to the unperturbed machine. Let \( M_0 \) be the transfer matrix for one period of the unperturbed machine:

\[ M_0 = \begin{pmatrix} \cos \mu_0 + \alpha_0 \sin \mu_0 & \beta_0 \sin \mu_0 \\ -\gamma_0 \sin \mu_0 & \cos \mu_0 - \alpha_0 \sin \mu_0 \end{pmatrix}. \] (A12)

where \( \mu_0 \) is the phase advance per period. If the perturbed betatron oscillations are stable, the transfer matrix \( M \) for one period of the perturbed machine and the corresponding normalized matrix \( \bar{M} \) can be written in the form:

\[ M = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}. \] (A13)

\[ \bar{M} = \begin{pmatrix} \cos \mu + \bar{\alpha} \sin \mu & \bar{\beta} \sin \mu \\ -\bar{\gamma} \sin \mu & \cos \mu - \bar{\alpha} \sin \mu \end{pmatrix}. \] (A14)
The relations between the elements of \( \mathbb{M} \) and those of \( \tilde{\mathbb{M}} \) are found by equating to \( \tilde{\mathbb{M}} \) the matrix \( N_0 M N_0^{-1} \), where \( N_0 \) is the matrix for normalization with respect to the unperturbed machine:
\[
N_0 = \sqrt{\frac{\beta}{\beta_0}} \begin{pmatrix} 1 & 0 \\ \alpha_0 & \beta_0 \end{pmatrix}.
\] (A15)

From Eqs. (A13), (A15) one obtains:
\[
N_0 M N_0^{-1} = \frac{1}{\beta_0} \begin{pmatrix} 1 & 0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ -\alpha_0 & 1 \end{pmatrix} =
\]
\[
\begin{pmatrix} \cos \mu + [\alpha - \alpha_0 (\beta/\beta_0)] \sin \mu & (\beta/\beta_0) \sin \mu \\ -\frac{1 + [\alpha - \alpha_0 (\beta/\beta_0)]^2}{\beta/\beta_0} \sin \mu & \cos \mu - [\alpha - \alpha_0 (\beta/\beta_0)] \sin \mu \end{pmatrix}.
\]

Therefore:
\[
\tilde{\beta} = \frac{\beta}{\beta_0}, \quad \tilde{\alpha} = \alpha - \alpha_0 \frac{\beta}{\beta_0}, \quad \psi = \int \frac{d\alpha}{\beta} \int \frac{d\psi}{\beta} \quad \gamma = \frac{1 + \tilde{\alpha}^2}{\tilde{\beta}},
\]

where \( \psi_0, \psi \) are the betatron phase functions of the unperturbed and of the perturbed machine.
A FORTRAN PROGRAM FOR COMPUTATIONS ON PERTURBED ALTERNATING GRADIENT SYNCHROTRONS

The program deals with the betatron motion of particles in each single transverse phase plane of an alternating gradient synchrotron which is perturbed by lenses. The deviation of the particle momentum from a central momentum value is assigned. In some parts the program is an extension of the program written by C. Bovet for computations on integer resonance slow extraction from the CPS.

The thin lens approximation is made. The dependence on the transverse displacement $x$ of the kick $\Delta x'$ given by a lens is of the type $\Delta x' = K (x - x_0)^j$, where $j$ is the order of the lens, $K$ is the strength and $x_0$ is the displacement from the reference line of the centre of the lens; one can impose that the kick $\Delta x'$ is given only if the displacement $x$ is larger and of the same sign as a displacement $x_s$, which corresponds to the position of a septum.

Observation points, i.e. points where some properties of the particle motion are output, can be inserted into the machine. Lenses and observation points are in general terms called insertions.

The unperturbed machine is treated as a linear device. Inside the program the pieces of machine between two successive insertions are represented by transfer matrices.

The following data must be given:

i) The nominal Q-value of the unperturbed machine for particles with zero momentum deviation, the variation of the Q-value with the momentum deviation $(dQ/Q)/(dp/p)$, the momentum deviation $\Delta p/p$ and the deviation $\Delta Q/Q$ of the Q-value for zero momentum deviation from the nominal value.

ii) For each insertion one must give: $j + 1$, $K$, $x_0$, $x_s$ and the values at the insertion point of the betatron functions $\beta$, $\alpha = - (d\beta/ds)/2$, $\psi = \int ds/\beta$, of the momentum compaction function $\alpha_p = \Delta x/(dp/p)$ and of the closed orbit displacement for zero momentum deviation. These values are those in the unperturbed machine. If $x_s = 0$ the kick
Δx' is given for any displacement. The observation points are denoted by j+i = 0; K, x_c, x_s are not used in this case. In a normalized system of units (Appendix I) β = 1, α = 0.

The data concerning the insertions must be given according to the azimuthal order of the insertions, starting from any of them.

iii) Data for the computations to be performed. After the data concerning the machine have been read and the transfer matrix of each piece of unperturbed machine has been worked out, or after a computation has been performed, a control card is read. This control card determines the computation which must be done next, and contains the data for performing this computation.

According to the control card the following computations are performed:

i) Tracking of a trajectory starting at the point immediately preceding the first insertion which has been introduced (point 1). The number of insertions to be traversed has to be given. It can correspond to more than one revolution. The coordinates of the trajectories are output at the points immediately preceding the insertions. An option exists to compute in addition:

a) the transfer matrix for the linearized motion in the neighbourhood of the computed trajectory, from point 1 to the final point of the trajectory;

b) the extreme displacements of the betatron oscillations of the computed trajectory and their azimuthal position;

c) the extreme displacements which a set of the trajectories has at the observation points.

ii) Computation of the linear properties of the phase plane in the neighbourhood of a one-revolution trajectory. This trajectory can be introduced or assumed to result from the last tracking executed.
iii) A search in the phase plane at point 1 of a fixed point of order
m ≤ 3 and computation of the linear properties of the motion in
its neighbourhood. The corresponding trajectory, which closes
itself after m revolutions, and the other m - 1 fixed points of
order m given by this trajectory are computed at the same time.

iv) Tracking in the phase plane at point 1 and at the observation
points of a divergent or closed separatrix, starting from the last
unstable fixed point which has been computed. The m branches of
the separatrix, each of them starting from one of the m unstable fixed
points of order m, are computed. In addition one can compute, if
required:

a) the particle density on one branch of the separatrix in the
phase plane at point 1 (in % per unit of the transverse displacement)
and the values and the azimuthal positions of the extremes of the
betatron oscillations which are performed by trajectories on the
separatrix;

b) the area included by the separatrix and the extreme displacements
of trajectories on the separatrix, at point 1 and at the observation
points.

The tracking is interrupted when an assigned upper limit for the
amplitude of the betatron oscillations is reached (outward-going
separatrix) or when the separatrix ends on an unstable fixed point
(closed separatrix).

v) Matching of the stable area in the phase plane to an assigned value.
The matching is done by varying in an assigned proportion the closed
orbit displacements due to orbit bumps, Δp/p and ΔQ/Q.
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5) P. Strolin, Integer resonance slow extraction (to be published).

6) C. Bovet, Programmes FORTRAN CERN pour le calcul des trajectoires perturbées et des systèmes d'éjection par résonance dans le CPS, MPS/DL Int. 65-5 (1965).
Fig. 1. Phase-plane diagram and coordinates of an unstable trajectory (1, 2, 3) of an s-point.

Theoretical computed: $k_3 = 0.02 	ext{mm}^{-1}$.
Fig. 2 Theoretical phase-plane in resonant conditions.
Fig 3: Vertical amplitude testing.
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