Stability of a relativistic rotating electron-positron jet: nonaxisymmetric perturbations

Ya.N.Istomin & V.I.Pariev
Lebedev Physical Institute, Leninsky Prospect 53, Moscow B-333, 117924, Russia

Abstract

We investigate the linear stability of a hydrodynamic relativistic flow of magnetized plasma in the simplest case where the energy density of the electromagnetic fields is much greater than the energy density of the matter (including the rest mass energy). This is the force-free approximation. We considered the case of light cylindrical jet in cold and dense environment, so the jet boundary remains at rest. Continuous and discrete spectra of frequencies are investigated analytically. An infinite sequence of eigenfrequencies is found near the edge of Alfvén continuum. Numerical calculations showed that modes having reasonable values of azimuthal wavenumber $m$ and radial number $n$ are stable and have attenuation increment $\gamma$ small. The dispersion curves $\omega = \omega(k_\parallel)$ have a minimum for $k_\parallel \xi 1/R$ ($R$ is the jet radius). This results in accumulation of perturbations inside the jet with wavelength of the order of the jet radius. The wave crests of the perturbation pattern formed in such a way move along the jet with the velocity exceeding light speed. If one has relativistic electrons emitting synchrotron radiation inside the jet, than this pattern will be visible. This provide us with the new type of superluminal source. If the jet is oriented close to the line of sight, than the observer will see knots moving backward to the core.

Key words: instabilities – MHD – galaxies: jets – quasars.


1 Introduction.

Possibly the most intriguing feature of a numerous extragalactic radio sources is the existence of a narrow well collimated radio jets. It is these jets that are believed to be responsible for the transportation of a great amount of energy from the central compact areas of the galaxies to their distant radio emitting parts. From the observations of the superluminal motions of bright knots along the jets one must conclude for the velocity of the flow to be relativistic with the Lorentz factor $\gamma$ being of the order of 5 to 10. One of the important problem in the physics of extragalactic jets is their stability over the large distances. Up to now there have been many works in which the stability of the jets is investigated under the different assumptions for the velocity of the matter in a jet and the influence of the magnetic field on the flow dynamics. Turland & Scheuer (1976) and Blandford & Pringle (1976) were the first who considered the Kelvin–Helmholtz instability of a plane relativistic hydrodynamic flow in vortex sheet approximation and applied the results to extragalactic jets. The perturbed modes are considered both for cylindrical geometry of a jet and for the plane boundary between the jet and the outside media if the wavelength is small enough (Hardee 1979, 1987; Birkinshaw 1984; Payne & Cohn 1985).

When it had been recognized that the magnetic field may be dynamically important for jet confinement, the problem of magnetohydrodynamics (MHD) stability of cylindrical flows naturally arose. This facet of the problem goes back to the investigations of the stability of magnetostatic plasma equilibria. Appert, Gruber & Vaclavik (1974) have pointed to the existence of Alfvén and slow wave continua in the case of the sheared magnetic field and (or) nonuniform plasma density. Cohn (1983) found the uniform jet with the ”top hat” velocity profile confined by the poloidal current flowing on the boundary, unstable for all modes. He has also investigated relative importance for instability fundamental and reflection modes (these modes differ by a radial wavenumber $n$, fundamental mode is one having $n = 0$, reflection modes have $n > 0$). The general system of the two first order differential equations on the radial Lagrangian displacement $\xi_r$ of the fluid element and the perturbation of the total pressure $P_1$, which describes the propagation of small disturbances along the cylindrical nonrelativistic MHD flow, were derived by Bondeson, Iacono & Bhattacharjee (1987) (thereafter referred to as BIB). Interesting information concerning the existence and location of point eigenvalues in the complex $\omega$ plane was obtained in BIB from a boundary layer analysis near the Suydam surface $kB = 0$. Recently, Torricelli-Ciamponi & Petrini (1990) and Appl & Camenzind (1992) investigated numerically instabilities of nonrelativistic
MHD cylindrical flows with nonzero poloidal magnetic field and poloidal current distributed over the jet cross section.

Next in order of complexity to relativistic hydrodynamic and nonrelativistic MHD approaches is relativistic MHD case, which is the generalization and unification both of them. When considering relativistic models with a significant magnetic field it is necessary to involve the electric force and, for non-stationary case, to add the displacement current. Indeed, at the speed of the hydrodynamic flow $v \simeq c$ the electric field in plasma with high conductivity is of the order of the magnetic one $E = -v \times B/c$. The charge density $\varrho$ is equal to $\nabla \cdot E/4\pi \simeq E/L$ and the current density $j$ in a stationary flow or for small nonstationarity is $j = c \cdot (\nabla \times B)/4\pi \simeq cB/L$. We see that the ratio of the electric force, $\varrho E$, acting on the unit volume, to the magnetic force, $j \times B/c$, is of the order of $E^2/B^2 \simeq v^2/c^2 \simeq 1$ for the relativistic case. So it would be, probably, better to use the word electro-magnetohydrodynamics instead of magnetohydrodynamics for relativistic flows. Many extragalactic and, possibly, stellar jets (see recent discovery of superluminal motions by Mirabel & Rodriguez, 1994) appear to be relativistic and magnetized, so there are numerous papers devoted to the construction of selfconsistent description of such flows (Camenzind 1987; Lovelace, Wang & Sulkanen 1987; Li, Chiueh & Begelman 1992; Contopoulos 1994). It is of interest to examine the stability of the solutions obtained.

We started this investigation with the simplest case where the energy density of the electromagnetic fields is much greater than the energy density of the matter (including the rest mass energy). This is the force-free approximation. This case is reversal to the pure hydrodynamics one when the electromagnetic field is absent. Using force-free approximation one can hope to take into account the influence of the electrodynamic effects on the stability and simultaneously obtain substantially simplified problem. The terms in the momentum equation which are proportional to the mass and the pressure of liquid, are therefore small compared to the electromagnetic force $\varrho E + j \times B/c$, so it is possible to set $\varrho E + j \times B/c = 0$ (we use units where $c = 1$). The force-free approximation together with the ideal hydrodynamics approximation (which means an infinite conductivity of plasma and consequently the absence of the electric field in the frame moving with the element of the medium) can be applied to the neighbourhood of the massive black hole, which is thought of as the central engine of active galactic nuclei. Such an approach was developed by Blandford & Znajek (1977) and Macdonald (1984) [see also chapter IV of the book "Black Hole: The Membrane Paradigm" by Thorne, Price & Macdonald (1986) and chapter VII of the book "Physics of Black Hole" by Novikov & Frolov (1986) and references therein].

In our previous paper (Istomin & Pariev 1994, thereafter referred to as IP) we consider the stability of a force-free jet with respect to axisymmetric perturbation. Present paper is the continuation of IP and deals with nonaxisymmetric perturbation. As in IP the equilibrium configuration of the jet has cylindrical symmetry. This means that all quantities describing the jet depend on the distance from the jet axis $r$ and do not depend on the coordinate along the jet $z$ and rotational angle $\phi$. The boundary of the jet has the shape of a cylinder. We suggest that the jet propagates in the medium which density is greater than that of the jet but temperature and pressure are small, so the condition of impenetrability is fulfilled and the boundary is at rest. The poloidal magnetic field $B_z$ is assumed to be uniform and parallel to the jet axis. The fluid moves along spirals because of the radial electric field. It has been proved analytically (IP) that under such conditions the relativistic flow is stable for axisymmetric modes ($m = 0$, where $m$ is azimuthal wavenumber). In section 2 we derive equations governing the problem, describe the procedure of finding eigenfrequencies based upon the Laplace transformation method, and discuss asymptotic behaviour of perturbations in a long time after initial excitation. In section 3 we present the results of numerical calculations. In section 4 we perform boundary layer analysis of our equation near the point $r_A$ in the complex plane $r$, which is the point of the coincidence of two Alfvén resonant points. Possible astrophysical implications of the results obtained is discussed in section 5.

## 2 Stability problem

Let us consider a flow of liquid in a force-free cylindrical jet. We use the units in which $c = 1$. The condition of an ideal flow is

$$E = -v \times B,$$

where $v$ is the plasma velocity. The force-free approximation is guided by the relation

$$\varrho E + j \times B = 0. \tag{2}$$

First we will review the stationary configuration of the jet described in IP. In this case $\nabla \times E = 0$ and the velocity $v$ can be written as

$$v = \kappa B + \Omega F r \hat{e}_\phi. \tag{3}$$
Here and below \( r, z \) and \( \phi \) are cylindrical coordinates, \( e_r, e_z \) and \( e_\phi \) are the unit vectors in the cylindrical coordinate frame, \( K = K(r) \), \( \Omega^F = \Omega^F(r) \). Then

\[
E = -\Omega^F r (e_\phi \times B)
\]

and \( \Omega^F \) can be treated as the angular rotation velocity of magnetic field lines (Thorne et al. 1986). In the cylindrical configuration \( B = B_z(r) e_z + B_\phi(r) e_\phi \), so using Maxwell equations, we obtain from (2)

\[
E(\nabla \cdot E) - B \times (\nabla \times B) = 0.
\]

According to (4) the only non–zero component of equation (5) is \( r \)–component. This implies

\[
\Omega^F B_z \frac{d}{dr} (\Omega^F r^2 B_z) = B_z \frac{dB_z}{dr} + \frac{1}{r} B_\phi \frac{d}{dr} (r B_\phi).
\]

This equation governs the force balance in radial direction and defines all possible solutions for the force-free electromagnetic fields in cylindrical configuration of the magnetic tubes. General solution of this equation was described in IP. For uniform poloidal magnetic field \( B_z = \text{const} \), we have

\[
B_\phi = \pm \Omega^F r B_z.
\]

If \( B_z \) is constant the stationary magnetic field structure is entirely determined by the function \( \Omega^F(r) \).

The case where the total charge of the jet is equal to zero is probably the most natural. If the jet has a charge not equal to zero, the electric field penetrates into the surrounding medium. This results in charge motion in the plasma and in a decrease of the charge of the jet. To avoid the problem of closing current loop somewhere outside the jet it is naturally to demand the total poloidal current through the jet to be equal to zero. These two requirements lead to \( \Omega^F(R) = 0 \), where \( r = R \) is the jet boundary (IP). The equilibrium stationary configuration of the jet is shown in Fig. 1, which is extracted from IP. We reproduce it here for illustration purpose.

### 2.1 Basic equations

We perform linear stability analysis using common method of small perturbations. In the subsequent formulae, the values referring to nonperturbed solution will be denoted by the subscript ‘0’, while ones referring to perturbation - by the subscript ‘1’. We will consider only nonaxysimmetric perturbations, so throughout the rest of the paper it is assumed \( m \neq 0 \) unless specified directly.

After removing the quantities \( \varrho, E, j \) from the initial system of Maxwell equations and equations (1)–(2) there remains only three resultant equations involving \( B \) and \( v \) only:

\[
\begin{align*}
\nabla \cdot B &= 0, \quad (8) \\
\frac{\partial B}{\partial t} &= \nabla \times (v \times B), \quad (9) \\
(\nabla \times B) \nabla \cdot (v \times B) - B \times (\nabla \times B) - B \times \\
&= \frac{\partial}{\partial t} (v \times B) = 0. \quad (10)
\end{align*}
\]

The quantities \( B \) and \( v \) can be represented as \( B = B_0 + B_1, v = v_0 + v_1 \). We consider the perturbations in the form

\[
\begin{align*}
B_1 &= b_1(r) \cdot \exp(-i\omega t + ikz + im\phi), \\
v_1 &= a_1(r) \cdot \exp(-i\omega t + ikz + im\phi),
\end{align*}
\]

where \( m \) is an integer. Substitution this expressions into linearized set of equations (8)–(10) and removal the components of the perturbed quantities with subscript ‘1’ in favour of \( B_{r1} \) lead us to the following second-order ordinary differential equation of the variable \( B_{r1} \), where the prime denotes differentiation with respect to \( r \):

\[
\begin{align*}
B_{r1}'' + B_{r1}' &\left(1 - \frac{2m^2}{r^3} \frac{1}{\omega^2 - k^2 - m^2/r^2} - \frac{m}{r^2} \frac{d}{dr} \frac{1}{k + m\Omega^F} \right) + B_{r1} \left[\frac{\omega^2 - k^2 - m^2}{r^2} \right] \\
&\quad m(k + m\Omega^F) \frac{d}{dr} \left( \frac{1}{(k + m\Omega^F)^2} \frac{d}{dr} \Omega^F \right) + \frac{1}{k - \omega + 2m\Omega^F - \Omega^F \omega} \times \\
(A_1 B_{r1}' + A_2 B_{r1}) &= 0. \quad (12)
\end{align*}
\]
The quantities $A_1$ and $A_2$ are

$$A_1 = -2\Omega^2 r(\omega + k) + \frac{d\Omega^F}{dr} \frac{\omega + k}{k + m\Omega^F} (m - m\Omega^F r^2 - 2kr^2\Omega^F),$$

(13)

$$A_2 = 2m \left( \frac{d\Omega^F}{dr} \frac{2m}{k + m\Omega^F} \right)^2 \frac{\Omega^F r^2(\omega + k) - m}{k + m\Omega^F} + \frac{d\Omega^F}{dr} \frac{1}{k + m\Omega^F} \left\{ 2m \frac{1}{r} \frac{1}{\omega - k^2 - m^2/r^2} \times \right.$$  

$$\times \left[ \frac{m^2}{r^2} (3m\Omega^F - \omega + k) - \Omega^F r^2(\omega^2 - k^2) \left( \Omega^F(\omega + k) + km \right) \right] + \frac{m}{r} (\omega - k) + \left\} + \frac{2m}{r^2} \Omega^F + \frac{1}{r^2}(\omega - k) + 3\Omega^2 (\omega + k)$$

$$+ 2m^2 \frac{1}{\omega + k} + \frac{2m^2}{r^2} \frac{1}{\omega^2 - k^2 - m^2/r^2} \left( \frac{m}{r^2(\omega + k)} - \Omega^2 (\omega + k) \right).$$

(14)

This equation has been obtained in IP. When deriving this equation it was accepted that $B_z = 0$ and $k$ of $\omega$ the conductivity of the outside medium, since there are no perturbations in that region. Equation (2), which is essentially the Euler momentum equation, the terms describing the contribution from the equation (12) with definitions (13) and (14) unchanged under reversal of the signs of $B_0$ and $k$. For definiteness we adopt the sign ‘+’ in (7) and consider arbitrary values of $k$. All other perturbed physical quantities can be readily expressed through $B_{r1}$. The expressions for some of them are given in the Appendix A. $B_{r1}(r)$ must fulfill the boundary conditions for $r = 0$ and $r = R$. For $r = 0$, $B_{r1}$ must be regular. If $|m| \neq 1$ this condition can be strengthened to become $B_{r1}|r=0 = 0$. The boundary condition for $r = R$ must be derived from the rigidity of the jet wall $v_{r1}|r=R = 0$ which has been assumed in the present investigation. It gives $B_{r1}|r=R = 0$. Thus, in order to find perturbed modes, we have to solve the edge problem for equation (12) with the boundary conditions

$$B_{r1}|r=0 = \text{regular},$$

$$B_{r1}|r=R = 0.$$  

(15)

As a result of calculation parameter $K(r)$, which determines the component of $v_0$ parallel to $B_0$, drops out from the equation (12). Therefore, the results of our investigation of stability do not depend on the values and profiles of the longitudinal velocity. The physical reason for that is the neglecting in the force–balance equation (2), which is essentially the Euler momentum equation, the terms describing the contribution from the inertia of the mass flow and the pressure. The results of the stability investigation do not depend also on the conductivity of the outside medium, since there are no perturbations in that region.

The primer goal of our investigation is to answer the question whether the jet is stable or not. So it is more convenient to use temporal approach for investigating stability, i.e. seek for complex values of $\omega$ for real $k$. The existence of only one eigenvalue $\omega$ with positive imaginary part would mean that the jet configuration is generally unstable. This is not the case for spatial approach, when the search is carried out for complex values of $k$ for real $\omega$. In this case the sign of imaginary part of $k$ is irrelevant for stability and more complicated analysis should be done to answer the question of stability (Lifshitz & Pitaevskii 1979). We adhere here to temporal approach.

The radial eigenvalue problem (12) has regular singularities in the following 4 cases:

1. $k + m\Omega^F = 0$ or $kB_0 = 0$. This is the resonant surface $r = r_c$. For nonrelativistic MHD consideration there exist local Suydam modes in the vicinity of $r_c$ (BIB). However this is not the case for the force-free approximation (we see it below). Expanding equation (12) to lowest order in $x = r - r_c$, substituting $B_{r1} \propto x^\nu$ we find the indicial equation $(\nu - 1)(\nu - 2) = 0$. This implies that the solution near $r = r_c$ is the linear combination of two linear independent solutions, $W_1(x)$ and $W_2(x)$, with some constants $k_1$ and $k_2$, $B_{r1} = k_1 W_1(x) + k_2 W_2(x)$. The first terms in expansion of these solutions are $W_1(x) = a_1 x^2 + a_2 x^3 + \ldots$, $W_2(x) = AW_1(x) \log x + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$. It can be shown directly by inserting the expression for $W_2(x)$ into equation (12) that $A = 0$. Therefore, this singular point does not lead to nonanalytic solutions in it’s neighbourhood and is only apparent.

2. $\omega^2 - k^2 - m^2/r^2 = 0$. This point can be interpreted as the resonance of the perturbation, having $k$ parallel to magnetic flux surfaces of $B_0$, with the fast magnetosonic wave which in the force-free approximation always has the speed equal to the speed of light. Indicial equation is $\nu(\nu - 2) = 0$. Similar to the type 1 singular point it can be shown that the coefficient of $\log x$ in the expansion of $B_{r1}$ near resonance vanishes, therefore this singular point is only apparent too.
3. $k - \omega + 2m\Omega^F - \Omega F^2 r^2 (\omega + k) = 0$. This is the resonance of the perturbation with the relativistic Alfvén wave $r = r_A$. When l.h.s. has a simple zero, the indicial equation will be $\nu^2 = 0$ and, in general, $B_{r1}$ has logarithmic singularity there. The expansion of other physical quantities near $r = r_A$ is given in the Appendix A.

4. $r = 0$. This is the singularity produced by the coordinate origin. Two linear independent solutions near $r = 0$ are $B_{r1} \propto r^{|m|-1}$ and $B_{r1} \propto r^{-|m|-1}$, where $m \neq 0$. In order to meet the boundary condition (15) one should choose the former solution only.

One can solve edge problem for equation (12) directly as it was done in IP for the case $m = 0$. However, the case $m \neq 0$ is more complex, particularly, because of the existence of the singularities in the coefficient of equation (12) mentioned above, two of them being only apparent. The numerical treatment of equation (12) can be greatly simplified and integration through 1-st and 2-nd type singular points can be avoided if we rewrite equation (12) as a system of the two first order differential equations in terms of the radial displacement $\xi$, and the disturbance of the total pressure $P_1$ instead of the radial component of the magnetic field perturbation $B_{r1}$. We need transformation into variables $\xi$, and $P_1$ to consider initial value problem in subsequent subsection and also in order to possible further generalization of the problem to include inertia of plasma. For the Lagrangian displacement (i.e. the displacement of a fluid element moving with the equilibrium flow) $\xi(r, t)$ one has well known expression relating it to the Eulerian disturbance of the velocity field $v_1$

$$\frac{\partial \xi}{\partial t} = v_1 + (\xi \nabla)v_0 - (v_0 \nabla)\xi. \quad (16)$$

The $r$-component of this relation is (bearing in mind the representation (11))

$$-i\omega \xi_r = v_r - ik v_z \xi_r - i \frac{m}{r} v_\theta \xi_r. \quad (17)$$

The $r$-component of the linearized induction equation (9) is

$$(-\omega + kv_z + \frac{m}{r} v_\theta) B_{r1} = \left( k B_z \frac{\partial}{\partial r} + \frac{m}{r} B_\theta \right) v_r. \quad (18)$$

Combining equations (17) and (18) one can readily obtain that $B_{r1} = i (kB_z + m/r B_\theta) \xi$, which in the force–free case with $B_z = \text{const}$ transforms to

$$B_{r1} = i \xi_r B_z (k + m \Omega^F). \quad (19)$$

We call the value $P_0 = \frac{1}{8\pi} \left( B_0^2 - E_0^2 \right)$ as a total pressure because this is the diagonal part of the electromagnetic stress tensor $\sigma_{ij} = \frac{1}{8\pi} \left( B_0^2 - E_0^2 \right) \delta_{ij} - \frac{1}{4\pi} (B_0 B_j - E_0 E_j)$. Next the perturbation of the total pressure will be

$$P_1 = \frac{1}{4\pi} \left( B_{z1} + B_\theta B_\phi - E_{r0} E_{r1} \right), \quad (20)$$

where the radial component of the perturbation of the electric field is $E_{r1} = -(B_z v_\phi - B_\phi v_z + v_\phi B_z - v_z B_\phi)$ and radially directed equilibrium electric field is $E_{r0} = v_\phi B_\theta - v_\theta B_z = -\Omega F r B_z$. Using formulas (A1)–(A3) from the Appendix A, which express $B_\phi$, $B_z$ and $E_r$ by means of $B_{r1}$ and $dB_{r1}/dr$, we find for $P_1$ expression by means of $B_{r1}$ and $dB_{r1}/dr$

$$4\pi P_1 = i B_z \frac{\omega + k}{S} \left[ \frac{A}{F} \frac{dB_{r1}}{dr} + B_{r1} \left( m \frac{d\Omega F}{dr} A \frac{dB_{r1}}{dr} + \frac{1}{rF} \left( \omega - k - \Omega F^2 r^2 (\omega + k) \right) \right) \right], \quad (21)$$

where

$$F = k + m \Omega^F, \quad (22)$$

$$S = \omega^2 - k^2 - m^2 / r^2, \quad (23)$$

$$A = k - \omega + 2m \Omega^F - \Omega F^2 r^2 (\omega + k). \quad (24)$$

Those values of $r$, which make $F$, $S$ or $A$ equal to zero, are 1-st, 2-nd or 3-d type singular points respectively. From (21) using (19) $dB_{r1}/dr$ is expressed by means of $P_1$ and $\xi_r$

$$\frac{dB_{r1}}{dr} = 4\pi P_1 \frac{iFS}{(\omega + k)B_z A} + i \left( \frac{F}{A} \left( \omega - k - \Omega F^2 r^2 (\omega + k) \right) + mr \frac{d\Omega F}{dr} \right) B_z \xi_r. \quad (25)$$
Differentiation of the equation (19) and substitution the result for $dB_{r_1}/dr$ in (25) lead after some cancellation to the first equation from the couple desired

$$\frac{d\xi_r}{dr} = \frac{4\pi P_1}{B_{z0}^2} \frac{S}{A(\omega + k)} + \xi_r \frac{1}{rA} \left( \omega - k - \Omega F^2 r^2 (\omega + k) \right).$$

(26)

To obtain the second equation, which relates $dP_1/dr$ to the linear combination of $P_1$ and $\xi_1$, we differentiate equation (21) on $r$, substitute for appeared in r.h.s. of (21) $d^2B_{r_1}/dr^2$ it’s value from the second order differential equation (12), and remove $dB_{r_1}/dr$ and $B_{r_1}$ in favour of $P_1$ and $\xi_r$ by means of (25), (19). Numerous cancellations occurred when this procedure was carrying on. Finally, we get the following equation

$$4\pi \frac{dP_1}{dr} = \frac{2\Omega F}{A} \left( \Omega F r(\omega + k) - \frac{m}{r} \right) 4\pi P_1 - (\omega + k) \left( A - \frac{4\Omega F^2}{A} \right) B_{z0}^2 \xi_r.$$

(27)

We introduce for convenience the dimensionless pressure disturbance $p_*= 4\pi P_1/B_{z0}^2$. Than pair of equations (26) and (27) can be rewritten as follows

$$A \frac{1}{r} \frac{d}{dr} (r\xi_r) = C_1\xi_r - C_2 p_*,
A \frac{dp_*}{dr} = C_3\xi_r - C_1 p_*,$$

(28)

where

$$C_1 = \frac{2}{r} \left( m\Omega F - \Omega F^2 r^2 (\omega + k) \right),$$

(29)

$$C_2 = -\frac{\omega^2 - k^2 - m^2/r^2}{\omega + k},$$

(30)

$$C_3 = -(\omega + k)(A^2 - 4\Omega F^2).$$

(31)

The system (28) has the same form as derived by Appert, Gruber & Vaclavik (1974) for nonrelativistic MHD stability investigation of the static plasma cylinder. The only difference is in the coefficients $A$, $C_1$, $C_2$ and $C_3$. The presence of the equilibrium flow does not influence the form of the set of the equations, but only modify the coefficients (e.g. BIB). Now we see that for relativistic force-free flows this result remains to be true. It is easier to integrate numerically set of equations (28) than the second order equation (12) and in our numerical computations we actually integrated (28). First, one should note that no derivatives of $\Omega F(r)$ enter into the (28). Second, which is more important, the only singularity in (28) is $A = 0$. The values of $r$ which makes $S = 0$ or $F = 0$ are regular points of the system (28). Because of the relation (19) it is clear that they will be regular points of the second order equation (12) on $B_{r_1}$ as well. This result is just that we have obtained above by considering explicit expansion $B_{r_1}$ in the neighbourhood of singular points.

Equations (28) can be readily converted into one second order, so called Hain–Lüst (Hain & Lüst, 1958) equation on $r\xi_r$

$$\frac{d}{dr} \left( \frac{A}{C_2} \frac{1}{r} \frac{d}{dr} (r\xi_r) \right) - r\xi_r \left[ \frac{d}{dr} \left( \frac{1}{C_2} \frac{C_1}{r} \right) + \frac{C_5 - C_2 C_4}{r C_2} \right] = 0,$$

(32)

where we made notice of a certain factorization

$$C_1^2 - C_2 C_3 = A(C_5 - C_2 C_4).$$

(33)

The $C_4$ and $C_5$ are

$$C_4 = -A(\omega + k),$$

(34)

$$C_5 = -4\Omega F^2 (\omega + k).$$

(35)

Equations (32) and (33) are identical in form to that derived in BIB for nonrelativistic MHD. As well as in equation (12) in $B_{r_1}$ the zero $C_2 = 0$ in (32) is only apparent singular point, $\xi_r$ is regular at this point. If the factorization (33) did not occur, equation (32) would have essential singularities when $A$ has a zero of quadratic or higher order.
2.2 Laplace transformation method

Now edge problem (12), (15) is reformulated for system (28) or equation (32) with evident boundary conditions: $\xi_r$ is finite at $r = 0$ and $\xi_r = 0$ at $r = R$. Thereafter we shall use dimensionless values $r' = r/R$, $\omega' = \omega R$, $\Omega^{F'} = \Omega^F R$, $k' = k R$ and the prime will be omitted. Thus, $r = 1$ will correspond to the jet boundary. When integrating equations (28) the problem arises how to treat the singularity $A = 0$. To answer this question it is necessary to remember that we use temporal approach, i.e. solving initial value problem. We seek for the solution $\xi_r(t, r)$ for all $t > 0$ having given the initial conditions, say, $v_1(0, r)$ and $B_1(0, r)$. Because of the all equations governing the problem is linear on perturbations and the explicit dependence of this equations from $\phi$ and $z$ is absent, it is enough to consider the behaviour of only one Fourier component of all disturbances with respect to $\phi$ and $z$. As well as in previous text we shall take the perturbation and initial conditions having the form $\xi_{rkm} \propto \exp(ikz + im\phi)$ and shall omit the subscripts $m$ and $k$ belonging to all the disturbances and initial conditions. Instead of doing Fourier transformation with respect to $t$ it is useful to apply the technique of Laplace transformation. We introduce function $\xi_{r\omega}(r)$ determined by

$$\xi_{r\omega}(r) = \int_0^\infty \xi_r(t, r)e^{i\omega t} dt.$$  

Then the reverse Laplace transformation is

$$\xi_r(t, r) = \int_{-\infty + i\sigma}^{+\infty + i\sigma} \xi_{r\omega}(r)e^{-i\omega t} \frac{d\omega}{2\pi}, \quad (36)$$

where the integration is performed along the line in the upper half of the complex plane $\omega$, which is parallel to the real axis and goes above all irregular points of $\xi_{r\omega}$. For the sake of resemblance to the Fourier transformation we use here the frequency $\omega$ related to the usual parameter $p$ of the Laplace transformation by $\omega = ip$. The Laplace transformation of other perturbed values is defined in the same manner. Carrying out the Laplace transformation of the equation (16) and the source equations of the problem (8)–(10), eliminating all disturbances in favour of $\xi_{r\omega}$, $v_1(0, r)$ and $B_1(0, r)$ we get finally the equation (32) with some r.h.s., which depends on the initial conditions.

Note, that we need to add to $v_1(0, r)$ and $B_1(0, r)$ the initial condition for the displacement $\xi(0, r)$, when processing with the (16). Contrary to that, if we did not introduce displacement and did restrict ourself to Laplace analogue of equation (12) in $B_{r1}$, the given $v_1(0, r)$ and $B_1(0, r)$ would perfectly determine the unique solution for $v_1(t)$ and $B_1(t)$. Thus, instead of (28), the equations for $\xi_{r\omega}$ and $p_{s\omega}$ will be

$$A \frac{1}{r} \frac{d}{dr}(r \xi_{r\omega}) = C_1 \xi_{r\omega} - C_2 p_{s\omega} + D_1,$$

$$A \frac{d}{dr} p_{s\omega} = C_3 \xi_{r\omega} - C_1 p_{s\omega} + D_2, \quad (37)$$

where $D_1$ and $D_2$ are linearly dependent on the initial conditions and do not contain any denominators except $\omega + k$ and Doppler shifted frequency $-\sigma = \omega - kv_0 z - \frac{m}{r} v_0 = -\omega - m\Omega^{F'} - KB_{0z}(k + m\Omega^{F'})$. Further, we derive second order differential equation on $r \xi_{r\omega}$ similar to (32) but with some nonzero r.h.s. Reduced to the normal form it will be

$$\frac{d^2}{dr^2}(r \xi_{r\omega}) + \frac{r C_2}{A} \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr}(r \xi_{r\omega}) \right) - r \xi_{r\omega} \left[ \frac{r C_2}{A} \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \left( \frac{1}{r} C_1 \right) \right) \right] + \frac{C_5 - C_2 C_4}{A} = 0.$$  

$$r \frac{d}{dr} \left( \frac{1}{C_2} \frac{d}{dr} \left( \frac{D_1}{C_2} \right) \right) + \frac{r}{A^2} (C_1 D_1 - C_2 D_2). \quad (38)$$

To solve edge problem for (38) with boundary conditions $r \xi_{r\omega}|_{r=0} \propto r^{-|m|}$, $r \xi_{r\omega}|_{r=1} = 0$ we do the following. First, integrate equation (38) (or, which is equivalently, system (37)) from $r = 0$ to $r = 1$ with the initial condition $r \xi_{r\omega}|_{r=0} = r^{-|m|}$. Thus, we obtain the solution $g_0(r, \omega, k, m)$ of nonuniform equation (38), which, in general, does not satisfy the boundary condition at $r = 1$. Then we integrate uniform equation (32) starting from $r = 0$ with the same initial condition $r \xi_{r\omega}|_{r=0} = r^{-|m|}$. As a result we obtain some solution $g_1(r, \omega, k, m)$ of (32). If $g_1$ is equal to 0 at $r = 1$, than this will be an eigenfunction of (32) with the frequency $\omega$ being the eigenfrequency of the problem $\omega_{nm}(k)$. The integer number $n$ counts different eigenfrequencies with the same $m$ and $k$. We assume the muchness of eigenfrequencies to be countable. This corresponds to the discrete spectrum. Usually there exists an infinite sequence of $\omega_{nm}$ for given $m$ and $k$. This sequence has either finite
or infinite accumulation point in $\omega$-plane or both (see numerical results below and section 3). The numbers $n$ can be chosen so that the higher $n$ the more oscillations has the corresponding radial eigenfunction. The least oscillating mode is referred to as a fundamental one in literature and the number $n=0$ is usually ascribed to this mode (Birkinshaw 1984, Payne & Cohn 1985, Appl & Camenzind 1992). All other modes are known as reflection.

In the case $\omega = \omega_{nm}(k)$ the edge problem for the equation (38) has no solutions unless $g_0(1, \omega, k, m) = 0$. If $g_0(0, \omega_{nm}, k, m) = 0$ than the number of the solutions would be infinite, because of all functions $g_0 + cg_1$ with arbitrary constant $c$ would be equal to 0 at $r = 1$. Then for $\omega$ not being the eigenfrequency the edge problem for (38) has unique solution

$$\xi_{r\omega} = g_0(r, \omega, k, m) - \frac{g_0(1, \omega, k, m)}{g_1(1, \omega, k, m)}g_1(r, \omega, k, m).$$  

(39)

To find out $\xi_r(t, r)$ one needs to make inverse Laplace transformation (36). The asymptotic behaviour of $\xi_r(t, r)$ for $t \to +\infty$ is not known a priori, therefore one can be sure that the definition of $\xi_{r\omega}$ is meaningful only for $\omega$ having $\text{Im}\omega \to +\infty$. It means that the integration in (36) must be done along the contour having $\sigma \to +\infty$ (long dashed line in Fig. 2). In this region of complex $\omega$-plane $\xi_{r\omega}$ must be an analytical function in $\omega$ and its values can be found from (39) by applying the procedure described above. Then, the contour integral (36) provides us with the full solution of the initial value problem. To answer the question of stability it is enough to find only the asymptotic behaviour of $\xi_r(t, r)$ for $t \to +\infty$. For this purpose it is necessary to continue analytically the expression (39) into the whole complex $\omega$-plane. Let us fixed some value of $r = r_*(0 < r_* < 1)$ and consider the singularities of $\xi_{r\omega}$ as a function of complex $\omega$. We assume that the initial conditions $v_1(r), B_1(r)$, the angular rotation velocity of magnetic field lines $\Omega^F$, and, therefore, $D_1(r, \omega), D_2(r, \omega)$ are entire functions in the whole complex $r$-plane. In $\omega$-plane the only singularities of $D_1$ and $D_2$ are the poles $\omega = -k$ and $\omega = m\Omega^F + KB_0(k + m\Omega^F)$, while the coefficients in l.h.s. of the equation (38) remain to be regular at these points.

First of all, $\xi_{r\omega}$ will be irregular if $g_0(r_*, \omega)$ or $g_1(r_*, \omega)$ have singularities. It is seen from (37) that this can happens when $A(r_*, \omega) = 0$, i.e. when Alfvén resonant point coincides with chosen $r_*$, when $\omega = -k$, and when Doppler shifted frequency $\sigma$ is equal to 0 at the point $r_*$. Notice, that from look at the equation (38) one might believe $C_2(r_*, \omega) = 0$ to be the singularity of $g_0$. However, it does not because of this is not the singularity of the original system (37). From expression (24) for $A$ one can readily conclude that for any complex $\omega$ the solution $r_\omega(A(\omega)) = 0$ must be complex too. Therefore, $r_\omega$ can be real only for real values of $\omega$, and $g_1, g_0$ are regular in the whole upper complex $\omega$ half-plane for any real $r$. By expanding (38) near the point $r = r_\omega(\omega)$ one can deduce the following behaviour of $g_0$ and $g_1$ in the neighbourhood of the Alfvén resonance

$$g_0 = c_{10}\log x + c_{20} + c_{30}\log^2 x + c_{40}x + c_{50}x^2 + \ldots,$$

$$g_1 = c_{11}\log x + c_{21} + c_{31}x + c_{41}x^2 + \ldots,$$  

(40)

where $x = (r - r_\omega)/r_{\omega}$, and $c_{ij}(\omega)$ are constants with respect to $\omega$. We see that $r = r_\omega$ is the logarithmic type branch point of $g_1, g_0$, and consequently, $\xi_{r\omega}$, considered them as the functions in the complex $r$-plane. On the other hand, for fixed $r = r_*$ and $\omega$ close to the Alfvén resonant frequency $\omega_A(r_*)$ (that is $A(r_*, \omega_A(r_*)) = 0$) the value $x$ can be expressed as

$$x = \frac{r^\prime_A(\omega - \omega_A)}{r_*} + \frac{r^\prime_A}{r_*}\left(\frac{r^\prime_A}{2r_*} - \frac{r^\prime_A}{r_*}\right)(\omega - \omega_A)^2 + \ldots,$$  

(41)

where

$$r_A' = \frac{dr_A}{d\omega}\Big|_{\omega = \omega_A(r_*)}, \quad r_A'' = \frac{d^2r_A}{d\omega^2}\Big|_{\omega = \omega_A(r_*)}.$$  

Inserting (41) into expressions (40) one can readily see that $\omega = \omega_A(r_*)$ is the logarithmic type branch point for $g_0, g_1$, and, therefore, for $\xi_{r\omega}(r_*, \omega)$. From expression (24) for $A$ we find

$$\omega_A(r_*) = \frac{k + 2m\Omega^F(r_*) - \Omega^F(r_*)r^2k}{1 + \Omega^F(r_*)r^2}.$$  

(42)

This equation indicates that for any real $r_*$ function $\xi_{r\omega}$ has unique branch point due to the Alfvén resonance $\omega_A(r_*)$ and this point lies on the real axis in the complex $\omega$-plane. At $\omega = -k$ function $\xi_{r\omega}$ has a pole. The reason for this pole is the resonance of the perturbation with the Alfvén wave, propagating in the negative $z$-direction. In the case of uniform poloidal magnetic field $B_{z,0} = \text{const}$ this wave has always the velocity equal to the light velocity irrelevant to the value of $r_*$, and to the curling angle of the magnetic field lines $\Omega^F(r_*)$ (see Appendix B). In the case of $B_{z,0} \neq \text{const}$ Alfvén waves propagating in both directions of $z$-axis become equal, both their
velocities do depend on \( r_\ast \) and the second Alfvén resonance point \( \omega_A(r_\ast) \) appears. It’s analytical properties are the same as for the first Alfvén resonance point: it is a logarithmic type branch point for \( g_0, g_1 \), and, therefore, for \( \xi_{r\omega}(r_\ast, \omega) \), with the same expansion of \( g_0, g_1 \) near it as the expansion (40). It can be shown that in the general case of nonuniform poloidal magnetic field \( B_{r0} \) both Alfvén resonant frequencies \( \omega_A(r_\ast) \) are always real for real values of \( r_\ast \) (see Appendix B). The singularity \( \sigma(r_\ast, \omega) = 0 \) or \( \omega = \omega_J(r_\ast) = m\Omega^F + KB_0(k + m\Omega^F) \) is also real for any real \( r_\ast \). It corresponds to the convection of the displacement \( \xi \) with the fluid velocity \( v_0 \).

To perform the analytic continuation we must deform therefore the path of integration in \( r \) of the system (37) never become real valued. Hence, the analytical continuation of \( \xi_{r\omega}(r_\ast) \) into the whole upper \( \omega \) half–plane (except poles \( \omega = \omega_{nm} \)) is provided by the formula (39) without any changes, i.e. when obtaining \( g_1 \) and \( g_0 \) integration of the system (37) should be performed along the real axis in the complex \( r \)-plane from \( r = 0 \) to \( r = 1 \). This is, however, not the case for further continuation of \( \xi_{r\omega}(r_\ast) \) into the lower \( \omega \) half–plane. For \( \Im \omega = 0 \) the \( r_A(\omega) \) cuts the real \( r \) axis and moves for \( \Im \omega < 0 \) into the opposite complex \( r \) half–plane. Note, that form expression (42) it follows that the real valued frequency always exists, the crossing point lying between 0 and 1. We want to stress that in generally not all the points \( r_A(\omega) \) cut the real \( r \) axis when \( \Im \omega \) becomes 0, some of them may always lie in the upper \( r \) half–plane, another — in the lower half–plane. But if it occurs that \( r_A(\omega) \) is real than the frequency \( \omega \), at which it does, is real too.

To perform the analytic continuation we must deform therefore the path of integration in \( r \) of the system (37) into a contour extending into the complex \( r \)-plane and going around the point \( r_A(\omega) \). If \( r_A(\omega) \) cuts the real \( r \) axis beyond the interval \( (0, 1) \) then integration path is not deformed. We also need to go around the points \( r_J(\omega) \) when calculating \( g_0 \). After determining the integration path in such a way we calculate \( \xi_{r\omega}(r_\ast) \) according to formula (39), where now the functions \( g_0 \) and \( g_1 \) are the integrals of the (37) (or, equivalently, second order equation (38)) along deformed path.

The procedure of analytical continuation is illustrated by Fig. 2. Initially integration in the formula of reverse Laplace transformation (36) is performed along long dashed line. Function \( \xi_{r\omega} \) is continued along the paths indicated by short dashed lines. Letters \( A, B, C \) and \( D \) denote the paths corresponding to 4 possible cases of the location of their intersection points with real \( r \) axis. Corresponding locations of Alfvén resonant points
The procedure of analytical continuation described does not give unique results for the lower \( \omega \) half-plane. The value of \( \xi_{\omega}(r_s) \) depends on the path in \( \omega \)-plane along which the continuation is performed. Indeed, we may construct the solution \( \xi_{\omega} \) given by the formula (39) not only for real values \( r \) but also for complex one. For this purpose when obtaining \( g_0(r) \) and \( g_1(r) \) it is necessary to perform integration along the contour in the complex \( r \)-plane, which connects three points: 0, the point \( r_s \), at which we are interested in the \( \xi_{\omega} \), and 1. By this way we obtain an analytical continuation of \( \xi_{\omega}(r) \) into the complex \( r \)-plane. The only singularities of such defined \( \xi_{\omega}(r) \) are logarithmic type branch points \( r = r_A(\omega) \) and \( r = r_f(\omega) \). Hence, for analytical continuation of \( \xi_{\omega}(r) \) to be unique, one should do branch cuts attached to the points \( r_A(\omega) \) and \( r_f(\omega) \). Once chosen when \( \text{Im} \omega \to 0 \), these branch cuts are drawn with the points \( r_A(\omega) \), \( r_f(\omega) \) when analytical continuation of \( \xi_{\omega}(r) \) is performed in \( \omega \). Contour of integration of equations (37) in \( r \)-plane must never cross them. These branch cuts are shown on the Fig. 3. If we find only \( g_1 \) (which is enough for determining eigenfrequencies \( \omega_{nm} \)), then we shall not pay attention to the points \( r_f(\omega) \), because they are only in the r.h.s. of equation (38). It is seen that the point \( r_s \) falls on different sides of the branch cut when \( \omega \) is changed along the paths \( B \) and \( C \). Hence, if we continue \( \xi_{\omega}(r_s) \) along the path \( C \) in \( \omega \)-plane, we will obtain one value, while along the path \( B \) the other. Continuations along paths \( B \) and \( A \) give different results also. The reason for this is that in the case \( B \) the contour of integration goes around the singular point \( r = r_A(\omega) \), while in the case \( A \) it does not, therefore, we obtain different values of \( g_0 \) and \( g_1 \) at the point \( r = 1 \) needed to be known in formula (39). Because of the same reason the results of the continuation of \( \xi_{\omega}(r_A) \) along the paths \( D \) and \( C \) are also different (but they are the same for paths \( D \) and \( A \)). Thus, for the sake of \( \xi_{\omega} \) being well determined in the lower \( \omega \) half-plane we need to cut it and keep the continuation paths from crossing the cuts. The cutting and choosing the continuation paths may be done in different ways. In our computations we choose the paths to be straight vertical lines in the \( \omega \)-plane, i.e. along the path \( \text{Im} \omega \) runs from \( +\infty \) to desired value, while \( \text{Re} \omega \) remains fixed. The branch cuts are also straight vertical lines going down to infinite negative \( \text{Im} \omega \). Our choice is reflected in Fig. 2.

Such determined \( \xi_{\omega} \) may have another singularities in the lower \( \omega \) half-plane. First of all, there can exist the poles at the discrete eigenfrequencies of the stability problem \( \omega = \omega_{nm} \) with \( \text{Im} \omega_{nm} \leq 0 \). Secondly, the singularity may arise at some \( \omega = \omega_c \) when the two points \( r_A(\omega) \) merge together. In this case contour of integration in \( r \)-plane becomes clutched between them and the integration procedure will be undetermined. If we consider two paths in \( \omega \)-plane with slightly different \( \text{Re} \omega \) such that \( \text{Re} \omega_1 < \text{Re} \omega_c < \text{Re} \omega_2 \) (paths \( C \) and \( E \) on Fig. 2) and plot corresponding trajectories of \( r_A(\omega) \) in \( r \)-plane (see Figs. 4 (c) and (e) for illustration) we find that certain "reconnection" of trajectories occurs near the point \( r_A(\omega_c) \). For \( \text{Im} \omega < \text{Im} \omega_c \) the contour of integration catch on different \( r_A(\omega) \) when they are moving away from merging point \( r_A(\omega_c) \) each on it’s own trajectory. Therefore, the values of \( g_0(1) \), \( g_1(1) \), \( \xi_{\omega}(r_s) \) for near \( \omega_1 \) and \( \omega_2 \) will in general strongly differ from each other, and \( \omega_c \) will be a singular point of \( \xi_{\omega} \) with the corresponding cut to be attached to (see Fig. 2). We shall see below that \( \omega_c \) is an accumulation point of the poles. It is necessary to stress that the singularity of \( \xi_{\omega} \) arises only if the contour of integration goes between the merging points \( r_A(\omega_c) \). If it does not go around no one of them or bypass both than \( \omega = \omega_c \) is a regular point. In particular, all merging points having \( \text{Im} \omega_c > 0 \) are regular. The singularity may arise only for those having \( \text{Im} \omega_c \leq 0 \).

2.4 Discrete spectrum and Alfvén continuum

Let us now consider the asymptotic behaviour of \( \xi_t(t, r) \) at \( t \to +\infty \). If \( \xi_{\omega}(r_s) \) is defined by an analytic continuation into the whole cut complex \( \omega \)-plane the contour of integration in (36) can be deformed down to lower values of \( \text{Im} \omega \) by going around all the poles of \( \xi_{\omega}(r_s) \) and alongside the branch cuts (see Fig. 2). The part of the integral along the closing line \( \text{Im} \omega \to -\infty \) vanishes for \( t > 0 \) and the integral in (36) will be equal to the sum of residues of the poles at \( \omega = \omega_{nm} \) and at \( \omega = -k \) and the contributions from both sides of each cut attached to the singular points \( \omega = \omega_b \) described above. Thus we obtain

\[
\xi_t(t, r) = i \sum_n \text{res}_{\omega = \omega_{nm}} (\xi_{\omega} e^{-i\omega t}) + i \text{res}_{\omega = -k} (\xi_{\omega} e^{-i\omega t}) + \frac{i}{2\pi} \sum_{\omega_b} \left[ \int_0^{+\infty} \xi_r(\omega_b - i\chi) e^{-x^2} d\chi - \int_0^{+\infty} \xi_r(\omega_b + i\chi) e^{-x^2} d\chi \right] e^{-i\omega_b t},
\]
where we have made the substitution $\omega - \omega_h = -i\chi$ and denoted the value of $\xi_{r\omega}$ on the left side of the branch cut as $\xi_{r\omega}^l$, while on the right side as $\xi_{r\omega}^r$. It is necessary to emphasize that the procedure of finding $\omega_{nm}$ having \( \text{Im} \omega_{nm} < 0 \) depends on the arbitrary choice of the branch cuts, i.e., if someone do another cutting (which corresponds to another bypassing procedure when integrating in $r$-plane, say, to go around $r_A(\omega)$ in the case A on Fig. 3), he will probably find out new eigenfrequencies and lose the old ones. This indicates that we are not able to decompose the integral in (36) into the sum (46) by the only way: with the cutting being changed, the contribution from poles may converted partially into the contribution from cuts and vice versa. Formula (46) is valid for all $t > 0$. Because of the factor $e^{-i\omega t}$ rapidly decaying with $t$, asymptotically for $t \to \infty$ the main contribution to the $\xi_r(t)$ will come from the poles having maximal imaginary part and from real valued $\omega_h$. Terms coming to expression (46) from the residues of the poles will be proportional to $\exp(-i\omega_{nm}t)$

$$\xi_r(t, r)|_{\text{discr}} = \sum_n E_{nm}(r, k)e^{-i\omega_{nm}(k)t}, \quad (47)$$

and corresponds to the discrete spectrum of $\omega$. From formula (39) and expansion (45) of $g_1(1, \omega)$ near the pole $\omega_{nm}$ it follows that the coefficients $E_{nm}(r)$ in (47) must be proportional to $g_1(r)$. Hence, $E_{nm}(r, k)$ are the eigenmodes for eigenfrequencies $\omega_{nm}(k)$. They are the solutions of the edge problem for uniform equation (32) with the integration along the contour bypassing Alfvén resonant points $r = r_A(\omega)$ in complex $r$-plane as described above. Second term in the expression (46) is proportional to $\exp(ikt)$, it’s radial profile is not an eigenmode and depends on the initial conditions. As it has been already mentioned, this term is the result of the degeneracy of one of the Alfvén resonant points arisen due to the $B_0 = \text{const}$. For real $r$ on the interval $(0, 1)$, which only are meaningful for physics, radial eigenmodes are continuous functions when $\text{Im} \omega_{nm} > 0$, have logarithmic type singularity at $r = r_A(\omega_{nm})$ when $\text{Im} \omega_{nm} = 0$, and have a discontinuity at the point $r = r_A(\text{Re} \omega_{nm})$ of intersection of the branch cut with real $r$ axis when $\text{Im} \omega_{nm} < 0$. In the last case the position of discontinuity depends on how we do the cutting of $r$-plane (which is related to the cutting of $\omega$-plane for each $0 < r < 1$), and is $r = r_A(\text{Re} \omega_{nm})$ only by our particular agreement on the cutting of $\omega$-plane. Radial eigenfunctions are well determined only for eigenfrequencies with nonnegative imaginary part, i.e. for unstable or neutrally stable modes. This is not surprising because, as has been already mentioned above, one can not even uniquely determine the whole multitude of stable eigenfrequencies $\omega_{nm}$. Such situation always takes place in the problem of the stability of force–free relativistic jet, because, as seen from (42), Alfvén resonant point lying in the interval $(0, 1)$ can be always found at some real frequency $\omega$ for any $k, m \neq 0$, and for any choice of the function $\Omega^F(r)$. All above treatment of the initial value problem involving Laplace transformation shows that this conclusion on eigenmodes have to be true for the stability problem of any ideal hydrodynamic flow with the equilibrium conditions depending only on one space variable (for the problem being reducible to the ordinary second order differential equation), whenever resonant surfaces of the perturbation with the flow or with the characteristic waves of the medium (sonic, Alfvén, slow magnetosonic) do exist for real $\omega$. Particularly, as pointed out in BIB, for cylindrical nonrelativistic MHD flow singularities with the Alfvén and slow magnetosonic waves (when $A = 0$ or $S = 0$ in the notations of BIB) exist only when $\omega$ is real. Moreover, these resonances are the logarithmic branch points for the solutions of radial differential equation. Therefore, all our consideration is directly applicable to that case, with the only correction for more complicated structure of the factor ahead of $d^2\xi_r/dr^2$ and $dp_r/dr$ in BIB equations (3) analogous to our equations (28), and, hence, more complicated structure of analytical continuation of $\xi_{r\omega}$ in the lower $\omega$ half–plane due to the enhanced number of possibilities for singular points to merge together. In all previous investigations known to us the authors either restricted themselves to finding only unstable modes in their numerical computations (Torricelli-Champoni & Petrini 1990, Appl & Camenzind 1992) or chose some particular uniform profiles for $v_0(r), B_0(r)$, equilibrium pressure $p_0(r)$ and density $\rho_0(r)$ such that the factor ahead of $d\xi_r/dr$ and $dp_r/dr$ became independent on $r$, and all resonances disappeared (Turland & Scheuer 1976; Blandford & Pringle 1976; Hardee 1979; Cohn 1983; Payne & Cohn 1985). Therefore, it could be possible to integrate radial equation along the real $r$ axis only, and in simple cases even obtain the dispersion relation $D(\omega, k, m) = 0$ expressed through well known mathematical functions. When finding stable modes it is necessary to bypass some of the singular points in the complex $r$-plane according to the procedure described above even if they are not on the real axis interval $(0, 1)$. Consider now the third term in (46) rising from integration along the branch cuts in $\omega$-plane. From (39), (40) and (41) it follows that the main terms of the $\xi_{r\omega}(\omega)$ expansion in $\omega - \omega_A$ in the vicinity of the branch point $\omega_A(r)$ are

$$\xi_{r\omega}(r) = c_{00}(\omega_A) \log^2 \left[ \frac{r'_{A}(r)}{r'_{A}(\omega - \omega_A(r))} \right] + \left( c_{10}(\omega_A) + \frac{g_0(1, \omega_A)}{g_1(1, \omega_A)} \right) \log \left[ \frac{r'_{A}(r)}{r'_{A}(\omega - \omega_A(r))} \right] + \ldots , \quad (48)$$

with the dropped terms giving nonzero contribution to (46) not greater than $(\omega - \omega_A) \log(\omega - \omega_A)$. Substitution
of (47) into (46) after performing calculations gives us two leading terms of $\xi_{rω}$ for $t \to \infty$

$$\xi_r(t, r) = \left( a_1(r) \log \left( \frac{r}{r_A(r)} \right) + a_2(r) \right) \frac{1}{t} e^{-iω_ω(r)t} + \ldots \quad (49)$$

At different points $r$ the perturbation has different frequencies $ω_A(r)$ and different phase velocities

$$v_{ph} = k \frac{ω_A(r)}{|k|^2}. \quad (50)$$

This corresponds to continuous spectrum of $ω$, since for fixed values $k$ and $m$ all frequencies $ω$, with $r_A(ω)$ lying between 0 and 1, are exited. As in the cases of the sheared noncompressible hydrodynamic flow under the presence of the critical surface (Timofeev 1970, for a review) and diffuse MHD pinch (Kadomtsev, 1988) the perturbation as a whole can be treated as slitted onto a number of localized, singular in $r$ perturbations propagating with the local Alfvén velocity $(50)$. The decaying of perturbation observed from (49) is due to the phase mixing of neighbouring localized perturbation when they propagate with different velocities $(50)$. Because for the flow considered in this paper continuous spectrum always present and is real the asymptotic for $t \to \infty$ behaviour of perturbations is determined either by an eigenfrequency with $\text{Re}ω_{nm} > 0$ (exponentially growing, unstable case) or by real eigenfrequencies if the former is absent (oscillations, neutrally stable case), or, at last, by continuous spectrum (49) (decaying with time, stable case) if there is no nonnegative eigenfrequencies.

3 Results of numerical computations

To find eigenfrequencies $ω_{nm}$ it is necessary to solve uniform equation (32). Even in the simplest case Ω$^k = \text{const}$ this equation already has too many singularities to be solved by the well known hypergeometric functions and we are led to either doing numerical computations or finding conditions and small parameters under which it can be simplified. The results of numerical computations are presented in this section.

In accordance with the properties of $ξ_{rω}$ described above we adopted the following procedure in numerical computations. We integrated in $r$ pair of equation (28) instead of (32) because they have simpler coefficients and do not contain additional singularity at $C_2 = 0$. Integration was based upon five order Dormand–Prince method. For solving eigenvalue problem we applied shooting technique. For every $k$, $m$ and chosen initial value of $ω$ we started from the point very close to $r = 0$ with the initial condition $ξ_r \propto r^{m|−1}$ and sought after the $ξ_{rω}| _{r=1}$ being equal to 0. If $\text{Im}ω > 0$ integration should be performed entirely along the real $r$ axis. For $\text{Im}ω$ negative, equal to zero or very small positive we passed round the zeroes of $A$ in the complex $r$-plane in accordance with the consideration in section 2: it is necessary to find the position of zeroes of $A$ in the $r$-plane for frequency $ω_{∞}$ having real part the same as $ω$, but the imagine part very large positive, then plot their paths in $r$-plane when diminishing $\text{Im}ω$ from $+∞$ to the value at hand. If the path of a zero intersects with real $r$ axis at a point in the interval $(0, 1)$ than this zero is needed to be passed round, and we have to integrate equations (28) along the contour going around the $r_A(ω)$ (see Figs. 4 for different cases of paths). In actual computations it occurred that the eigenfrequencies $ω_{nm}$ have small negative imaginary part (module of it is less than 0.1). If we consider only $ω$ with $|\text{Im}ω| < 1$ than the points $r_A(ω)$, which are needed to be passed round, will lie close to the real $r$ axis. The intersection point $r_A(ω_0)$ of a path of $r_A(ω)$ with the real $r$ axis, when changing $\text{Im}ω$, will be located close to the $r_A(ω)$. Therefore, numerical procedure can be simplified because there is no necessity to follow full path of each $r_A(ω)$ when $\text{Im}ω$ runs from $+∞$ to the value at hand. We actually did the following: took real $ω_0 = \text{Re}ω$, found all $r_A(ω_0)$ belonging to the interval $(0, 1)$, then found all $r_A(ω)$ and pass round only those of them, which are the nearest to $r_A(ω_0)$. In doubtful cases we did the full procedure of finding which points from all multitude of $r_A(ω)$ must be passed round, being provided for the separate code. The value of $ξ_r$ at $r = 1$, which was being achieved to be 0, does not depend on the particular contour of integration, provided the points $r_A(ω)$ are passed round properly. Hence we can suite the contour of integration for programming convenience. We chose the rectangular contour when going around the points $r_A(ω)$, and, in order to save the accuracy of computations in the vicinity of singularity $r = r_A$, used to passed round also the points $r_A(ω)$ when $\text{Im}ω$ was very small positive value.

We see that for small $\text{Im}ω_{nm}$ the rule for bypassing singular points in equations (28) coincide with that used in calculations of Landau damping of waves in plasma medium when doing integration in velocity space (Lifshitz & Pitaevskii, 1979). There are two differences, however. First, in our case integration is performed in the interval $(0, 1)$, while in the case of Landau damping along the whole real $r$ axis from $−∞$ to $+∞$. Second, not all singular points of $ξ_{rω}$ in $ω$ plane are poles, but the branch points as well. As described in section 2 these lead to the existence of continuous spectrum for the problem considered in present work.

It occurred that $\text{Im}ω_{nm}(k) ≤ 0$ for all $k$, $m$, and 3 functions of Ω$^k(r)$ involved in calculations, so the jet is stable with respect to helical perturbations as well as with respect to axisymmetric perturbations. Actually,
computations were performed for $-10 < k < 10$, $-3 < m < 3$, and for the first 3 radial modes for each $k$ and $m$. The dependencies
\[
\Omega^F (r) = \Omega (1-r^2), \quad \Omega^F (r) = \Omega e^{-r} \cos \left( \frac{5\pi}{2} r \right), \quad \text{and}
\]
\[
\Omega^F (r) = \Omega \left[ \frac{1}{3} (r^3 - 1) - \frac{a+b}{2} (r^2 - 1) + ab(r-1) \right] \left( \frac{a+b}{2} - ab - \frac{1}{3} \right)^{-1},
\]
where $\Omega$, $a$, $b$ are constants, $0 < a < 0.5$, $0.5 < b < 1$, were tried out. Constant $\Omega$ in the expressions for $\Omega^F$ was ranged in the interval from 0.1 to 20.

Our model does not itself provide us any information about the function $\Omega^F (r)$ except that $\Omega^F (1) = 0$. The angular rotational velocity of magnetic field lines has to be found from the consideration of the jet origin. We adhere the viewpoint that the part of the jet closest to the symmetry axis is connected by magnetic field lines directly to the black hole and can be described in the frame of force–free approximation (jets emerging from the accretion disks seem to be not the force–free, because of the kinetic energy of the mass flow is comparable to the Pointing flux, Pelletier & Pudritz (1992)). Therefore, to determine $\Omega^F (r)$ one should solve the equations governing stationary two dimensional axisymmetric structure of magnetic fields in the vicinity of a rotating black hole taking into account the unavoidable process of particle creation there (Blandford & Znajek 1977, Takahashi et al. 1990, Nitta, Takahashi & Tomimatsu 1991, Beskin, Istomin & Pariev 1992(b)). This is still unresolved problem. Simple model describing the force–free magnetic field in the vicinity of a slowly rotating black hole was developed disregarding the effects of e$^+$e$^-$ pair creation in Beskin, Istomin & Pariev, 1992a. The dependence
\[
\Omega^F (r) = \Omega (1-r^2),
\]
which resembles the function (51) for $0 < r < 1$, but is the entire function in complex $r$-plane, contrary to (51).

Following the procedure outlined we have calculated the dispersion curves $\omega_{nm} = \omega_{nm}(k)$. First three branches (n=0,1,2) of them for $\Omega^F$ given by (52) and $m = 2$ are shown on fig. 5 and fig. 6. On fig. 7 we show the dependence of the real part of $\omega_{nm}(k)$ for $m = -1$. In this case, because of the absence Alfvén resonance surface inside the jet, imaginary part of $\omega_{nm}(k)$ is always equal to 0 (continuum spectrum does present, of course). If some 3 values $k$, $\omega$, and $m$ are a solution of the eigenvalue problem than the values $-k, -\omega^*$, and $-m$ will be a solution too but for the complex conjugated function $\xi^*$, so we depicted only the branches of $\omega_{nm} = \omega_{nm}(k)$ having Re$\omega > 0$. Those having Re$\omega < 0$ can be obtained by the reflection of fig. 5 and fig. 7 with respect to the coordinates origin. On fig. 8(a) we plotted an example of radial eigenmode $\xi$ when there is no Alfvén resonant point on the interval 0 < $r$ < 1, on fig. 8(b) the same for the case when resonant point exists. Note the localized character of the mode in the last case. In order to find physical reason for decaying of the discrete modes having Alfvén resonant point we calculated Pointing flux $S = 1/4\pi (E \times B)$ for each mode. On fig. 8(c) the radial component of energy flux $rS_r$, calculated for mode shown on fig. 8(a), is depicted as a dependence from $r$. It is seen, that $S_r < 0$, i.e. the energy flows to the jet axis, and, which is more important, Alfvén resonant surface is a drain of electromagnetic energy of the mode. This energy is converted into the energy of the mean magnetic and electric fields, so one should expect that near Alfvén resonant surface strong amplification of the magnetic and electric fields does occur. However, the consideration of this process is based on the second, nonlinear approximation and is beyond the scope of the present work.

4 Boundary layer analysis

In this section we consider the spectrum of eigenfrequencies $\omega_{nm}$ near the merging point $\omega_c$ of two $r_A(\omega)$. At this point $A(r, \omega)$ has at least quadratic zero in $r$. We assume that $A' (\omega_c) \neq 0$ and the point $r_A(\omega_c)$ does not coincide with 0 and 1 (prime denotes partial differentiation with respect to $r$ at the point $r = r_a$). To find points $\omega_c(k,m)$ it is necessary to solve the couple of equations $A(r, \omega_c) = 0$, $\omega' (r_c, \omega_c) = 0$. The solution can be represented in implicit form as
\[
k = -m\Omega^F (r_c) + \frac{m\Omega^F (r_c)}{\Omega^F (r_c)^2 |_{r=r_c}} (1 + \Omega^F (r_c) r_c^2),
\]
\[
\omega_c = m\Omega^F (r_c) + \frac{m\Omega^F (r_c)}{\Omega^F (r_c)^2 |_{r=r_c}} (1 - \Omega^F (r_c) r_c^2).
\]
There are two cases of the solution of the system (53), (54) for each $k$ and $m$. In the first case, when the solution $r_c$ of (53) is real, $\omega_c$ determined by (54) will be real. In the second case, when (53) has a pair of complex conjugated solutions $r_c$ and $r^*_c$, equation (54) will give us a pair of complex conjugated values $\omega_c$ and $\omega^*_c$. As described in Subsection 2.3, contour of integration of equation (32) can never being clamped between merging points $r_A(\omega)$ when $\text{Im}\,\omega_c > 0$, so we are able to move it away out of the neighbourhood of $\omega_c$. Hence, $\omega_c$ with $\text{Im}\,\omega_c > 0$ is regular point of $\xi_{\omega}$ and can not be an accumulation point of the poles. Perturbations with $\text{Im}\,\omega < 0$ are exponentially decaying with time and do not contribute to the asymptotic of $\xi_c(t)$ at $t \to \infty$. Consequently, the spectrum $\omega_{nm}$ near the real points $\omega_c$ is of primary importance for the stability problem. If $\omega_c$ and $r_c$ are real, the frequency $\omega_c$ coincides with the edge of the Alfvén continuum, i.e. $\frac{\text{d}\omega_A(r)}{\text{d}r}\bigg|_{r=r_c}=0$.

Below we perform boundary layer analysis of equation (32) in the vicinity of $r_c$, similar to that was done in BIB for the edge of the slow wave continuum in the nonrelativistic MHD stability problem of nonrotating cylindrical flow. But in contrast to BIB, who always integrated radial eigenvalue equation along real $r$ axis, we adhere the rules of contour deformation in the complex $\nu$-plane, which are described in Section 2. At the point $r=r_c$, the eigenvalue problem (32) has a singularity such that to lowest order in $x=r-r_c$

$$A \simeq A^r x^2/2,$$

$$C_1, C_2, C_5 \simeq \text{const}, \quad C_4 \simeq -A''(\omega+k)x^2/2,$$

$$(C_1/rC_2)^\nu \simeq \text{const}.$$ 

Thus, to lowest order in $x$, equation (32) becomes

$$\frac{d}{dx} \left(x^2 \frac{d}{dx}(r\xi_r)\right) + D(r\xi_r) = 0,$$  

(55)

where

$$D = -\frac{2}{A^r} \left[C_5 + rC_2 \left(\frac{C_1}{rC_2}\right)\right].$$  

(56)

Setting $r\xi_r \propto x^n$, we obtain for the characteristic exponent

$$\nu = -1/2 \pm \sqrt{1/4 - D}.$$  

(57)

If

$$D > 1/4,$$  

(58)

than $\xi_r$ will have infinite number of oscillations in the vicinity of $r=r_c$ and $|\xi_r| \propto r^{-1/2}$. If $D \leq 1/4$, than two linear independent solutions will be of power type without oscillations. As we will see below, in the case $D > 1/4$ there exist an infinite sequence of eigenfrequencies $\omega_{nm}$. Asymptotically, for $n \to \infty$, the eigenfrequencies converge geometrically toward the marginal point $\omega = \omega_c$. The eigenfunctions corresponding to these eigenfrequencies have the higher number of oscillations in the vicinity of $r=r_c$, the higher is the number $n$, and the limiting solution for $\xi_r$ has an infinite number of oscillations. In the other case, $D \leq 1/4$, the picture outlined fails to be true, and the lowest–order boundary analysis can not provide any information on eigenfrequencies. Using definition (56) and relations (53), (54) oscillations condition (58) can be transformed to the form

$$\frac{d}{dr} \left[\frac{\Omega'^F}{(\Omega'^F x^2)^\nu}\right] \frac{d}{dr} \left[\frac{x^8}{\Omega'^F x^2}\right] < 0,$$  

(59)

where all derivatives are evaluated at the point $r=r_c$. We see that this condition depends only on the position of the point $r_c$, and does not depend on $k$ and $m$ separately.

Let us now consider the frequency $\omega$ slightly different from $\omega_c$

$$\omega = \omega_c + \Delta,$$  

(60)

where $|\Delta| \ll 1$. In the vicinity of $\omega = \omega_c$, $r = r_c$ the main terms of expansion $A$ in $\omega$ and in $r$ are

$$A = -(1 + \Omega'^F x^2)\Delta + \left[2m\Omega'' - (\omega_c + k)(\Omega'^F x^2)''\right] x^2/2.$$  

(61)

Note, that the equation (55) is invariant under rescaling of $x$. Furthermore, from (61) it follows, that the rescaling of $x$ leaves the lowest–order in $\Delta$ and in $x$ approximation of radial mode equation (32) invariant if $\Delta$ is scaled the same as $x^2$. Therefore, boundary layer of thickness $x \propto \Delta^{1/2}$ arises near the point $r = r_c$ in
the equation (32). To obtain "inner layer" equation we introduce inner rescaled variable $X$ by the following definition

$$X = \frac{x}{\sqrt{\Delta}} \left( -\frac{2}{D r_c} \right) \left( -\frac{m \Omega^F}{1 + \Omega^F r_c^2} \right)^{1/2},$$

(62)

where we determine $\sqrt{\Delta}$ such that the $-\pi/2 < \arg \sqrt{\Delta} < \pi/2$ and accept for the argument of the square root from the expression enclosed by parentheses the value 0, if this expression is positive, and the value $\pi/2$, if this expression is negative. With such definition the lower-order inner-layer equation becomes

$$\frac{d}{dX} \left[ (1 - X^2) \frac{d}{dX} (r \xi) \right] - D(r \xi) = 0.$$  

(63)

This is Legendre’s equation with singular points $X = 1$ and $X = -1$. At these points $r \xi$ has logarithmic singularities, as they are merely two closely spaced first-order zeroes of $A$ in $r$. Equation (63) shows that within the "inner layer", $|X| \ll 1$, the finite frequency shift is essential. Furthermore, for $|X| \gg 1$ and $|X| < 1$, there exists an "intermediate" region where the frequency shift becomes negligible, and where the inner-layer equation (63) approaches the small $|x|$ limit of the external equation (55). The boundary layer analysis consists of matching the solutions of the external equation (55) for $|x|$ small with those of the complete inner-layer equation (63) for $|X|$ large, using (62) to relate $|x|$ and $|X|$, to obtain an equation for $\Delta$. Figs. 9 (a), (b) show the complex $x$ and $X$ planes for the case when $\text{Im} \Delta < 0$. On Fig. 9 (a) $m \Omega^F/D < 0$, on Fig. 9 (b) $m \Omega^F/D > 0$. For definiteness, we assume that the contour of integration of equation (32) approaches the two merging points $r A(\omega)$ along the real $x$ axis from $x < 0$ and moves away them also along the real $x$ axis to $x > 0$.

Let us introduce the notation for characteristic exponent (57)

$$\nu = -1/2 \pm is, \quad s = \sqrt{D - 1}/4.$$  

(64)

Parameter $s$ is real for oscillatory solutions and imagine if the otherwise. Then, the solution of the external equation (55) in the vicinity of $r = r_c$ is for $x < 0$

$$r \xi = a_+ x^{1/2 + is} + a_- x^{-1/2 - is},$$

(65)

where $a_+/a_-$ is fixed by the external solution in order that the boundary condition at $r = 0$ be satisfied.

Similarly, for $x > 0$,

$$r \xi = b_+ x^{1/2 + is} + b_- x^{-1/2 - is},$$

(66)

and $b_+/b_-$ is fixed by the external solution in order that the boundary condition at $r = 1$ be satisfied. Matching the solution of the internal equation with the solution (65), (66) of the external equation leads to asymptotical, when $|X| \to \infty$, expressions for inner-layer solution

$$r \xi = \begin{cases} 
  b_+ \left( \sqrt{\Delta} X \right)^{-1/2 + is} + b_- \left( \sqrt{\Delta} X \right)^{-1/2 - is} & \text{when } \text{Re} X \to +\infty, \\
  a_+ \left( \sqrt{\Delta} X \right)^{-1/2 + is} + a_- \left( \sqrt{\Delta} X \right)^{-1/2 - is} & \text{when } \text{Re} X \to -\infty.
\end{cases}$$

Here we denote

$$\tilde{\Delta} = -\Delta \frac{1}{D} \frac{2}{r_c} \frac{m \Omega^F}{1 + \Omega^F r_c^2}.$$  

Since the scaled inner-layer equation (63) is independent of $\Delta$, so is the connection matrix between the expansion coefficients for $r \xi$ in the two asymptotic regions $\text{Re} X \to +\infty$ and $\text{Re} X \to -\infty$ divided from each other by branch cuts attached to the points $X = +1$ and $X = -1$ (if the contour of integration goes between these points). Thus,

$$\begin{pmatrix} b_+ \left( \sqrt{\Delta} \right)^{is} \\
 b_- \left( \sqrt{\Delta} \right)^{-is} \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\
 v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} a_+ \left( \sqrt{\Delta} \right)^{is} \\
 a_- \left( \sqrt{\Delta} \right)^{-is} \end{pmatrix}.$$  

(67)

Here $v_{ij}$ only depend on $D$. The explicit form of $v_{11}$, $v_{12}$, $v_{21}$, $v_{22}$ follows from the solution of the inner-layer equation (63). The general solution of this equation is

$$r \xi = A P_v(X) + B Q_v(X),$$
where \( P_\nu(X) \) and \( Q_\nu(X) \) are Legendre’s functions of the first and second order respectively with \( \nu \) from (64). The main terms of the asymptotic expansion of \( P_\nu(X) \) and \( Q_\nu(X) \), when \( |X| \to \infty \), are (Abramowitz & Stegun, 1970)

\[
P_\nu(X) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(-is)}{\Gamma(1/2-is)} (2X)^{-1/2-is} + \frac{\Gamma(is)}{\Gamma(1/2+is)} (2X)^{-1/2+is},
\]

\[
Q_\nu(X) = \sqrt{\pi} \frac{\Gamma(1/2+is)}{\Gamma(1+is)} (2X)^{-1/2-is}.
\]

The form of these expansions just match the expansion (65), (66) of the solution of the outer–layer equation. Therefore, considering branch cuts as shown on Figs. 9, involving the relations between \( P_\nu \) and \( Q_\nu \) on the different sides of a branch cut, and properly choosing the branches of functions \( X^{-1/2\pm is} \) in the asymptotic expressions, one can obtain the coefficients of connection matrix (if the contour of integration goes between the points \( X = +1 \) and \( X = -1 \))

\[
v_{11} = -e^{-2\pi s}(1 + \coth \pi s), \quad v_{22} = e^{2\pi s}(\coth \pi s - 1), \quad v_{12} = \sqrt{\pi} e^{2\pi s} \left( \frac{\Gamma(is)}{\Gamma(1/2 + is)} \right)^2,
\]

\[
v_{21} = -\frac{8e^{-2\pi s}}{\sqrt{\pi}} \left( \frac{\Gamma(-is)}{\Gamma(1/2 - is)} \right)^2.
\]

Then, taking into account that \( R_+ = b_+/b_- \) and \( R_- = a_+/a_- \) are fixed by boundary conditions, we find that in order for the system (67) to have nontrivial solution for \( a_+, a_-, b_+, b_- \), the following dispersion relation must be fulfilled

\[
v_{21} R_+ R_- \left( \tilde{\Delta}^{is} \right)^2 + (v_{22} R_+ - v_{11} R_-) \tilde{\Delta}^{is} - v_{12} = 0.
\]

This is a quadratic equation on \( \tilde{\Delta}^{is} \) with the solutions

\[
\tilde{\Delta}^{is}_\pm = \frac{1}{2v_{21}\sqrt{R_+ R_-}} \left[ -\frac{e^{\pi s} R_+ + e^{-\pi s} R_-}{\sqrt{R_+ R_-} \sinh \pi s} \pm \sqrt{\left( \frac{e^{\pi s} R_+ + e^{-\pi s} R_-}{\sqrt{R_+ R_-} \sinh \pi s} \right)^2 - 4 \cosh^2 \pi s} \right].
\]

Because of the multivaluedness of the raising to a complex power there exist two infinite sequences of the solutions \( \tilde{\Delta} \)

\[
\tilde{\Delta}_n(\pm) = \exp \left[ \frac{1}{1} \arg \tilde{\Delta}_n - \frac{2\pi n}{s} \right] = e^{i \log |\tilde{\Delta}_n|} \exp \left[ \frac{i}{s} \log |\tilde{\Delta}_n| \right], \quad n = \text{integer number}.
\]

5 Possible astrophysical implications

We believe that strongly magnetized inner parts of the jets can be directly connected to the central supermassive black hole by magnetic field lines. Let us estimate whether such jet can be force–free. We assume the black hole have typical values of its mass and rotating parameter: \( M \simeq 10^8 M_\odot, a = J/M \leq 1 \). On the distances from black hole comparable to the Schwarzschild radius strong magnetic field is expected to be of the order of \( 10^4 G \) (Begelman, Blandford & Rees, 1984). Than the electron density of \( n_c \simeq 1 cm^{-3} \) is enough to screen
the longitudinal (along the magnetic field) electric field component so that the MHD approximation becomes possible. In the inner part of the flow connected with the black hole by magnetic field lines the particle density \( n \) cannot much exceed the value \( n_c \) because the particles constrained to move along the magnetic surfaces do not escape the black hole and the only source for replenishing them is \( e^+e^- \) pairs production. However, particles creation in the black hole magnetosphere seems to be possible only in the presence of the longitudinal electric field, which vanishes for \( n \gg n_c \). Therefore, an equilibrium between the particles outflow fed the jet and their creation is established for particle densities of the order of \( n_c \). The necessity of particles creation in the magnetosphere of supermassive black hole was first pointed out by Blandford & Znajek (1977). In this case the energy density of the magnetic and electric fields is \( 10^{13} \) times greater than the rest energy density of the \( e^+e^- \) pairs. That makes the force-free approximation adequate in the vicinity of supermassive black hole.

Apparently, the force-free approximation is also valid for the inner parts of the jet, which are close to the axis of symmetry and are connected with the black hole. To show this consider the conservation of the particles flow and the conservation of the current in the process of possible recollimation of this inner part of the jet from the sizes of the order of the black hole event horizon to much greater. Continuity of the particles flow implies that \( \gamma n \propto 1/R^2 \), where \( R \) is the radius of the jet, \( \gamma \) is the Lorentz factor of the plasma flow, \( n \) is the particles concentration in the reference frame comoving with the plasma flow and the velocity of the particles is relativistic everywhere.

Conservation of the current flowing inside the jet leads to that the magnetic field after recollimation will be predominantly toroidal and scale as \( B \propto 1/R \). We see therefore that the ratio \( B^2/4\pi nn^2 \propto 1/\gamma \), so after recollimation to larger radii (say, parsecs) the jet remains to be force-free unless the Lorentz factor of the flow will not reach an unbelievably high value \( 10^{13} \). Therefore, we hope to apply our consideration for those jets, which are believed to be electron–positron, and are likely to be force-free.

The remarkable feature of the dispersion curves \( Re\omega(k) \) is that they have a minimum at some \( k = k_{\text{min}} \) and \( k_{\text{min}} \neq 0 \) (see Figs. 5 and 7). At the same time waves damping \( Im\omega(k_{\text{min}}) \) is either small (it never exceeded 0.1 in our computations) or even equal to 0 for modes with \( m < 0 \). Because of these, the perturbation with \( k \approx k_{\text{min}} \) do not propagate since the group velocity \( d\omega/dk \) vanishes for \( k = k_{\text{min}} \). In contrast to the waves having \( k \neq k_{\text{min}} \) this wave packet undergoes only diffuse broadening due to the finite value of \( d^2\omega/dk^2 \) for \( k = k_{\text{min}} \). It means that such oscillations form the "standing wave" with the wave vector \( k_{\text{min}} \). Such "standing wave" does not transmit any energy because of vanishing group velocity. We use quotation marks to name this phenomena in order not to confuse it with the well known in many fields of physics standing waves, which are the linear superposition of two progressive waves. The amplitude of the "standing wave" will be larger than the amplitudes of other waves because it experiences a dispersion spreading only. This phenomena is caused by the fact that the oscillations with wave vectors less and greater then \( k_{\text{min}} \) propagate in the opposite directions.

The phenomenon of "standing wave" takes place for axisymmetric perturbations as well (IP).

After a long time after initial excitation the pattern of disturbance is formed with the wave crests moving with the phase velocity \( \omega(k_{\text{min}})/k_{\text{min}} \). In our numerical calculations this velocity was always greater than the velocity of light. A "standing wave" perturbation occupies progressively growing with time as \( \propto t^{1/2} \) part of the jet. The edges of this pattern move with velocity \( \propto t^{-1/2} \), which is less than the velocity of light. At the same time amplitude of the perturbation inside the "standing wave" region is decaying with time as \( \propto t^{-1/2} \). It is impossible to transmit any information by moving wave crests faster than the speed of light. If one has relativistic electrons emitting synchrotron radiation inside the magnetic configuration dealing with (which is the case for extragalactic jet), than this pattern will be visible. This provide us with the new type of superluminal source. Now according to well known formula one can calculate the observable velocity of such superluminal source in the projection onto the plane of the sky

\[
V_{\text{obs}} = \frac{v \sin \theta}{1 - v/c \cos \theta},
\]

where \( \theta \) is the angle between the jet axis and the line of sight of the observer, \( v \) is the velocity of the superluminal source. In Fig. 10 the dependence of \( V_{\text{obs}} \) from \( \theta \) is depicted. If \( \theta < \theta^* = \arccos c/v \) than \( V_{\text{obs}} < 0 \), i.e. the apparent motion of knots will be reversal, in the direction opposite to the wave vector \( k_{\text{min}} \). Observer will see superluminal motion (\( | V_{\text{obs}} | > c \)) if

\[
\arccos c/\sqrt{2v} - 45^\circ < 135^\circ - \arcsin c/\sqrt{2v}.
\]

Note, that the velocity, which enters into the effect of relativistic beaming, is the velocity of matter flow inside the jet rather than the velocity \( v \). Consequently, the brightness of superluminally moving knots will not be affected by their motion.

The source of perturbations may be instability of accretion disk around the central black hole, which affects the magnetic field in the vicinity of a black hole, and, therefore, lead to the excitation of the disturbances at the base of the jet. If the jet is oriented close to the line of sight (\( \theta < \theta^* \)), the observer will see chain of knots
moving backward to the core. In the counterjet, if observable despite the fading due to the effect of relativistic beaming, knots will move outward from the core, as seen from Fig. 10. Another possibility is the excitation of perturbations in the jet at the hot spot, where the jet is ended. Though we consider only stationary equilibrium jet structure, the velocity of the advancing of the jet into the extragalactic medium is believed to be only mildly relativistic (~ 0.2 c) (see, e.g. Begelman, Blandford & Rees, 1984), so one can admit that the phenomenon of "standing wave" may take place in that region as well. Here knots in the jet, provided that $\theta < \theta^*$, will move from the hot spot in the direction of the core, while in the counterjet they will move to the hot spot out of the core. If $\theta > \theta^*$ then the observable direction of motion for both jet and counterjet coincides with the projection of the real motion of knots onto the plane of sky, i.e. outward from the core and outward from the hot spot.

From expression (69) it follows that if $\theta \to \theta^*$, then $|V_{\text{obs}}| \to \infty$ having different signs for $\theta < \theta^*$ and $\theta > \theta^*$. The equality of $V_{\text{obs}}$ to infinity means that the whole jet splashes simultaneously throughout all its length. In reality this can not come true. The jet radius changes along it's length, jet are usually slightly bent. These two reasons limit the value $|V_{\text{obs}}|$ by some high but finite value. If the viewing angle of the jet $\theta$ is near $\theta^*$ and changes along the jet such that there exist pieces of the jet viewed at an angle less than $\theta^*$ and greater than $\theta^*$, then pairs of knots can be observed which either collide and vanish or emerge and move in the opposite directions, depending on whether $\theta$ changes along the direction of real motion of wave crests from being greater than $\theta^*$ to being less than $\theta^*$ (in the case of merging knots) or from being less than $\theta^*$ to greater than $\theta^*$ (in the case of newly born pairs of knots). When the wave crest of a "standing wave" pattern passes that piece of the slightly bent jet, where $\theta \approx \theta^*$, observer will see that the knot, corresponding to that wave crest, is stretched along the jet and it’s total luminosity (not the brightness) becomes considerably larger. Then the next wave crest passes through that place of the jet, and observer sees the next flash. This can be the reason for the quasi periodical splashing of the innermost parts of the jets with modest viewing angles $\theta \approx \theta^*$ of the order of 45°, while usually such bursts are explained by relativistic beaming of the moving knots along slightly curved trajectory in the jet having $\theta$ no more than a few degrees (see recent paper by Camenzind & Krockenberger, 1992). However, careful examination of these effects needs considering physical equation describing disturbances propagation inside curved jet for finding dependence $v(\theta)$, not only kinematic picture. This is beyond the scope of the present paper.

It is the task for radioastronomers to detect such "natural" (not due to the effect of projection) superluminal motions. Bååth (1992) reported about three epoch observation of one component in 3C345 moving inward to the core, but he writes that the significance of this observation still remains to be verified. Hardee (1990) proposed another scenario which can lead to observation of backward motions of the intersection points of the shocks in nonmagnetized jet. In the frame of our model periodical structures moving backward to the core may be observable while Hardee’s model predicts an isolated knots.

### 6 Summary

A considerable amount of extragalactic jets are extremely well collimated and extends over the distances tens times longer than their radii. Bright well known example of such jet is one emerging from the galaxy M87. The problem of extraordinary stability is long standing problem. In this work we have shown numerically that a jet with a longitudinal current is stable within the force–free approximation for all velocities of longitudinal motion and for wide range of the velocities of rotation. We consider initial value problem for linearized set of relativistic equations describing disturbances propagation inside cylindrical jet. By using Laplace transformation we find asymptotical behaviour of perturbations over long time since initial excitation. The stability problem is reduced to eigenvalue problem for radial modes. It was shown that there exist Alfvén continuous spectrum of eigenfrequencies and discrete spectrum having the accumulation point where two Alfvén resonances coincide. Numerical calculation shows that all eigenfrequencies have been computed have negative or equal to zero imaginary part. This means the stability of the jet with respect to helicoidal perturbations as well as to axially symmetrical or pinch ones. The physical reason for stability is that there is a shear of the magnetic field because of changing the curling of the magnetic field lines with the radius (the absence of the shear would imply $\Omega^F(r) \propto 1/r$, which leads either to the value of rotational velocity $\Omega^F r + KB_0 \phi$ being undefined on the symmetry axis of the jet, if $1 + KB_{0z} \neq 0$, or flow velocity of the matter being equal to the speed of light on the symmetry axis, if $1 + KB_{0z} = 0$, both cases have no physical sense). Because of the fluid pressure is low compared to that of the electromagnetic field, even a small shear stabilizes the motion perpendicular to the magnetic field lines and prevent the development of instability.

We also find that the dispersion curves $\omega = \omega_{nm}(k)$ have minima for certain values of $k = k_{\text{min}}$. This means that such oscillations form a "standing wave" with a wave vector $k_{\text{min}}$. The wave crests of this "standing wave" are spirals moving along the jet with the velocity exceeding the speed of light. The amplitude of the "standing wave" will be larger than the amplitudes of other waves because it experiences a dispersion spreading only. An example, illustrating this behaviour of perturbation, was presented in IP. Relativistic particles emitting
components are the velocity first order perturbation parallel to zero order magnetic field remains also to be free. Another two does not enter into the basic equations (1), (2) of the main text and can be free. Therefore the component of In the approximation of ideal force-free plasma the component of the velocity parallel to the magnetic field line

\[
B_{\phi_1} = \frac{i}{S} \left[ \frac{dB_{r_1}}{dr} \left( \frac{1}{F} \omega (\omega + k) \Omega^F r - \frac{m}{r} \right) + B_{r_1} \frac{d\Omega^F}{dr} \frac{r}{F^2} k \times \right] \\
(\omega + k)(\omega - k - m\Omega^F) - \frac{m}{r^2} B_{r_1} + B_{r_1} \frac{\Omega^F}{F}(\omega^2 - \omega k - 2k^2),
\]

\[A1\]

\[
B_{z_1} = \frac{i}{S} \frac{\omega + k}{F} \left[ \frac{dB_{r_1}}{dr} (\omega - k - m\Omega^F) - B_{r_1} \frac{d\Omega^F}{dr} m \frac{\omega - k - m\Omega^F}{F} \right] \\
+ \frac{1}{r} B_{r_1} (\omega - k + m\Omega^F).
\]

\[A2\]

For electric field

\[
E_{r_1} = \frac{i}{S} \left[ \frac{dB_{r_1}}{dr} \left( -\frac{m}{r} + \Omega^F r k (\omega + k) + \Omega^F \frac{m^2}{r} \right) + B_{r_1} \frac{d\Omega^F}{dr} \frac{r (\omega + k)}{F^2} \times \right] \\
(-k \omega + k^2 - m\Omega^F \omega + m^2 / r^2) + B_{r_1} \frac{1}{F} \left( -\frac{m \omega}{r^2} + \Omega^F \left( -2 \omega^2 - \omega k + k^2 + \frac{m^2}{r^2} \right) \right),
\]

\[A3\]

\[
E_{\phi_1} = -B_{r_1} \frac{\omega - m\Omega^F}{k + m\Omega^F},
\]

\[A4\]

\[
E_{z_1} = B_{r_1} \Omega^F r \frac{\omega + k}{k + m\Omega^F}.
\]

\[A5\]

In the approximation of ideal force-free plasma the component of the velocity parallel to the magnetic field line does not enter into the basic equations (1), (2) of the main text and can be free. Therefore the component of the velocity first order perturbation parallel to zero order magnetic field remains also to be free. Another two components are

\[
v_{r_1} = -B_{r_1} \left( \frac{\omega - m\Omega^F}{k + m\Omega^F} - KB_{z_0} \right) \frac{1}{B_{z_0}},
\]

\[A6\]

\[
B_{z_0} v_{\phi_1} - B_{\phi_0} v_{z_1} = \frac{i}{S} \left[ \left\{ \frac{dB_{r_1}}{dr} \left( \Omega^F r (\omega + k) - \frac{m}{r} \right) - B_{r_1} \left( \frac{m}{r^2} + \Omega^F (\omega + k) \right) \right\} \times \right] \\
\left\{ \frac{1}{F} (m \Omega^F - \omega) + KB_{z_0} \right\} + \\
B_{r_1} \frac{d\Omega^F}{dr} \frac{r}{F} (\omega + k) \left\{ \omega - k + \frac{1}{F} \left( \omega m \Omega^F - \omega^2 \Omega^F^2 - \frac{m^2}{r^2} \right) + KB_{z_0} (\omega - k - m\Omega^F) \right\}.
\]

\[A7\]
To obtain expressions for all components of the electric and magnetic fields and velocity we substitute in (A1)–(A7) for \( B_{r_1} \) it’s value from equation (19) of the main text, which express \( B_{r_1} \) via \( \xi_r \). Thus, we lead to the following

\[
B_{r_1} = iB_{z_0}F\xi_r, \quad (A8)
\]

\[
B_{\phi_1} = -B_{z_0} \left\{ \Omega^F r \frac{d\xi_r}{dr} + \xi_r \frac{d}{dr}(\Omega^F r) + \frac{k}{S} \left[ \frac{d\xi_r}{dr} \left( \Omega^F(\omega + k) - \frac{m}{r^2} \right) - \xi_r \left( \Omega^F(\omega + k) + \frac{m}{r^2} \right) \right] \right\}, \quad (A9)
\]

\[
B_{z_1} = B_{z_0} \left\{ -\left( \frac{d\xi_r}{dr} + \frac{1}{r} \xi_r \right) + \frac{m}{rS} \left[ \frac{d\xi_r}{dr} \left( \Omega^F(\omega + k) - \frac{m}{r^2} \right) - \xi_r \left( \Omega^F(\omega + k) + \frac{m}{r^2} \right) \right] \right\}, \quad (A10)
\]

\[
E_{r_1} = B_{z_0} \left\{ \Omega^F r \frac{d\xi_r}{dr} + \xi_r \frac{d}{dr}(\Omega^F r) - \frac{\omega}{S} \left[ \frac{d\xi_r}{dr} \left( \Omega^F(\omega + k) - \frac{m}{r^2} \right) - \xi_r \left( \Omega^F(\omega + k) + \frac{m}{r^2} \right) \right] \right\}, \quad (A11)
\]

\[
E_{\phi_1} = -iB_{z_0}(\omega - m\Omega^F)\xi_r, \quad (A12)
\]

\[
E_{z_1} = iB_{z_0}\Omega^F(\omega + k)\xi_r, \quad (A13)
\]

\[
v_{r_1} = -iB_{z_0}(\omega - m\Omega^F - KB_{z_0})\xi_r, \quad (A14)
\]

\[
B_{z_0}v_{\phi_1} - B_{\phi_0}v_{z_1} = -B_{z_0} \left\{ r\xi_r \frac{d\Omega^F}{dr}(1 + KB_{z_0}) + \frac{1}{S}(m\Omega^F - \omega + KB_{z_0}) \times \left[ \frac{d\xi_r}{dr} \left( \Omega^F(\omega + k) - \frac{m}{r^2} \right) - \xi_r \left( \Omega^F(\omega + k) + \frac{m}{r^2} \right) \right] \right\}, \quad (A15)
\]

The function \( \xi_r(r) \) has logarithmic singularity at the point \( r = r_A \), which is the simple zero of \( A \) (the case of a second order zero is considered in section 4), and is regular everywhere in the remaining part of complex plane \( r \), except infinite point. Therefore, one can conclude from equations (A8)–(A15) the following behaviour for disturbances of different physical quantities near Alfvén resonant point \( r = r_A \) \( (x = (r - r_A)/r_A) \)

\[
B_1 \propto \log x, \quad B_{\phi_1} \propto \frac{1}{x}, \quad B_{z_1} \propto \frac{1}{x}; \quad E_{r_1} \propto \frac{1}{x}, \quad E_{\phi_1} \propto \log x, \quad E_{z_1} \propto \log x; \quad v_{r_1} \propto \log x, \quad B_{z_0}v_{\phi_1} - B_{\phi_0}v_{z_1} \propto \frac{1}{x}.
\]

It might appear from equations (A9)–(A11), (A15) that the quantities \( B_{\phi_1}, B_{z_1}, E_{r_1}, B_{z_0}v_{\phi_1} - B_{\phi_0}v_{z_1} \) are singular at the \( r = r_S \), which is the zero of \( S \). However, this is not the case. Being regular at \( r = r_S \) function \( \xi_r \) can be expanded into power series in the neighbourhood of \( r = r_S \). By expanding second order differential equation (32) on \( \xi_r \) one can find that the combination \( rd\xi_r/dr(\Omega^F(\omega + k) - m/r^2) - \xi_r(\Omega^F(\omega + k) + m/r^2) \), which enters into all expressions (A9)–(A11), (A15), becomes equal to 0 when \( S = 0 \). Therefore, all physical quantities regular there. The only singularity may arise when \( A = 0 \), i.e. for modes from Alfvén continuum.

**Appendix B**

In the general case, when \( B_{z_0} \neq \text{const} \), one can perform calculations analogous to that was done in section 2 when deriving the radial eigenmode equations (28). The coefficient by the derivatives \( \frac{1}{r} \frac{d}{dr}(r\xi_r) \) and \( \frac{dp_x}{dr} \) is of special interest, because it contains the singular points of the system of two first order differential equations on the displacement and the disturbance of the total pressure. It occurred to be the following

\[
A = B_{z_0}^2(\omega - m\Omega^F)^2 + (\omega B_{\phi_0} + \Omega^F r KB_{z_0})^2 - \left( \frac{m}{r} B_{\phi_0} + KB_{z_0} \right)^2. \quad (B1)
\]

Zeroes of \( A(\omega_A, r) = 0 \) give us Alfvén resonant frequencies \( \omega_A \) at a given value \( r \). The expression (B1) is quadratic in \( \omega \), therefore, for each \( r \) the equation \( A = 0 \) has two solutions for \( \omega_A(r) \). If we suppose \( B_{z_0} = \text{const} \) and use for \( B_{\phi_0} \) the expression (7) with the sign ‘+’, the equation (B1) will become

\[
A = B_{z_0}^2(\omega + k) \left[ \omega - k - 2m\Omega^F + \Omega^F r(\omega + k) \right] = B_{z_0}^2(\omega + k)A. \quad (B2)
\]

So \( A \) factorizes into two coefficient, one of which do not contain the \( r \)-dependence. These two coefficient are present in system (28).
Next, let us show that at any real $r$ the quadratic equation $A = 0$ has only real solutions $\omega_A$. The discriminator of this equation is

$$D = (-m\Omega^F + \Omega^F r\kappa)^2 - (1 + \tau^2)\Omega^{E^2} r^2 \left( k^2 + \frac{m^2}{r^2} \right) + (1 + \tau^2) \left( \frac{m}{r} \tau + k \right)^2,$$

where $\tau = B_\phi / B_z$. The solutions $\omega_A$ will be complex if and only if $D < 0$. The condition $D < 0$ transforms to

$$\Omega^{E^2} r^2 \left( k + \tau \frac{m}{r} \right)^2 > (1 + \tau^2) \left( k + \tau \frac{m}{r} \right)^2,$$

which, in turn, means that $E_{\tau 0}^2 > B_0^2$. In stationary ideal MHD configuration electric field can never exceed magnetic field. This would imply the velocity of the fluid $v$ being greater that the velocity of light. Naturally, it can be easily shown that any solution of the equation (6), governing the stationary jet configuration, satisfy the requirements $|E| < |B|$ (see IP). Therefore, $D$ must be nonnegative value and the solutions $\omega_A$ are always real. Thus, in the general case of force–free cylindrical equilibrium Alfvén continua are always real and do not lead to instability.
References


Appert K., Gruber R., Vaclavik J., 1974, Phys. Fluids, 17, 1471


Bååth L.B., 1992, Physica Scripta, T43, 57

Begelman M.C., Blandford R.D., Rees M.J., 1984, Rev. Mod. Phys., 56, 255


Beskin V.S., Istomin Ya.N., Pariev V.I., 1992(b), AZh, 69, 1258


Novikov I.D., Frolov V.P., 1986, Physics of the Black Holes, Nauka Press, Moscow


Figure 1. Schematic representation of the equilibrium stationary configuration of a jet with a uniform poloidal magnetic field $B_z$. The frequency of rotation in the dimensionless units described in the beginning of Subsection 2.1 is $\Omega^F = 10(1 - r^2)$. The jet boundary for $r = 1$ and three magnetic tubes for $r = 1/4$, $2/3$ and $9/10$ are shown. The magnetic field lines are spiralling on a magnetic tube. Since $\Omega^F (1) = 0$, the total current through the jet is equal to zero and the magnetic field is purely poloidal both at the boundary and at the axis of symmetry. The curling of magnetic field lines is maximum for $r = 1/\sqrt{3}$, decreasing for smaller and larger radii. The density of the poloidal current $j_z$ is negative when $r < 1/\sqrt{2}$ and positive when $1 > r > 1/\sqrt{2}$. The electric field $E$ induced by jet rotation is radial. The plasma velocity $v$ along the magnetic tube consists of two components: rotation with angular velocity $\Omega^F (r)$, and motion along the magnetic field lines with a speed $v_\parallel = KB$. We see that the rotation velocity $r\Omega^F$ can exceed the speed of light $c$ (in our case the maximum value of $r\Omega^F$ is $20/3\sqrt{3}c$ at $r = 1/\sqrt{3}$); nevertheless, the quantity $v$ is restricted by $c$ due to the existence of a predominantly toroidal magnetic field. The dispersion curves $\omega = \omega_{nm} (k)$ for perturbations of this equilibrium state are plotted in the Figs. 5,6,7.

Figure 2. Schematic picture, illustrating analytical properties of $\xi_{r\omega}$ in the complex $\omega$-plane. See explanation in the text. Branch cuts are shown by thin solid line, the contour of integration shifted into lower $\omega$ half-plane breaks up into the circles around the poles $\omega = \omega_{nm} (k)$ and the part going round the branch points and along the attached branch cuts. This contour is shown by bold solid line with arrows.

Figure 3. Schematic picture of the complex $r$-plane. The trajectories of $r_A (\omega)$ when changing $\omega$ along the paths shown in Fig. 2, are plotted by dashed lines, deformed contours of integration in cases $C$ and $B$ are plotted by bold solid lines, branch cuts (one for each case $A$, $B$, $C$ or $D$) are shown by thin solid lines.

Figure 4. The thin curves with arrows show the trajectories of 6 points $r_A (\omega)$ when changing $\text{Im} \omega$ from $+\infty$ to some large negative value. The endpoints are marked by crosses. In all figures $\Omega^F = 10(1 - r^2)$, $m = 1$, $k = -3$. Figures (a), (b), (c), (d) and (e) correspond to the continuation paths $A$, $B$, $C$, $D$, and $E$ in Fig. 2 respectively. $\omega_A (0) = 17$, $\omega_A (1) = -3$, two complex conjugated $\omega_c$ are $\omega_c = 3.3117 \pm 0.1564 i$, $r_c$ is placed at $0.5$. In Fig. (a) $\text{Re} \omega = -5$, in (b) $\text{Re} \omega = 5$, in (c) $\text{Re} \omega = 3.305$, in (e) $\text{Re} \omega = 3.315$, in (d) $\text{Re} \omega = 17.5$. On figure (d) 4 lateral loops are enlarged, because their dimensions are less than $0.01$ in reality.

Figure 5. The dependence of the real part of $\omega_{nm} (k)$, $m = 2$, $\Omega^F = 10(1 - r^2)$. Three lowermost branches of the dispersion relations are shown. They are distinguished from each other by an appropriate number. Straight lines are $\omega = k$ and $\omega = -k$.

Figure 6. The dependence of the imaginary part of $\omega_{nm} (k)$. $m = 2$, $\Omega^F = 10(1 - r^2)$. Three curves correspond to those shown on Fig. 5 and are indicated by appropriate numbers.

Figure 7. The dependence of the real part of $\omega_{nm} (k)$. $m = -1$, $\Omega^F = 10(1 - r^2)$. Three lowermost branches of the dispersion relations are shown. They are distinguished from each other by an appropriate number. Straight lines are $\omega = k$ and $\omega = -k$.

Figure 8. Radial dependence of the second mode for (a) $m = -1$, $k = -0.4$, $\Omega^F = 10(1 - r^2)$, $\omega_{1-1} = 6.86$ and (b) $m = 2$, $k = 1$, $\Omega^F = 10(1 - r^2)$, $\omega_{22} = 8.54 - 5 \cdot 10^{-3} i$. The solid curve shows $\text{Re} \xi_r$ and the dashed line $\text{Im} \xi_r$. The solution was started so that $\xi_r$ is real as $r \to 0$. Unit of $\xi_r$ is arbitrary. Fig. (c) shows radial dependence of the radial component of energy flux, calculated for mode (a).

Figure 9. Complex $x$ and $X$ plane, shown for the case $\text{arg} \Delta = -\pi/3$. In Fig. (a) $m \Omega^F / D < 0$, in Fig. (b) $m \Omega^F / D > 0$. Contour of integration is plotted by bold solid line, branch cuts are shown by thin solid lines.

Figure 10. The dependence of the observable velocity of the standing wave pattern $v_{\text{obs}}$ on the angle $\theta$ between the jet axis and the line of sight of the observer. Phase velocity of the perturbation $v$ is chosen equal to $3c$. If the jet is pointed directly to the observer than $\theta = 0$, if it is pointed directly from the observer than $\theta = 180^\circ$. 