Stationary Drift-Rossby Vortices of Elliptic Boundary in a Background Shear Flow

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Abstract

The Charney-Hasegawa-Mima equation is a two-dimensional nonlinear equation describing the geostrophic Rossby flow and the plasma drift flow. An analytic solution of the equation is given which represents a stationary vortex of elliptic boundary in a background shear flow. The solution is obtained by assuming a suitable form for the arbitrary function in the stationary equation.

Keywords

Charney-Hasegawa-Mima equation, drift-Rossby wave,
elliptic vortex, background shear flow

The geostrophic Rossby flow in the atmosphere of a rotating planet and the electrostatic drift flow in a magnetized plasma may be treated by a two-dimensional nonlinear equation for the stream function, known as the Charney-Hasegawa-Mima (C.H.M.) equation.^{1),2)} Among many papers devoted to the study of vortical motion described by the C.H.M. and the related equations,³⁾ Horiata et al⁴⁾ investigated the stationary vortex solutions of the C.H.M. equation in the presence of a background shear flow. In their work, the entire two-dimensional region was divided by a circular boundary into the internal and external regions, in each of which an appropriate linear equation for the stream function being assumed. This treatment was a straightforward generalization of the method of Larichev and Reznik⁵⁾ to the shear flow case, and the multipolar vortices obtained were confined inside the circular boundary.

The vortices observed in natural shear flows are, however, often elongated in the direction of the flow as shown e.g. by the Great Red Spot of Jupiter. In the present letter, we give a stationary vortex solution of the C.H.M. equation having an elliptic core.

We use the terminology of the geostrophic case: the plasma case may be treated in a similar way. Taking the local Cartesian coordinates on the northern hemisphere of the planet with the x -axis toward east, the y -axis north and z -axis local zenith, we express the Coriolis parameter as $f(y) = f_0 + \beta y$ in the so-called β -plane approximation with

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f_0 and β constants, and the depth of the atmosphere as $H = H_0(1 + h)$ where H_0 is the average depth and $h(x, y, t)$ the relative displacement. Hereafter the time t and the space variables (x, y) are normalized by f_0^{-1} and the Rossby radius $\sqrt{gH_0}/f_0$, respectively, where g is the gravitational acceleration. Using the same notations for the dimensionless variables as the physical ones before the normalization, the C.H.M. equation is given by⁴⁾

$$\frac{\partial q}{\partial t} + [H, q] = 0, \quad (1)$$

where $q = \nabla^2 h - h + \beta y$ is the potential vorticity, with h playing the role of a stream function, and the Poisson bracket is defined by $[A, B] = \hat{z} \cdot (\nabla A \times \nabla B)$.

We consider stationary solutions of eq. (1) which move in the x direction with a constant velocity u . Then h depends on x , y and t as $h(x - ut, y)$, and eq. (1) yields $[h + uy, q] = 0$ showing that q is an arbitrary function of $h + uy$, which is just the stream function in the moving frame. Denoting it by $\psi = h + uy$, we obtain the equation for ψ :

$$\nabla^2 \psi - \psi + (u + \beta)y = F(\psi), \quad (2)$$

where $F(\psi)$ is an arbitrary function of ψ . In the following, we rewrite $(x - ut, y)$ as (x, y) for simplicity. The components of flow velocity in the moving frame are given by $v_x = -\partial\psi/\partial y$ and $v_y = \partial\psi/\partial x$.

Now we take an ellipse centered at the origin and with the major axis lying along the x -axis. Let the equation of the ellipse E be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b. \quad (3)$$

In what follows, we take advantage of the arbitrariness of the function F in eq. (2), and assume that it is given by

$$F(\psi) = \begin{cases} -\psi + \omega_1 : & \text{inside } E \\ -\psi + \omega_0 : & \text{outside } E, \end{cases} \quad (4)$$

where ω_1 and ω_0 are constants and $\omega_1 \neq \omega_0$. Then eq. (2) becomes

$$\nabla^2 \psi = \begin{cases} -\lambda y + \omega_1 : & \text{inside } E \\ -\lambda y + \omega_0 : & \text{outside } E \end{cases} \quad (5)$$

where we have put

$$\lambda = u + \beta. \quad (6)$$

When $\lambda = 0$, eqs. (5) reduce to the two-dimensional Navier-Stokes equations with the constant vorticity ω_1 and ω_0 inside and outside E , respectively. The case was treated by Moore and Saffman⁶⁾ who obtained solutions which consist of an elliptic monopolar core inside, and flows with uniform strain and vorticity outside E . When applied to the present case of the C.H.M. equation, however, it should be noted that the condition $\lambda = 0$ corresponds to the wave velocity $u = -\beta$, that is, just the long wavelength limit of the

phase velocity of linear Rossby waves. Then the vortex may lose the energy by radiating the Rossby wave, as emphasized by Nycander.³⁾

Therefore, we will investigate the case $\lambda \neq 0$ in greater detail. Since the boundary E itself must be one of the stream lines, $\psi = \text{const}$, we put

$$\psi = 0 \quad \text{on } E, \quad (7)$$

taking the arbitrary constant to be null. The inner solution of eq. (5) satisfying the condition (7) is given by

$$\psi_{\text{in}} = -A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)(y + B), \quad (8)$$

with

$$A = \frac{1}{2} \frac{a^2 b^2}{3a^2 + b^2} \lambda, \quad B = -\frac{3a^2 + b^2}{a^2 + b^2} \frac{\omega_1}{\lambda}. \quad (9)$$

For $|B| < b$, eq. (8) shows that $\psi_{\text{in}} = 0$ on the line $y = -B$ besides on the boundary E .

The external solution is most conveniently expressed in terms of the elliptic coordinates

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta, \quad (10)$$

where

$$c = (a^2 - b^2)^{1/2} \quad (11)$$

is the distance from the origin to one of the foci of E . The curves $\xi = \text{constant}$ are a family of confocal ellipses, and one of them, $\xi = \xi_0$ say, will coincide with E provided that

$$a = c \cosh \xi_0, \quad b = c \sinh \xi_0, \quad (12)$$

and the region $\xi > \xi_0$ corresponds to the outside of E . The exterior equation (5) now becomes

$$\begin{aligned} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\psi &= \frac{1}{2}c^2(\cosh 2\xi - \cos 2\eta)(\omega_0 - \lambda c \sinh \xi \sin \eta) \\ &= \frac{1}{2}\omega_0 c^2 \cosh 2\xi - \frac{1}{4}\lambda c^3 \sinh 3\xi \sin \eta \\ &\quad - \frac{1}{2}\omega_0 c^2 \cos 2\eta + \frac{1}{4}\lambda c^3 \sinh \xi \sin 3\eta, \end{aligned} \quad (13)$$

and the solution, ψ_{ex} say, must satisfy together with the condition (7), the condition of continuity of the normal derivative across E :

$$\frac{\partial \psi_{\text{in}}}{\partial \xi} = \frac{\partial \psi_{\text{ex}}}{\partial \xi} \quad \text{at} \quad \xi = \xi_0. \quad (14)$$

Since the right-hand side of eq. (13) as well as the internal solution ψ_{in} , rewritten in terms of the (ξ, η) coordinates, have the η dependence like $1, \sin \eta, \cos 2\eta$ and $\sin 3\eta$, we can solve eq. (13) by assuming that ψ_{ex} has the same η dependence. Putting

$$\psi_{\text{ex}} = G_0(\xi) + G_1(\xi) \sin \eta + G_2(\xi) \cos 2\eta + G_3(\xi) \sin 3\eta \quad (15)$$

in eq. (13), we have the equations for $G_n(\xi)$, $n = 0, 1, 2, 3$, which are to be solved under the boundary conditions (7) and (14), that is

$$G_n(\xi_0) = 0, \quad G'_n(\xi_0) = F'_n(\xi_0), \quad n = 0, 1, 2, 3 \quad (16)$$

where $F_n(\xi)$ are the corresponding functions when ψ_{in} is expanded similar to eq. (15). The result is given as follows:

$$\begin{aligned} G_0(\xi) &= \frac{1}{2}(\omega_1 - \omega_0)ab(\xi - \xi_0) + \frac{1}{8}\omega_0\{c^2 \cosh 2\xi - (a^2 + b^2)\}, \\ G_1(\xi) &= \frac{1}{32}\lambda c^3\left\{\frac{3a^2 + b^2}{c^2} \sinh \xi - \sinh 3\xi\right\}, \\ G_2(\xi) &= \frac{1}{8}\omega_0 c^2 + \frac{1}{4}(\omega_0 - \omega_1)ab \sinh 2\xi + \left[\frac{a^2 b^2}{2(a^2 + b^2)}\omega_1 - \frac{1}{8}(a^2 + b^2)\omega_0\right] \cosh 2\xi, \\ G_3(\xi) &= -\frac{c^2}{3a^2 + b^2}G_1(\xi) \end{aligned} \quad (17)$$

where use has been made of eqs. (11) and (12).

An alternative treatment of the external equation (5), allowing a more “physical” interpretation, would be to write it

$$\psi_{\text{ex}} = \psi_0(y) + \varphi$$

with

$$\begin{cases} \psi_0''(y) = -\lambda y + S_0, \\ \nabla^2 \varphi = \gamma_0 (= \omega_0 - S_0), \end{cases} \quad (18)$$

and regard $\psi_0(y)$ as giving the assumed background shear flow, $-\psi_0'(y) = \frac{1}{2}\lambda y^2 - S_0 y + v_0$, where v_0 is a constant. Solving eqs. (18) in terms of the elliptic coordinates, we obtain the same result as eq. (17). Examples of the flow patterns given by eqs. (8), (15) and (17) are shown in Fig. 1 and Fig. 2.

In conclusion, we have given an analytic solution for the stationary vortex of elliptic boundary of the Charney-Hasegawa-Mima equation in a background shear flow. The stability of the flow obtained is left to be examined.

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Fig.1 Stream lines, $\psi = \text{const}$, in the moving frame for $a = 2$, $b = 1$, $u + \beta = -0.6$, $\omega_0 = 0.2$ and $\omega_1 = -0.05$. $|B| = 0.22 < b$.

Fig.2 Stream lines, $\phi = \text{const}$, in the moving frame for $a = 2$, $b = 1$, $u + \beta = -0.6$, $\omega_0 = 0.2$ and $\omega_1 = -0.5$. $|B| = 2.2 < b$.