LONGITUDINAL BLOW-UP OF THE BUNDLES AT TRANSITION,

CAUSED BY COUPLING BETWEEN SYNCHROTRON OSCILLATIONS AND BETATRON OSCILLATIONS

by

H.G. Hereward and A. Sørensen
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LONGITUDINAL BLOW-UP OF THE BUNCHES AT TRANSITION,

CAUSED BY COUPLING BETWEEN SYNCHROTRON OSCILLATIONS AND BETA TRON OSCILLATIONS

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INTRODUCTION

Particles with large betatron amplitudes, radial or vertical, have a larger circumference - at the same kinetic energy - than a particle with no betatron oscillations. This extra circumference decreases their rotation frequency. Even with beam control, the R.F. system cannot behave suitably for two kinds of particles simultaneously (probably the R.F. system will adjust itself to suit a particle with some kind of average betatron amplitude), and particles with large betatron amplitude will go through transition differently from particles with small betatron amplitude. This looks like a non-Liouvillean blow-up, i.e. a loss of particle density, if one looks only at the longitudinal projection of the six-dimensional phase space.

A further change of circumference with betatron amplitude can arise from a non-linearity, such as a sextupole term, in the focusing field. This is not treated in the present paper, but our calculation of the longitudinal dynamics, up to the section "The change in circumference", is general enough to cover it too. Any betatron-dependent term in the equation for synchrotron oscillations must be accompanied by a synchrotron-dependent term in the betatron oscillations, and this we have not treated either.

THE EQUATION OF MOTION

We start by writing down the equations (5.5a) and (5.5b) from Courant and Snyder [6]. These equations were also the starting point from the perturbation-theoretical treatment [2] of the longitudinal space-charge forces; where they appear as equations (1.1) and (1.2).

\[
\frac{\Delta E}{\omega_s} \frac{d}{dt} \left( \frac{\Delta E}{\omega_s} \right) = \frac{\omega}{2\pi} \left( \sin \phi - \sin \phi_s \right) \tag{1}
\]

\[
\frac{d\phi}{dt} = \frac{\eta h \omega_s}{\beta^2} \frac{\Delta E}{E} \tag{2}
\]
The betatron oscillations will increase the circumference of the orbit from $C$ to $C + \Delta C$, therefore $\frac{d\varphi}{dt}$ will be different from zero even for a particle with $\Delta E = 0$. Eq. (1) is unchanged, while eq. (2) will have an extra term:

$$\frac{d\varphi}{dt} = \frac{\eta h \omega_s}{\beta^2} \frac{\Delta E}{E} + \lambda \quad (3)$$

The part of $\frac{d\varphi}{dt}$ which comes from $\Delta E/E$ goes to 0 and changes sign at transition with $\eta$; the part $\lambda$ which is added due to betatron oscillations does nothing special at transition. It will be slightly $\beta$ and $\gamma$ dependent but this can be neglected for the region near transition.

We use the same convention for the phase angle $\varphi$ as was used in the previous transition reports [1], [2], [3], [4] and also in Courant and Snyder [6]: one imagines an R.F. wave travelling around the machine and $\varphi$ is then taken to mean the phase in R.F. radians by which the particle in question is lagging behind this wave, measured from a cross-over. The number of R.F. radians paced out per unit time by the wave is:

$$\omega_1 = h \omega_s \quad (4)$$

and the number of R.F. radians paced out per unit time by a particle with $\Delta E = 0$ is:

$$\omega_2 = h \frac{2\pi v C + \Delta C}{C} = h \omega_s \frac{1}{1 + \frac{\Delta C}{C}} \sim h \omega_s \left( 1 - \frac{\Delta C}{C} \right) \quad (5)$$

Therefore, with $\Delta E = 0$

$$\frac{d\varphi}{dt} = \omega_1 - \omega_2 = h \omega_s \frac{\Delta C}{C} \quad (6)$$

and we have:

$$\lambda = h \omega_s \frac{\Delta C}{C} \quad (7)$$

The sign is the same before and after transition: a particle with larger than average betatron oscillations has $\Delta C > 0$ and therefore $\lambda > 0$.

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We now perform exactly the same manipulations on the pair of equations (1), (3) as was done on the pair (1), (2) in ref. [2], part 1. In equation (3) we substitute

\[ \beta = \frac{R G \omega_g}{c} \]  

(8)

\[ \eta = 2 \left( \frac{m c^2}{E_{tr}} \right)^2 \frac{1}{E_{tr}} \left( \frac{\partial E}{\partial t} \right)_{tr} \]  

(9)

and we find :

\[ \Delta E = \frac{1}{2} \left( \frac{E_{tr}}{m c^2} \right)^2 \frac{E_{tr}}{c^2} \frac{E^2}{\hbar c^2} \omega_g \frac{1}{t} \left( \frac{d\varphi}{dt} - \lambda \right) \]  

(10)

Now combine (1) and (10), introduce linearization and the characteristic time parameter \( T \) etc., and we have :

\[ \frac{d}{dt} \left\{ \frac{1}{t} \left( \frac{d\varphi}{dt} - \lambda \right) \right\} + \frac{\text{sgn} (t - t_0)}{T^3} \varphi = 0 \]  

(11)

Put \( t = xT \) and take \( t_0 = 0 \) :

\[ \frac{d}{dx} \left\{ \frac{1}{x} \left( \frac{d\varphi}{dx} - \Lambda \right) \right\} - \frac{\text{sgn} (x)}{x} \varphi = 0 \]  

(12)

with

\[ \Lambda = \lambda T = \hbar \omega_g T \frac{\Delta C}{c} \]  

(13)

We note that \( \Lambda \) is a dimensionless quantity. Like \( \frac{d\varphi}{dx} \), it is measured in R.F. radians per unit interval of \( x \).

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SOLUTION OF THE EQUATION OF MOTION

An unexpected problem

As we know the solution of equation (12) for the special case \( A = 0 \) (see ref. [2]), the most natural thing to do is to solve the same equation with \( A \neq 0 \) by means of "The method of variation of parameters", in the same way as was done with the space-charge equation in ref. [2], part 2. This method is described in any textbook on differential equations, see for instance Langer [7], p. 193. To do this we must write eq. (12) in the form:

\[
p_0(x) \frac{d^2 \theta}{dx^2} + p_1(x) \frac{d \theta}{dx} + p_2(x) \theta = f(x)
\]  

(14)

with the perturbing term on the right. But this presents a problem because we find ourselves using

\[
\frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}
\]

(15)

which is only valid for \( x \neq 0 \).

If we go ahead with:

\[
\frac{d^2 \theta}{dx^2} + \frac{1}{x} \frac{d \theta}{dx} + x \text{ sgn}(x) \theta = -A \frac{1}{x}, \quad x \neq 0
\]

(16)

and then put

\[
\theta = |x| \left\{ j(x) \frac{J_2(y)}{J_3} (y) + n(x) \frac{N_2(y)}{N_3} (y) \right\}
\]

(17)

with

\[
y = \frac{2}{3} |x|^{3/2}
\]

(18)
we find that the integral giving $j(x)$ diverges at $x = 0$. This is connected with the fact that:

$$|x| \int_0^y j(y) \to 0$$  \hspace{1cm} (19 a)

for $x \to 0$. So at $x = 0$ we are trying to express a finite solution $\phi$ by an infinite coefficient $j(x)$.

We shall therefore solve eq. (12) in a somewhat different way, where divergence problems do not occur.

How to get around it

Let us take a look at eq. (10). Here, everything in front of the multiplication sign is a constant, therefore

$$\frac{1}{t} \left( \frac{d\phi}{dt} - \lambda \right)$$  \hspace{1cm} (20)

is a "good" variable in the same way as $\Delta E/E$, that is, a variable on which continuity requirements may be made. The same must also be the case with the quantity

$$p = \frac{1}{x} \left( \frac{d\phi}{dx} - A \right)$$  \hspace{1cm} (21)

as this is the same as (20) except for a factor $T^2$.

* The other coefficient, $n(x)$, does not diverge, and this is because:

$$|x| \int_0^y n(y) \to \frac{3^{2/3} \rho(z)}{\pi} \neq 0$$  \hspace{1cm} (19 b)

for $x \to 0$.

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Let us introduce $p$ as a variable and we get the following pair of first-order equations:

\[
\frac{d\phi}{dx} = x \ p + A \tag{22}
\]

\[
\frac{d\theta}{dx} = - \text{sgn}(x) \ \theta \tag{23}
\]

This pair is equivalent to eq. (12). It can also be written as a single second-order equation in $p$ only: assume $x \neq 0$, but otherwise positive or negative at will. Then from (23):

\[
\frac{d^2 p}{dx^2} = - \text{sgn}(x) \ \frac{d\phi}{dx} \tag{24}
\]

We now substitute (22) and here

\[
\frac{d^2 p}{dx^2} + \text{sgn}(x) \cdot x \ p = - \text{sgn}(x) \ A \tag{25}
\]

This equation is equivalent to (12) and to (22), (23) for all $x \neq 0$. We note that this equation is slightly simpler than the corresponding equation (12) for $\theta$, but the important thing is not that the first-order term is lacking, but that the perturbation term is finite and that no divergence problem occurs.

Our problem is now to solve eq. (25). From "Tables of Bessel functions," ref. [5], eq. (31), we find that the homogeneous equation

\[
\frac{d^2 p}{dx^2} + \text{sgn}(x) \cdot x \ p = 0 \tag{26}
\]

has the solution

\[
p = x^{1/2} \left\{ B_+ \ J_{\frac{1}{3}} \left( \frac{2}{3} \sqrt{\text{sgn}(x)} \ x^{3/2} \right) + B_- \ J_{-\frac{1}{3}} \left( \frac{2}{3} \sqrt{\text{sgn}(x)} \ x^{3/2} \right) \right\} \tag{27}
\]

for an interval in which $\text{sgn}(x)$ is a constant, that is, an interval which does not
contain \( x = 0 \). We cannot expect the same \( B \)'s to apply on both sides of \( x = 0 \), but when we impose continuity on \( p \) and \( \theta \) by the methods of ref.\([2]\), we find that the solution can be written:

\[
p = |x|^{1/2} \left\{ B_+ J_{\frac{1}{3}} (y) + B_- J_{-\frac{1}{3}} (y) \right\} \tag{28}
\]

with

\[
y = \frac{2}{3} |x|^{3/2} \tag{29}
\]
as before. This we can also write in the Bessel-Neumann form:

\[
p = |x|^{1/2} \left\{ B_+ J_{\frac{1}{3}} (y) + B_- N_{\frac{1}{3}} (y) \right\} \tag{30}
\]

"The method of variation of parameters" then gives the solution of eq.\((25)\)

\[
p = |x|^{1/2} \left\{ j(x) J_{\frac{1}{3}} (y) + n(x) N_{\frac{1}{3}} (y) \right\} \tag{31}
\]

with

\[
j(x) = \int_x^\infty \frac{|x'|^{1/2} N_{\frac{1}{3}} (y')}{W \left( |x'|^{1/2} J_{\frac{1}{3}} (y'), |x'|^{1/2} N_{\frac{1}{3}} (y'), x' \right)} \cdot (-\text{sgn}(x') \Lambda) \, dx'
\]

\[
n(x) = \int_x^{\infty} \frac{|x'|^{1/2} J_{\frac{1}{3}} (y')}{W \left( |x'|^{1/2} J_{\frac{1}{3}} (y'), |x'|^{1/2} N_{\frac{1}{3}} (y'), x' \right)} \cdot (-\text{sgn}(x') \Lambda) \, dx'
\]

We work out the Wronskian and have then:

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\[ j(x) = \frac{\pi}{3} \lambda \int_{-\frac{1}{3}}^x |x'|^{1/2} N_{\frac{1}{3}}(y') \, dx' \quad (34) \]

\[ n(x) = -\frac{\pi}{3} \lambda \int_{-\frac{1}{3}}^x |x'|^{1/2} J_{\frac{1}{3}}(y') \, dx' \quad (35) \]

We note that these integrals converge at \( x = 0 \) and also at \( x = \pm \infty \); Furthermore, the integrands are even functions of \( x' \). If we therefore put:

\[ j(-\infty) = B_j, \quad n(-\infty) = B_N \]
\[ j(\infty) = B'_j, \quad n(\infty) = B'_N \]

we can write:

\[ B'_j = B_j + \frac{2\pi}{3} \lambda \int_0^\infty |x|^{1/2} N_{\frac{1}{3}}(y) \, dx \quad (38) \]

\[ B'_N = B_N - \frac{2\pi}{3} \lambda \int_0^\infty |x|^{1/2} J_{\frac{1}{3}}(y) \, dx \quad (39) \]

Inside the integration interval there is now a monotonic relationship between \( x \) and \( y \). We can therefore introduce \( y \) as an integration variable and we arrive at the simpler expressions:

\[ B'_j = B_j + \frac{2\pi}{3} \lambda \int_0^\infty N_{\frac{1}{3}}(y) \, dy \quad (40) \]

\[ B'_N = B_N - \frac{2\pi}{3} \lambda \int_0^\infty J_{\frac{1}{3}}(y) \, dy \quad (41) \]
Gröbner and Hofreiter [8] give the integrals. We quote their equation 521 1):

\[
\int_0^\infty J_\nu (x) \frac{dx}{x^k} = \frac{\Gamma \left( \frac{\nu - k + 1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{\nu + k + 1}{2} \right)} \quad , \quad -\frac{1}{2} < k < \Re(\nu) + 1
\]  

For \( k = 0 \), \( \nu = \frac{1}{3} \), the condition is fulfilled and we have:

\[
\int_0^\infty J_{\frac{1}{3}} (x) \ dx = 1
\]  

Their equation 521 2) is:

\[
\int_0^\infty N_\nu (x) \ dx = -\tan \frac{\nu\pi}{2} \quad , \quad |\Re(\nu)| < 1
\]  

For \( \nu = \frac{1}{3} \) we have:

\[
\int_0^\infty N_{\frac{1}{3}} (x) \ dx = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}
\]  

The values of the integrals are substituted into (40), (41) and we have:

\[
\frac{B'}{J} = B - \frac{2\pi}{3\sqrt{3}} \ A
\]  

\[
\frac{B'}{N} = B - \frac{2\pi}{3} \ A
\]  

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The constants \( B_j, B_n, B'_j, B'_n \), enter into expressions like (30):

\[
P = |x|^{1/2} \left\{ B_j \frac{J}{\frac{1}{3}}(y) - B_n \frac{N}{\frac{1}{3}}(y) \right\}, \quad x \ll 0 \tag{48}
\]

\[
P = |x|^{1/2} \left\{ B'_j \frac{J}{\frac{1}{3}}(y) + B'_n \frac{N}{\frac{1}{3}}(y) \right\}, \quad x \gg 0 \tag{49}
\]

Eq. (21) defines \( p \) as

\[
p = \frac{1}{x} \left( \frac{\partial \varphi}{\partial x} - A \right) \tag{50}
\]

For \( |x| \) large it is a good approximation to drop the perturbation term \( A \). Then it is straightforward to show that with:

\[
\vartheta = |x| \left\{ A_j \frac{J}{\frac{1}{2}}(y) - A_n \frac{N}{\frac{1}{2}}(y) \right\}, \quad x \ll 0 \tag{51}
\]

\[
\vartheta = |x| \left\{ A'_j \frac{J}{\frac{1}{2}}(y) + A'_n \frac{N}{\frac{1}{2}}(y) \right\}, \quad x \gg 0 \tag{52}
\]

the relationships between \( A' \)'s and \( B' \)'s are:

\[
\begin{pmatrix}
A_j \\
A'_j
\end{pmatrix} = \begin{pmatrix}
S, -C \\
C, S
\end{pmatrix} \begin{pmatrix}
B_j \\
B'_j
\end{pmatrix} \tag{53}
\]

\[
\begin{pmatrix}
A_n \\
A'_n
\end{pmatrix} = \begin{pmatrix}
S, -C \\
C, S
\end{pmatrix} \begin{pmatrix}
B_n \\
B'_n
\end{pmatrix} \tag{54}
\]

where

\[
S = \sin \frac{\pi}{6} = \frac{1}{2}, \quad C = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \tag{55}
\]
Using now the relationships (41), (42) between primed and unprimed B's we find:

\[
A'_J = A_J + \frac{2\pi}{3\sqrt{3}} A
\]

\[
A'_N = A_N - \frac{2\pi}{3} A
\]

This is an exact solution of the problem (the problem dealt with in [2], in contrast, obliged us to use the unperturbed solution in the perturbing term and so only gave a first approximation result).

THE BLOW-UP

In order to get an estimate of the relative blow-up of the longitudinal phase space, we need some kind of information on the unperturbed bunch size. The piece of information which we shall use is the value given for the unperturbed bunch length at transition \( \hat{\theta}_0(0) \) at p. 15 in ref. [1].

\[
\hat{\theta}_0(0) = \pm 0.127 \text{ R.F. radians}
\]

So far we have only considered the motion of a single particle, starting with a certain \( A_J \) and \( A_N \), ending with a certain \( A'_J \) and \( A'_N \). As in ref. [2] equations (1.74), let us define two quantities \( A \) and \( \psi \):

\[
A_J = A \cos \psi, \quad A_N = A \sin \psi
\]

It is shown in ref. [2] that the particles at the outline of a non-oscillating bunch all have the same \( A = \hat{A} \), but various values of \( \psi \). Equation (2.18) of ref. [2] gives the relationship between this \( \hat{A} \) and the unperturbed bunch length at transition:

\[
\hat{\theta}_0(0) = \frac{3^{2/3}}{\pi} \Gamma\left(\frac{2}{3}\right) \hat{A}
\]

This can also be seen from eq. (19 b) of the present paper.

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Imagine now a non-oscillating bunch well before transition. The particles have all values of $A$ between 0 and $\hat{A}$, and all possible values of $\psi$, see Fig. 1.

**FIG. 1**

Before transition

Two particles with the same $A$ and $\psi$ may have different values of $\hat{A}$, therefore their $A'$ and $\psi'$ after transition will not be the same. Let us assume that the synchrotron oscillations and the betatron oscillations are uncorrelated, so that for every $A$, $\psi$ there is a spectrum of $\hat{A}$'s ranging from $-\hat{A}$ to $+\hat{A}$, where $\hat{A}$ is independent of $A$ and $\psi$. The phase-space area occupied by particles after transition will then be the race-track shaped area shown in Fig. 2.
FIG. 2

After transition

The maximum value of \( A' = \left( A_j^2 + A_N^2 \right)^{1/2} \) for any particle is:

\[
\hat{A}' = \hat{A} + \sqrt{\left( \frac{2\pi}{3\sqrt{3}} \hat{A} \right)^2 + \left( -\frac{2\pi}{3} \hat{A} \right)^2} = \hat{A} + \frac{2\pi}{3\sqrt{3}} \hat{A}
\]  

(61)

The blow-up ratio has no unequivocal meaning: one could for instance take the ratio between areas, between maximum particle densities in A-space, in phase-space (that is, in \((\theta, p)\)-space) or in \(\theta\)-space, etc. In order to calculate particle densities we need information on the detailed spectra of A's
and A's. As this information is not available, let us simply define the blow-up ratio as:

$$D = \frac{\hat{A}}{\hat{A}}$$

(62)

this probably give a reasonable estimate for the other ratios as well. From (60) and (61) we then have:

$$D = 1 + \frac{4\pi \sqrt{3}}{9} \frac{\hat{A}}{\hat{A}}$$

$$= 1 + \frac{4 \Gamma \left( \frac{2}{3} \right)}{3^{5/6}} \frac{\hat{A}}{\hat{\theta}(0)} \approx 1 + 2.168 \frac{\hat{A}}{\hat{\theta}(0)}$$

(60)

THE CHANGE IN CIRCUMFERENCE

Take the machine to be linear, so that betatron oscillations of a particle are symmetrical about its equilibrium orbit. (sextupoles could alter this).

Let us first look at the vertical oscillations. They are roughly of the form:

$$Z = A_Z \left[ 1 + w \cos(M\varphi) \right] \cos(Q\varphi + \varphi_o)$$

(64)

where

Q is the betatron number $\approx 6.25$ for the CPS,

w is the relative wiggle amplitude $\approx 0.15$,

M is the number of magnet periods = 50,

$\varphi$ is the azimuth angle,

$\varphi_o$ is an arbitrary phase,

$A_Z$ is the vertical betatron amplitude, mean of F and D.
The element of orbit length is:

\[
dC = \left[ (R \, d\varphi)^2 + (dZ)^2 \right]^{1/2}
\]

\[
= R \left[ 1 + \frac{1}{2} \left( \frac{1}{R} \frac{dZ}{d\varphi} \right)^2 \right] d\varphi + \text{higher order terms}
\]

(65)

Dropping the higher order terms, the total circumference is:

\[
C + \Delta C = \int_0^{2\pi} R \left[ 1 + \frac{1}{2} \left( \frac{1}{R} \frac{dZ}{d\varphi} \right)^2 \right] d\varphi = 2\pi R \left( 1 + \frac{1}{2} \frac{1}{R^2} \left< \left( \frac{dZ}{d\varphi} \right)^2 \right> \right)
\]

(66)

Because \( Q \) is not integer, we take the average circumference of many successive orbits.

From (64) we get:

\[
\left( \frac{dZ}{d\varphi} \right)^2 = \frac{\pi^2}{4} Z^2
\]

\[
\left\{ \begin{array}{l}
Q^2 \sin^2 (Q\varphi + \varphi_0) \\
+ \frac{U^2}{4} (M + Q)^2 \sin^2 \left[ (M + Q)\varphi + \varphi_0 \right] \\
+ \frac{U^2}{4} (M - Q)^2 \sin^2 \left[ (M - Q)\varphi - \varphi_0 \right] \\
+ Qw (M + Q) \cdot \frac{1}{2} \left\{ - \cos \left[ (M + 2Q)\varphi + 2\varphi_0 \right] + \cos \left[ M\varphi \right] \right\} \\
+ Qw (M - Q) \cdot \frac{1}{2} \left\{ - \cos \left[ M\varphi \right] + \cos \left[ (2Q - M)\varphi + 2\varphi_0 \right] \right\} \\
+ \frac{U^2}{4} (M + Q) (M - Q) \cdot \frac{1}{2} \left\{ - \cos \left[ 2M\varphi \right] + \cos \left[ 2Q + 2\varphi_0 \right] \right\}
\end{array} \right.
\]

(67)
This looks rather complicated, but using

\[ \left\langle \sin^2 x \right\rangle = \left\langle \cos^2 x \right\rangle = \frac{1}{2} \]

\[ \left\langle \sin x \right\rangle = \left\langle \cos x \right\rangle = 0 \]

we get simply

\[ \left\langle \left( \frac{dZ}{d\phi} \right)^2 \right\rangle = \frac{a^2}{Z} \cdot \frac{1}{2} \left\{ Q^2 + \frac{1}{2} (Mw)^2 + \frac{1}{2} (Qw)^2 \right\} \]

With the values of \( Q, M, \) and \( w \) substituted we have

\[ \left\langle \left( \frac{dZ}{d\phi} \right)^2 \right\rangle \approx 3.43 \frac{a^2}{Z} \]

The quantity we need is the relative circumference change \( \Delta C/C \):

\[ \frac{\Delta C}{C} = \frac{1}{2} \frac{1}{R^2} \left\langle \left( \frac{dZ}{d\phi} \right)^2 \right\rangle \approx 17.15 \left( \frac{a^2}{R} \right)^2 \]

For radial betatron oscillations the result is the same. There are additional terms in the integrand, but the first-order ones (in \( y \) and \( \frac{dy}{d\phi} \)) average out and the extra second order ones are negligible. With radial as well as vertical betatron oscillations present we have:

\[ \frac{\Delta C}{C} \approx 17.15 \frac{a^2 + b^2}{R^2} \]

The numerical coefficient must be regarded as an approximate estimate, because (64) is not completely accurate. If a better figure is needed one would probably make numerical integrations of orbit length on a computer. It is also possible to evaluate the effect of betatron oscillations on the revolution frequency by way of a Hamiltonian that includes the longitudinal/transverse coupling terms [9], but it is not easy to apply this method to an A.C. machine.
NUMERICAL RESULTS

We shall now put numbers into the formulae. The numbers we need are all relatively accurately known, except the betatron amplitudes $A_Z$ and $A_R$. As the effect depends upon the square of these numbers, they ought to be relatively accurately known as well. In order to have an easy way of improving the calculation when more accurate values are known, let us just put

$$
\left( \frac{A_R^2}{A_R^2 + A_Z^2} \right)^{1/2} = 10 \text{ mm ,}
$$

(74)

as a standard value which is at least of the correct order of magnitude, and scale the answer afterwards. As $R = 100 \text{ m}$, we get with this standard value

$$
\frac{\Delta C}{C} = 1.715 \times 10^{-7}
$$

(75)

which is a rather small value.

This is to be substituted into (13) in order to have a numerical value of $\Lambda$. We have

$$
h w_s = 2\pi r_{RF} = 2\pi \times 9.42 \times 10^6 \text{ s}^{-1} = 5.919 \times 10^7 \text{ s}^{-1}
$$

(76)

The time constant $T$ giving the duration of the transition region is (ref. [1])

$$
T = 1.35 \times 10^{-3} \text{ s}
$$

(77)

We then find:

$$
\Lambda = 1.378 \times 10^{-2}
$$

(78)

The reason why a very small $\Delta C/C$ can give a relatively large $\Lambda$ deserves some comment: during the time in which the R.F. system turns one radian, a perturbation of $\theta$ is produced; this perturbation is equal to $\Delta C/C$ R.F. radians. But the transition lasts for a time $T$ which is long compared with the time in which the R.F. system turns one radian, in fact $1.095 \times 10^2$ times longer, and $\Lambda$ gives the perturbation in $\theta$ during the time $T$. 
We assume that the beam control reacts at some kind of average particle; so let us put:

\[ \hat{\Lambda} = \frac{1}{2} \Lambda = 9.39 \times 10^{-3} \]  

(79)

This number is to be substituted into (63) together with (58). We find then

\[ D = 1.160 \]  

(80)

for the standard value (74).

As already said, the betatron amplitudes \( \tilde{A}_R \) and \( \tilde{A}_Z \) are not very accurately known. According to Y. Baeconnier [10] the best values are:

\[ \tilde{A}_R = 8.8 \text{ mm}, \quad \tilde{A}_Z = 7.1 \text{ mm} \]  

(81)

which gives

\[ \left( \tilde{A}_R^2 + \tilde{A}_Z^2 \right)^{1/2} = 11.3 \text{ mm} \]  

(82)

With these numbers we get

\[ D = 1 + 0.160 \left( \frac{11.3}{10.0} \right)^2 = 1.204 \]  

(83)

When the bunch rotates in phase-space, bunch-length oscillations will result, and on the pick-up stations one will observe a signal which is indicated, very roughly, in Fig. 3. The signal will oscillate between 100% and about 80% of what it would be with this effect and other perturbations absent. Therefore, the average reduction in signal strength is about 10%.
CONCLUSION

The coupling between betatron oscillations and synchrotron oscillations causes an apparent blow-up of the bunches if one only looks at the longitudinal phase-space, with a resulting loss of particle density. The average reduction of phase-space density is of the order of 10%. This is probably somewhat less than the blow-up observed in the C.P.S., but even so, it is too large to be neglected.

Experiments are in preparation.
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