ON CRITERIA FOR THE EXISTENCE OF BACKWARD HYBRID WAVES IN CYLINDRICAL STRUCTURES

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In some recent papers (1-7), the properties of hybrid waves at cut-off conditions in cylindrical waveguides have been considered. A close correlation was found between the coincidence in cut-off frequency of neighbouring hybrid modes and the appearance of backward waves in this region. Below, we prove mathematically that the coincidence conditions are sufficient for the existence of backward hybrid waves, at least in a definite class of cylindrical structures.

We shall consider the lossless cylindrical waveguide model, filled with two concentric layers of anisotropic dielectric (see Fig. 1). Each anisotropic medium is characterized by the following tensors:

\[ \hat{\varepsilon} = \begin{pmatrix} \varepsilon_{ir} & 0 & 0 \\ 0 & \varepsilon_{ir} & 0 \\ 0 & 0 & \varepsilon_{iz} \end{pmatrix}, \tag{1} \]

where \( i = 1, 2 \).

The given model includes, as special cases, all structures considered in Refs. (1-7), as well as plasma waveguides with an extreme axial magnetic field \( (H_0 = 0 \text{ and } H_0 \to \infty) \).

Dispersion relations of the chosen model have the following form:

\[ F = SR + \kappa^2 (\varepsilon_{2r} - \varepsilon_{1r}) \frac{\nu}{ka} F = 0, \tag{2} \]

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where $S = \frac{\varepsilon_{2z}}{\varepsilon_{1z}} \psi_1 \phi_0 - \frac{\varepsilon_{12}}{\varepsilon_{12}} J_{\nu+1}(\Gamma_1 ka),$

$P = \frac{\varphi_1}{\gamma_2 \phi_0} - \frac{J_{\nu+1}(\gamma_1 ka)}{J_{\nu}(\gamma_1 ka)},$

$R = \gamma_1^2 \gamma_2^2 P + (\varepsilon_{2r} - \varepsilon_{1r}) \frac{\nu}{ka},$

$\psi_1 = J_{\nu}(\Gamma_2 kb) N_{\nu+1}(\Gamma_2 ka) - N_{\nu}(\Gamma_2 kb) J_{\nu+1}(\Gamma_2 ka),$

$\psi_0 = J_{\nu}(\Gamma_2 kb) N_{\nu}(\Gamma_2 ka) - N_{\nu}(\Gamma_2 kb) J_{\nu}(\Gamma_2 ka),$

$\varphi_1 = J_{\nu}^1(\gamma_2 kb) N_{\nu+1}(\gamma_2 ka) - N_{\nu}^1(\gamma_2 kb) J_{\nu+1}(\gamma_2 ka),$

$\varphi_0 = J_{\nu}^1(\gamma_2 kb) N_{\nu}(\gamma_2 ka) - N_{\nu}^1(\gamma_2 kb) J_{\nu}(\gamma_2 ka),$

$\gamma_1^2 = \varepsilon_{1r} - \bar{k}^2, \quad \gamma_2^2 = \varepsilon_{2r} - \bar{k}^2, \quad \Gamma_1^2 = \frac{\varepsilon_{12}}{\varepsilon_{1r}} \gamma_1^2,$

$k$ - is the wave number in vacuum : $\kappa$ - is the propagation constant and $\nu$ - is the azimuthal number.

One of the characteristic features of hybrid waves appears to be their transformation into transverse $E$- and $H$-waves at cut-off ($k = 0$). Then $E$-waves have non-vanishing $E_{z1}, E_{r1}, E_{\theta}$ components and $H$-waves have non-vanishing $H_{z1}, H_{r1}, H_{\theta}$ components.

Dispersion relations (2) at $k = 0$ split into two equations:

$S = \sqrt{\frac{\varepsilon_{2z}}{\varepsilon_{1z}}} \psi_1 \phi_0 \left(\sqrt{\varepsilon_{2r} kb}, \sqrt{\varepsilon_{2r} ka}\right) - \frac{J_{\nu+1}(\Gamma_1 ka)}{J_{\nu}(\Gamma_1 ka)} = 0 \quad \text{for E-waves (4a)}$

and

$R = \sqrt{\frac{\varepsilon_{1r}}{\varepsilon_{2r}}} \frac{J_{\nu}(\Gamma_2 ka)}{\phi_0 \left(\sqrt{\varepsilon_{1r} kb}, \sqrt{\varepsilon_{1r} ka}\right)} \frac{\partial \varphi_0(\sqrt{\varepsilon_{2r} kb}, \sqrt{\varepsilon_{2r} ka})}{\partial (\sqrt{\varepsilon_{2r} ka})}$

$= \frac{\partial J_{\nu}(\Gamma_2 ka)}{\partial (\sqrt{\varepsilon_{1r} ka})} = 0 \quad \text{for H-waves (4b)}$

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Notice that Equation (4a) depends only on the axial components $\varepsilon_{1z}$ of the tensor, and Equation (4b) only on the transverse ones ($\varepsilon_{2r}$).

Let us consider now Equation (2) in the neighbourhood of $\kappa = 0$, and expand it into a series as a function of the small parameters $\kappa$ and $u = \kappa a - (\kappa a)_0$, where $(\kappa a)_0$ is the root of one of the equations $(4a,b)$. As a result, limiting ourselves to second order terms, we obtain:

$$
F(u, \kappa) = F(0, 0) + \frac{\partial F}{\partial \kappa} \kappa + \frac{\partial F}{\partial u} u + \frac{1}{2} \left[ \frac{\partial^2 F}{\partial \kappa^2} \kappa^2 + 2 \frac{\partial^2 F}{\partial \kappa \partial u} \kappa u + \frac{\partial^2 F}{\partial u^2} u^2 \right] = 0. \tag{5}
$$

If equations (4) are incompatible, then we have at $\kappa = 0$

$$
\frac{\partial F}{\partial \kappa} = 0, \quad \frac{\partial F}{\partial u} = \begin{cases} R \frac{\partial S}{\partial u}, & \text{if } (\kappa a)_0 \text{ is the root of Eq. } (4a) \\ S \frac{\partial R}{\partial u}, & \text{if } (\kappa a)_0 \text{ is the root of Eq. } (4b) \end{cases}
$$

and, respectively, in first order, $\frac{d(\kappa a)}{d\kappa} = 0$ (see Fig. 2), i.e., the dispersion curve $(\kappa a)$ tends to be normal to the frequency axis.

If the Equations (4a) and (4b) are compatible (as in the case of coincidence of cut-off frequencies), at $\kappa = 0$, we have

$$
\frac{\partial F}{\partial \kappa} = \frac{\partial F}{\partial u} = \frac{\partial^2 F}{\partial \kappa \partial u} = 0. \tag{6}
$$

In the neighbourhood of $\kappa = 0$, condition (6) gives, for the dispersion relations, the following form

$$
\bar{\kappa}^2 = Au^2, \tag{7}
$$

where

$$
A = -\frac{e_{1r} e_{2r}}{(e_{2r} - e_{1r})^2} \left( \frac{\partial R}{\partial u} \frac{\partial S}{\partial u} \right)_{\kappa=0} \frac{\nu}{(\kappa a)^2}.
$$

As the expansion of the dispersion equation (2) was carried out in the region of the real solution, the factor $A > 0$. Therefore in the neighbourhood of cut-off frequencies, equation (7) has two real branches ($\bar{\kappa} = \pm \sqrt{Au}$). One of these solutions ($\bar{\kappa} = \sqrt{Au}$) applies to forward waves, and the other one ($\bar{\kappa} = -\sqrt{Au}$) to backward waves.
As an example, let us consider the dispersive characteristics of hybrid waves in the disc-loaded waveguide, in small pitch approximation. In such an approximation, field II (in the region of discs) can be described by some equivalent dielectric tensor, namely

\[
\mathbf{\varepsilon}_2 = \begin{pmatrix}
-\frac{\lambda}{D} & 0 & 0 \\
0 & -\frac{\lambda}{D} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(8)

where \( \frac{\lambda}{D} \gg 1 \), \( \lambda \) is the wave length and \( D \) is the period of structure (we suppose the disc thickness to be much smaller than the period \( D \)).

When \( \frac{\lambda}{D} \to \infty \), the dispersion equation, the field components, as well as other characteristics of the waveguide with tensor (8) in field II, and the free space in field I, are in complete agreement with the expressions for a disc-loaded waveguide obtained with the small pitch approximation. This permits us to consider the disc-loaded waveguide as a special case of the chosen model. It is not difficult to obtain, from equation (2) the dispersion relations of disc-loaded waveguides. For this, one has to put \( \varepsilon_{1r} = \varepsilon_{2r} = \varepsilon_{2s} = 1 \), \( \varepsilon_{2r} \to -\infty \) and to normalize the \( F \) function to \( \varepsilon_{2r} \):

\[
\gamma_1^2 \text{ SP} + \frac{\nu}{ka} \left[ S + \bar{z}^2 P \right] = 0,
\]

(9)

where

\[
S = \frac{\psi_0}{\psi_1} (k_0, ka) + P , \quad P = -\frac{J_{\nu+1}(\gamma_1 ka)}{\gamma_1 J_{\nu}(\gamma_1 ka)} \cdot \
\gamma_1^2 = 1 - \bar{k}^2.
\]

At cut-off, relation (9) splits into two different equations:

a) \( J_0^\prime (ka) = 0 \),

b) \( J_\nu (ka) = 0 \).

(10)

The system of equations (10) is compatible when the a/b parameter has the following values:

\[
\left( \frac{a}{b} \right)_{\nu mn} = \frac{\alpha_{\nu m}}{\beta_{\nu n}},
\]

(11)

where \( \alpha_{\nu m} \) is the m-root, and \( \beta_{\nu n} \) is the n-root of equations (10a,b), respectively.

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The dispersion equation (9), in the neighbourhood of the cut-off coincidence of the two modes corresponding to the $\alpha_{\nu m}$ and $\beta_{\nu n}$ roots, has a form similar to (7), where the $A$ coefficient, in this particular case, is equal to

$$A = \frac{(\frac{\nu}{\alpha_{\nu m}})^2 - 1}{(\frac{a}{b})_{\nu m n} (\frac{\nu}{\alpha_{\nu m}})^2} \cdot \frac{J'_{\nu} (\beta_{\nu n}) N'_{\nu} (\alpha_{\nu m})}{N_{\nu} (\beta_{\nu n}) J_{\nu} (\alpha_{\nu m})}.$$  \hspace{1cm} (12)

It is easy to show, using the properties of the roots of Bessel functions, that the $A$ coefficient is always positive.

In Fig. 3 are given the exact and the approximate dispersion curves obtained respectively from equations (7) and (12), for $\nu = 1$, $m = n = 1$. One can thus see the agreement between exact and approximate calculations in the neighbourhood of $\kappa = 0$. In such a case also the right-hand branches correspond to forward waves, and the left-hand ones have sharply defined regions with anomalous dispersion.

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FIGURE CAPTIONS

Fig. 1. Model for a cylindrical waveguide structure with concentric anisotropic dielectric layers.

Fig. 2. Dispersion curves of two lower hybrid modes at \( \nu = m = n = 1 \) and \( A_b = 0.55 \). (A case of cut-off frequencies non-coincidence)

Fig. 3. Dispersion curves in the region of coincidence of cut-off frequencies for the following case: \( \nu = 1, m = n = 1 \) and \( A_b = 0.480 \).

Full lines: calculated with the help of the exact equation (9).
Dotted lines: calculated with the help of the approximate equation (7).

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