Abstract

A class of $G$-invariant Einstein-Yang-Mills (EYM) systems with co-

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Einstein-Yang-Mills-systems
A certain class of
1 Introduction

In recent years there has been a lot of interest in static globally regular finite energy solutions (so called soliton solutions) and black hole solutions to Einsteins equations with Yang-Mills-, Higgs- and other nonlinear sources. In 1988 Bartnik and McKinnon [1] found a numerical solution to the $SU(2)$ Einstein-Yang-Mills (EYM) system. This was quite surprising, because neither the vacuum-Einstein [2] equations, nor the pure Yang-Mills [3] equations on Minkowski space have nontrivial soliton solutions. Hence, the very weak gravitational interaction can change qualitatively the spectrum of soliton solutions of a theory in which gravity was neglected. Other authors discovered numerically black hole solutions [4] to the $SU(2)$ EYM-system and the existence of both types of solutions was established rigorously [5]. Unfortunately, all solutions to EYM-systems with arbitrary gauge group turned out to be unstable in the sense of linear stability [6]. Subsequently several authors have investigated other models, such as $SU(2)$ EYM-Higgs [7], EYM-dilaton [8] or Einstein-Skyrme systems [9] and found in some cases linearly stable solutions. Another line of research is to look for solutions of the EYM-equations in arbitrary spacetime dimension, which are invariant under some symmetry group $K$ [10]. Then one has the possibility to apply the powerful theory of dimensional reduction and spontaneous compactification [11, 12]. Some authors [13, 14, 15] discussed an analytical $SU(2)$ EYM-system in $(3 + 1)$ dimensions with cosmological constant $\Lambda$. The aim of this paper is to show that this EYM-system is a special case of a general construction of EYM-systems with arbitrary gauge group.

The paper is organized as follows: In section 2 we discuss a general construction, which yields on each homogeneous space $G/H$ a $G$-invariant EYM-system with cosmological constant. If $G/H$ is a symmetric space, we can use this solution to obtain a static $G$-symmetric EYM-system on the spacetime $\mathbb{R} \times G/H$. This is shown in section 3. In section 4 we apply the construction to an arbitrary Lie group $\Gamma$, considered as a symmetric space and in section 5 we discuss the special case $\Gamma = SU(2)$. We recover the known analytical solution mentioned above [13, 14, 15]. Within our geometric framework we are able to give an intrinsic proof that the configuration under consideration is of sphaleron type. In the last section we show that this EYM-system appears also in the theory of parallel transport along mixed states.

2 General construction

We consider a semisimple compact Lie group $G$ and an arbitrary subgroup $H \subset G$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ resp.. On $\mathfrak{g}$ let their be given a positive definite $Ad(G)$-invariant bilinear form $\gamma$, for instance the negative of the Killing form $K$. The
form $\gamma$ defines by left transport a $G$-biinvariant metric $g$ on $G$

$$
g(l_g'X, l_g'Y) = \gamma(X, Y), \quad X, Y \in \mathfrak{g}. \tag{1}$$

We denote by $l_g$ and $r_g$ the left resp. right multiplication in the group $G$ with group element $g$. The prime denotes the corresponding tangent map, i.e. $l_g'X \in T_gG$. In terms of the canonical left invariant Lie-algebra-valued 1-form $\theta$ on $G$ we can write symbolically

$$
g = \gamma(\theta, \theta). \tag{2}$$

Now we consider $G$ as a principal bundle over $G/H$ with structure group $H$ and canonical projection $\pi : G \rightarrow G/H$. Obviously $g$ is invariant under the right action of the structure group $H$. Therefore, we can use the fact that every $H$-invariant metric on an $H$-principal bundle $P$ with base space $M$ defines and is defined by, three geometrical objects: A connection $\Gamma$ in the bundle $P$, a metric $g_M$ on the base space $M$ and, for every point $x \in M$, an $r_H$-invariant metric on $E_x$, the fiber over $x$ [12].

We discuss these geometrical objects in our case. Let $P = G$, $M = G/H$ and $g$ be the $H$-invariant metric on $P$. We define for every $g \in G$ the horizontal subspace $\text{Hor}_g$ of the tangent space $T_gG$ as the orthogonal complement of the canonical vertical subspace $\text{Ver}_g \subset T_gG$ of the bundle $P = G$ with respect to the metric $g$. The vertical subspace $\text{Ver}_g$ is obviously given by $\text{Ver}_g = l_g'\mathfrak{h}$. Thus, from (1) and the $\text{Ad}(H)$-invariance of $g$ we see

$$
\text{Hor}_g = l_g'\mathfrak{m},
$$

where $\mathfrak{m} \subset \mathfrak{g}$ is the orthogonal complement of $\mathfrak{h}$ with respect to $\gamma$. The decomposition

$$
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{4}
$$

is because of the $\text{Ad}(H)$ invariance of $\gamma$ reductive, i.e. $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The horizontal subspaces $\text{Hor}_g$ define a connection $\Gamma$, which is known as the canonical connection in the bundle $G(G/H, H)$ [16]. It’s connection form $\omega$ is given by the $\mathfrak{h}$-component of $\theta$ with respect to the decomposition (4):

$$
\omega = \theta_\mathfrak{h}. \tag{5}
$$

The metric $g_{G/H}$ on the base space $M = G/H$ can be obtained in the following way: Take two tangent vectors $U_1, U_2 \in T_x(G/H)$ at the point $x \in G/H$ and lift them to horizontal tangent vectors $\hat{U}_1, \hat{U}_2 \in \text{Hor}_g$ at an arbitrary point $g \in E_x$, i.e. $\pi'(U_i) = U_i, i = 1, 2$. We put

$$
g_{G/H}(U_1, U_2) := g(\hat{U}_1, \hat{U}_2). \tag{6}
$$

This definition of $g_{G/H}$ is, because of the $r_H$-invariance of $g$, independent of the point $g \in E_x$. If we have a (local) section $s : G/H \rightarrow G$, then $g_{G/H}$ reads locally

$$
g_{G/H} = \gamma\left(\left(s^*\theta_\mathfrak{m}\right), \left(s^*\theta_\mathfrak{m}\right)\right), \tag{7}
$$
where $\theta_\mathcal{M}$ is the $\mathcal{M}$-component of $\theta$ and the star denotes pull back. The $r_H$-invariant metric on $E_x$ is given by restriction of $g$ to $E_x$. Due to the $l_G$-invariance of $g$, every such a metric on $E_x$ defines and is defined by the same $Ad(H)$-invariant scalar product $\gamma_\mathcal{H}$ on $\mathfrak{h}_1$, namely

$$\gamma_\mathcal{H}(V_1, V_2) = g(l^*_g V_1, l^*_g V_2),$$

where $V_1, V_2 \in \mathfrak{h}_1$ and $g \in G$ and $\gamma_\mathcal{H}$ is the restriction of $\gamma$ to the subspace $\mathfrak{h}_1 \subset \mathfrak{g}$. One more consequence of the $l_G$-invariance of $g$ is that the connection $\Gamma$ and the metric $g_{G/H}$ are invariant under the left action of $G$, compare with equations (5) and (7).

Conversely, if we have a connection $\Gamma$ with connection form $\omega$ in the bundle $G(G/H, H)$ and a metric $g_{G/H}$ on $G/H$, both not necessarily $l_G$-invariant, and if we have an $Ad(H)$-invariant scalar product $\gamma_\mathcal{H}$ on the Lie algebra $\mathfrak{h}_1$ then these three geometrical objects define a $r_H$-invariant metric $g$ on $G$ by [17]

$$g(X, Y) := \gamma_\mathcal{H}(\omega(X), \omega(Y)) + g_{G/H}(\pi'(X), \pi'(Y)), X, Y \in T_e G. \quad (9)$$

In the next step we will write down the scalar curvature $R_G$ of the Levi-Civita connection on $G$ in terms of $\omega$, $\gamma_\mathcal{H}$ and $g_{G/H}$. The $Ad(H)$-invariant scalar product $\gamma_\mathcal{H}$ determines a bi-invariant metric on $H$, similarly as in equation (1). The scalar curvature of $H$ calculated with respect to this metric is constant and is denoted by $R_H$. The curvature $\Omega$ of the connection $\Gamma$ is given by

$$\Omega = D \omega = d \omega + \omega \wedge \omega. \quad (10)$$

If we choose a local coordinate system $(x^\mu)$ on $G/H$ and a local section $s : G/H \to G$ we can write

$$F = s^* \Omega = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu,$$

where $F_{\mu\nu}$ takes values in $\mathfrak{h}_1$. Note that the quantity $\gamma_\mathcal{H}(F_{\mu\nu}, F^{\mu\nu})$ considered as a function on $G/H$ is independent of the section $s$ and the coordinate system $(x^\mu)$. Now the scalar curvature $R_G$ reads, see [12, 11]:

$$R_G = \pi^* (R_{G/H}) + R_H - \frac{1}{4} \pi^* (\gamma_\mathcal{H}(F_{\mu\nu}, F^{\mu\nu})), \quad (12)$$

where $R_{G/H}$ denotes the scalar curvature on $G/H$ and $\pi^*$ the pull back under $\pi$. Equation (12) looks very simple, but this is due to the $Ad(H)$-invariance of $\gamma_\mathcal{H}$ and the $l_G$-invariant construction of the metric in the fibers $E_x$, see equation (9). In general the splitting of $R_G$ is much more complicated.

It is clear that $R_G$ is constant on each fiber $E_x$. Therefore we can integrate the Einstein action

$$S = \int_G (R_G - \Lambda_1) \, dv_G \quad (13)$$
over the fibers $E_x$ and we get

$$S = V_H \int_{G/H} \left( R_{G/H} - (\Lambda_1 - R_H) - \frac{1}{4} \gamma_8 \left( F_{\mu\nu}, F^{\mu\nu} \right) \right) dv_{G/H} .$$

(14)

Here $dv_G$ resp. $dv_{G/H}$ denote the volume forms on $G$ resp. $G/H$, $V_H$ is the volume of the structure group $H$, which is equal to the volume of each fiber $E_x$ and $\Lambda_1$ has the meaning of a cosmological constant. Equation (14) gives the action of a coupled Einstein-Yang-Mills-system on $G/H$ with cosmological constant $\Lambda = \Lambda_1 - R_H$.

Hence, we arrive at the following result. If the metric $g$, given by (9), is a solution of the Einstein equations with cosmological constant $\Lambda_1$, then the metric $g_{G/H}$ and the connection $\Gamma$ form an Einstein-Yang-Mills-system with cosmological constant $\Lambda = \Lambda_1 - R_H$. This is clear, because every variation of $g_{G/H}$ and $\omega$ yields a variation of $g$. But $g$ is a solution of a variation principle with action (13). Therefore $g_{G/H}$ and $\omega$ are solutions of a variation principle with action (14).

It is a known fact that the bi-invariant so called Killing metric $g_K$ on a semisimple compact Lie group $G$ arising from the negative of the Killing form $K$, see equation (1), is a solution of the Einstein equations with some cosmological constant. One can easily verify this statement by calculating the Ricci tensor $Ric$. Doing this one gets [16]

$$Ric = \frac{1}{4} g_K .$$

(15)

Another consequence of this equation is that the scalar curvature $R_G$ of $G$ calculated with respect to $g_K$ is

$$R_G = \frac{1}{4} D_G ,$$

(16)

where $D_G$ is the dimension of $G$. If we choose $g = \alpha g_K$, $\alpha > 0$, as the metric on $G$, the Einstein equations on $G$ will be fulfilled. The corresponding cosmological constant is given by

$$\Lambda = \frac{D_G - 2}{4\alpha} .$$

(17)

Thus, we can use the above construction to find $G$-invariant EYM-systems on homogeneous spaces $G/H$ with compact semisimple Lie group $G$.

3 EYM-systems on $\mathbb{R} \times G/H$

Let $G$ be a compact semisimple Lie group with Killing metric $g_K$ and $H$ be a subgroup of $G$, so that $G/H$ is a symmetric space. Then we can construct a $G$-invariant static EYM-system on spacetime $N = \mathbb{R} \times G/H$ using the construction described in the foregoing section, with the first component of $N$ playing the role of time.
We use the Killing metric \( g_K \) on \( G \) to obtain the EYM–system \( (g_{G/H}, \omega_{G/H}) \) on \( G/H \). Moreover, we have the projection \( \rho : N \to G/H \), which projects \( (t, x) \in \mathbb{R} \times G/H \) onto \( x \in G/H \). On \( N \) we consider the static metric
\[
g_N = -dt \otimes dt + \beta \rho^*(g_{G/H}), \quad \beta \in \mathbb{R},
\] (18)
and on the \( H \)-bundle \( (\rho^*G) \) over \( G/H \) the static connection form
\[
\omega_N = \rho^*(\omega_{G/H}).
\] (19)
Because the Yang-Mills equations are fulfilled on \( G/H \), one easily shows that the Yang-Mills equations on \( N \) are independently of \( \beta \) fulfilled, too. It remains to check the Einstein equations. If \( G/H \) is a symmetric space, it is easy to calculate the Ricci tensor \( \text{Ric}_{G/H} \) and the energy-momentum-tensor \( T_{G/H} \) on \( G/H \):
\[
\text{Ric}_{G/H} = \frac{1}{2} g_{G/H},
\] (20)
\[
T_{G/H} = \frac{4 - D_{G/H}}{8} g_{G/H},
\] (21)
where \( D_{G/H} \) is the dimension of \( G/H \). Here we used
\[
(T_{G/H})_{\mu \nu} = \gamma_0(F_{\mu \tau}, F_{\nu}^{\tau}) - \frac{1}{4} (g_{G/H})_{\mu \nu} \gamma_0(F_{\tau \rho}, F_{\rho \tau}^{\rho}),
\] (22)
with the components \( F_{\mu \nu} \) of the field strength given by equation (11).

The Ricci tensor \( \text{Ric}_N \) on \( N \) calculated with respect to \( g_N \) is, because of the simple structure of the metric \( g_N \) (18), given by
\[
\text{Ric}_N = \frac{1}{2} \rho^*(g_{G/H}),
\] (23)
i.e. \( \text{Ric}_N \) has no time components. The energy-momentum-tensor \( T_N \) on \( N \) takes the form
\[
T_N = \frac{1}{2} \rho^*(g_{G/H}) - \frac{D_{G/H}}{8\beta^2} g_N.
\] (24)
Now it is a matter of fine tuning the constants to fulfill the Einstein equations on \( N \). The Einstein equations on \( N \) read
\[
\text{Ric}_N - \frac{1}{2} g_N (R_N - \Lambda_N) = \kappa T_N,
\] (25)
where \( \Lambda_N \) is the cosmological constant and \( \kappa \) is the gravitational constant. Using equations (18), (23) and (24) we calculate
\[
\text{Ric}_N = \frac{1}{2} g_N (R_N - \Lambda_N) = \frac{1}{2} \rho^*(g_{G/H}) - \frac{1}{2} g_N \left( \frac{D_{G/H}}{2\beta} - \frac{\Lambda_N}{2} \right),
\]
\[
= \left( \frac{\rho \Lambda_N}{2} + \frac{2 - D_{G/H}}{4} \right) \rho^*(g_{G/H}) + dt \otimes dt \left( \frac{D_{G/H}}{4\beta} - \frac{\Lambda_N}{2} \right),
\]
\[
\kappa T_N = \left( -\frac{4 - D_{G/H}}{8\beta^2} \right) \rho^*(g_{G/H}) + dt \otimes dt \left( \kappa \frac{D_{G/H}}{8\beta^2} \right)
\]
Hence, we obtain two equations, which are equivalent to
\[ \beta = \kappa, \quad \Lambda = \frac{D_{G/H}}{4\kappa}. \] (26)

Therefore, if the cosmological constant and the gravitational constant are related by equation (26), then the system \((g_N, \omega_N)\) is a solution of the EYM-equations on \(N\).

This solution can also be obtained by solving the equations of spontaneous compactification [11] on \(\mathbb{R} \times G/H\).

4 The symmetric space \((F \times F)/F_{\text{diag}}\)

Let \(F\) be a semisimple compact Lie group with Lie algebra \(\mathfrak{g}\) and Killing form \(K_{\mathfrak{g}}\). We can consider \(F\) as a symmetric space on which \(G = F \times F\) acts transitively by
\[ (f_1, f_2) \ast f := f_1 f(f_2)^{-1}, \quad f, f_1, f_2 \in F. \]
The stabilizer of the unit element \(e \in F\) is given by
\[ F_e = \{(f, f); f \in F\} = F_{\text{diag}} \equiv H \]
and we can write
\[ F \equiv (F \times F)/F_{\text{diag}} = G/H. \]

The canonical splitting of the Lie algebra \(\mathfrak{g}\) of \(G\) into a direct sum of the Lie algebra \(\mathfrak{h}\) of \(H\) and a vector space \(\mathfrak{m}\) is given by
\[ \mathfrak{g} \cong (X, Y) = \left(\frac{X + Y}{2}, \frac{X + Y}{2}\right) + \left(\frac{X - Y}{2}, -\frac{X - Y}{2}\right), X, Y \in \mathfrak{g}, \]
i.e.
\[ \mathfrak{h} = \{(X, X); X \in \mathfrak{g}\}, \quad \mathfrak{m} = \{(X, -X); X \in \mathfrak{g}\}. \] (27) (28)

The Lie algebra \(\mathfrak{g}\) is the direct sum of two semisimple Lie algebras namely \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\). Therefore, the Killing form \(K\) on \(\mathfrak{g}\) is completely defined by \(K_{\mathfrak{h}}\), namely
\[ K((X_1, X_2), (Y_1, Y_2)) = K_{\mathfrak{h}}(X_1, Y_1) + K_{\mathfrak{h}}(X_2, Y_2). \] (29)

One easily obtains that \(\mathfrak{h}\) and \(\mathfrak{m}\) are orthogonal with respect to \(K\).

In the next step we use \(\gamma = -\alpha K, \alpha > 0\), to construct an EYM–system on \(F\) as described in section 2. We denote the projection from \(G = F \times F\) onto the first resp. second component by \(p_1\) resp. \(p_2\) and we write \(\theta_i := p_i \circ \theta_F, i = 1, 2\), where \(\theta_F\) is the canonical left invariant 1-form on \(F\). It is clear that \(G\) has
the structure of a principal bundle over $F$ with structure group $H = F_{\text{diag}}$ and projection $\pi : G \to F$ given by
\[ \pi(f_1, f_2) = f_1(f_2)^{-1}, f_1, f_2 \in F. \tag{30} \]
We choose the following section $s : F \to G$ in this bundle
\[ s(f) := (f, e), f \in F. \tag{31} \]
With these notations the canonical left invariant 1-form $\theta$ on $G$ reads
\[ \theta = (\theta_1, \theta_2) \]
and its projection onto $\mathfrak{g}$ resp. $\mathfrak{m}$ has the form
\[ \theta_{\mathfrak{g}} = \left( \frac{\theta_1 + \theta_2}{2}, \frac{\theta_1 + \theta_2}{2} \right), \tag{32} \]
\[ \theta_{\mathfrak{m}} = \left( \frac{\theta_1 - \theta_2}{2}, -\frac{\theta_1 - \theta_2}{2} \right). \tag{33} \]
Moreover, we get
\[ s^*\theta_{\mathfrak{g}} = \left( \frac{\theta_F}{2}, \frac{\theta_F}{2} \right), \tag{34} \]
\[ s^*\theta_{\mathfrak{m}} = \left( \frac{\theta_F}{2}, -\frac{\theta_F}{2} \right), \tag{35} \]
where we used $pr_1 \circ s = id_F$ and $pr_2 \circ s = e$. Now the gauge potential $A_F = s^*\omega$ and the metric $g_F$, see equations (5) and (7), can be expressed by
\[ A_F = s^*\omega = s^*\theta_{\mathfrak{g}} = \left( \frac{\theta_F}{2}, \frac{\theta_F}{2} \right), \tag{36} \]
\[ g_F = \gamma ((s^*\theta_{\mathfrak{m}}), (s^*\theta_{\mathfrak{m}})) = -\frac{\alpha}{2} K_{\mathfrak{g}}(\theta_F, \theta_F). \tag{37} \]
To derive the last equation one has to take into consideration equations (35) and (29). The scalar product $\gamma_{\mathfrak{g}}$ in the Lie algebra $\mathfrak{g}$ of the structure group is given by the restriction of $\gamma$ to $\mathfrak{g}$. Identifying $H = F_{\text{diag}}$ with $F$ and using equation (29) we get
\[ \gamma_{\mathfrak{g}} = -2\alpha K_{\mathfrak{g}}. \tag{38} \]
Comparing with equation (16), we obtain
\[ R_H = \frac{1}{8\alpha} D_F, \tag{39} \]
where $R_H$ is the scalar curvature of the structure group and $D_F$ is the dimension of $F$. The metric $g$ on $G = F \times F$ fulfills the Einstein equations on $G$ with
cosmological constant \( \Lambda_1 = \frac{D_F - 1}{2} \), see equation (17). Thus, the cosmological constant \( \Lambda \) of the EYM–system consisting of \( g_F \) and \( A_F \) reads

\[
\Lambda = \Lambda_1 - R_H = \frac{3D_F - 4}{8\alpha}.
\]

(40)

The physical EYM-action has the form

\[
S = \int \left( \frac{1}{16\pi\chi} (R - \Lambda) - \frac{1}{4} (F_{\mu\nu}, F^{\mu\nu}) \right) dv,
\]

where \( (.,.) \) is the negative of the Killing form and \( 8\pi\chi = \kappa \) is the gravitational constant, which appears in the Einstein equations (25). It is obtained by dividing the action (14) by \( 2\alpha \) and identifying \( \alpha \) with the gravitational constant \( \kappa \).

Let’s summarize our results. On every compact semisimple Lie group \( F \) exists an \( F \)-symmetric EYM–system with gauge group \( F \), consisting of \( g_F \) and \( A_F \), see equations (37) and (36). The gravitational constant \( \alpha \) and the cosmological constant \( \Lambda \) are related by equation (40) (fine tuning).

We considered \( F \) as a symmetric space. Therefore, we can apply the construction described in section 3 to get an EYM–system on \( N = \mathbb{R} \times F \). We obtain the gauge potential and the metric on \( N \) from equations (19) and (18), where the EYM–system on \( F = G/H \) and the scalar product in the Lie algebra of the structure group \( F \) are given by equations (36) and (37) resp. (38) with \( \alpha = 1 \). The resulting relations between the occurring constants \( \beta, \kappa \) and \( \Lambda_N \) can be obtained from equation (26).

5 The case \( SU(2) \times SU(2)/SU(2) - \) a relation to instantons

For \( F = SU(2) \) it is easy to write down the explicit form of the metric \( g_{SU(2)} \) and the gauge potential \( A_{SU(2)} \) in local coordinates. We parametrize \( SU(2) \) by stereographic coordinates \( z_\beta \)

\[
SU(2) \ni x = \frac{z_1}{|z|^2 + 1} + \frac{z_2}{|z|^2 + 1} i \sigma^\beta, \quad z_\beta \in \mathbb{R}, \beta = 1, 2, 3, |z|^2 = \sum_{\beta=1}^3 (z_\beta)^2, \quad (42)
\]

where \( \sigma^\beta \) denote the Pauli matrices and \( \mathbb{1} \) is the \( 2 \times 2 \) unit matrix. The Killing form \( K_{SU(2)} \) on the Lie algebra \( su(2) \) is given by

\[
K_{SU(2)}(X, Y) = 4tr(XY), \quad X, Y \in su(2). \quad (43)
\]

To get the metric \( g_{SU(2)} \) one has to use equation (37)

\[
g_{SU(2)} = -\frac{\alpha}{2} K_{SU(2)}(\theta_{SU(2)}, \theta_{SU(2)}) = -2\alpha tr \left( \theta_{SU(2)} \otimes \theta_{SU(2)} \right). \quad (44)
\]
Here \( \otimes \) denotes the symmetrized tensor product and in what follows we write \( \bar{x} \) for the hermitian conjugate of \( x \). With \( \theta_{su(2)} = x^{-1}d x = - d(x^{-1})x = - d\bar{x}x \) we get

\[
g_{SU(2)} = -2\alpha tr \left( d\bar{x} \otimes dx \right) = \frac{16\alpha}{(|\vec{r}|^2 + 1)^2} \sum_{\beta=1}^{3} dz^{\beta} \otimes dz^{\beta}.
\] (45)

It is obvious that \( SU(2) \) endowed with this metric is a 3-sphere with radius \( 2\sqrt{\alpha} \).

The gauge potential \( A_{su(2)} = \frac{1}{2}\theta_{su(2)} \) takes an especially simple form, if we perform a gauge transformation

\[
A' = u^{-1}A_{su(2)}u + u^{-1}du
\] (46)

with

\[
u = \frac{\bar{x} - 1}{|x - 1|}.
\] (47)

After a simple calculation we get

\[
A' = \frac{1}{|\vec{r}|^2 + 1}z_{\alpha}dz_{\beta}\varepsilon^{\alpha\beta\gamma}i\sigma_{\gamma} \, ,
\] (48)

where \( \varepsilon^{\alpha\beta\gamma} \) is totally antisymmetric and \( \varepsilon^{123} = 1 \). The cosmological constant \( \Lambda \) follows from equation (40) and has the value

\[
\Lambda = \frac{3D_{F} - 4}{8\alpha} = \frac{5}{2\alpha}.
\] (49)

If we apply the construction described in section 3, then we obtain a static EYM–system on \( N = \mathbb{R} \times S^3 \equiv \mathbb{R} \times SU(2) \). In local coordinates \((t, z_{\alpha})\) this solution reads

\[
A_{N} = \frac{1}{|\vec{r}|^2 + 1}z_{\alpha}dz_{\beta}\varepsilon^{\alpha\beta\gamma}i\sigma_{\gamma}
\] (50)

\[
g_{N} = -dt \otimes dt + \frac{16\kappa}{(|\vec{r}|^2 + 1)^2} \sum_{\beta=1}^{3} dz^{\beta} \otimes dz^{\beta}.
\] (51)

Here \( \kappa \) is the gravitational constant. From equations (38) and (43) we get the scalar product in the Lie algebra of the structure group

\[
\gamma_{su(2)}(X, Y) = -8tr(X, Y), \, X, Y \in su(2)
\] (52)

The cosmological constant \( \Lambda_{N} \) we obtain from equation (26) and \( D_{SU(2)} = 3 \)

\[
\Lambda_{N} = \frac{3}{4\kappa}.
\] (53)

The same results were obtained in [14, 15]. But in these papers the geometric structure of the solution was left in the dark. We hope that our considerations clarified this point completely.

10
In paper [13] there was mentioned a relation of the gauge potential $A_{SU(2)} = \frac{1}{2} \theta_{SU(2)}$ to the BPST instanton solution [18], but only in terms of a local coordinate chart. In the bundle language, this relation looks as follows: Let us consider the principal bundle $P = SU(2) \times SU(2) \rightarrow SU(2)$ as a subbundle of the quaternionic Hopf bundle $P_H$; we show that the connection form $\omega$ on $P$, see equation (5), is the pull back of the instanton connection form $\omega_{inst}$ on $P_H$. The quaternionic Hopf bundle is given by pairs $(a, b)$, $a, b \in \mathbb{H}$ with
\[ \bar{a}a + \bar{b}b = 1 \] (54)
and by the right action of unimodular quaternions
\[ \psi_u(a, b) := (au, bu), \bar{u}u = 1. \] (55)
Here bar denotes quaternionic conjugation. The set of unimodular quaternions is isomorphic to the group $SU(2)$. Therefore, the bundle $P = SU(2) \times SU(2)$ is naturally embedded by a bundle homomorphism $i$ into the bundle $P_H$
\[ i : P \ni (a, b) \rightarrow (\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}b) \in P_H. \] (56)
On $P_H$ the instanton connection is defined by
\[ \omega_{inst} = \bar{a}da + \bar{b}db. \] (57)
Taking into account $\bar{u} = u^{-1}$, for $u$ unimodular, we obtain
\[ i^*\omega_{inst} = \frac{1}{2}(\theta_1 + \theta_2), \] (58)
where $\theta_i = \pi \theta_{SU(2)}, i = 1, 2$ as defined in the foregoing section. Comparing with equations (5) and (32) it is clear that $\omega$ is the pull back of $\omega_{inst}$ under $i$.
This gives us the possibility to calculate the Chern-Simons index $k$ of the gauge potential $A_{SU(2)}$, see equation (36), in a very simple geometrically intrinsic way.
If we represent the base space $S^4$ of the quaternionic Hopf bundle as the set of quaternions plus one point, we can choose the local section
\[ s_I : \mathbb{H} \ni x \rightarrow \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{1}{\sqrt{1 + |x|^2}} \right) \in P_H \] (59)
in the bundle $P_H$. We denote by $A_I$ the instanton gauge potential and by $F_I$ its field strength, i.e. $A_I = s_I^*\omega_{inst}$. Notice that the section $s$ in the bundle $P$, see equation (31), is the restriction of $s_I$ under the embedding $i$. Therefore, $A_{SU(2)}$ is the pull back of $A_I$ under $i$. The embedding $i$ induces an embedding of the base space of $P$ into the base space of $P_H$, which we denote by the same letter $i$. The image $i(M)$ of the base space $M = SU(2)$ of $P$ is an equator of $S^4$. We
denote one of the two half-spheres of $S^4$ whose boundary is $i(M)$ by $N$. Now we can calculate

$$
k = \frac{1}{8\pi^2} \int_{\mathcal{S}U(2)} tr(A_{SU(2)} \wedge dA_{SU(2)} + \frac{2}{3} A_{SU(2)} \wedge A_{SU(2)} \wedge A_{SU(2)})
$$

$$
= \frac{1}{8\pi^2} \int_{\partial N} tr(A_I \wedge dA_I + \frac{2}{3} A_I \wedge A_I \wedge A_I)
$$

$$
= \frac{1}{8\pi^2} \int_N d \left( tr(A_I \wedge dA_I + \frac{2}{3} A_I \wedge A_I \wedge A_I) \right)
$$

$$
= \frac{1}{8\pi^2} \int_N tr(F_I \wedge F_I) .
$$

(60)

One easily checks that the connection $A_I$ is up to gauge transformations invariant under the natural action of $SO(5)$ on $S^4$. Therefore, $tr(F_I \wedge F_I)$ is up to a factor the volume form on $S^4$ and we get

$$
\frac{1}{8\pi^2} \int_N tr(F_I \wedge F_I) = \frac{1}{2} \frac{1}{8\pi^2} \int_{S^4} tr(F_I \wedge F_I) .
$$

The topological index of the basic instanton is 1, hence the Chern-Simons index $k$ of our solution has to be $1/2$. This is an intrinsic proof that the solution found is of sphaleron type. The calculation of the Chern-Simons index in terms of local gauge potentials is much more complicated and may yield incorrect results, if one chooses a singular gauge. Therefore the authors in [14] had to perform a gauge transformation before they had got the correct result.

6 A relation to Berry’s phase and the distance of Bures

The RYM-system $(g_{SU(2)}, A_{SU(2)})$, described in the foregoing section, is well known from the study of parallel transport along $2 \times 2$- density matrices, see [19, 20, 21].

We consider the trivial $U(2)$ principal bundle

$$GL(2, \mathbb{C}) \to GL(2, \mathbb{C})/U(2) \equiv: D_2(2)
$$

(61)

with the projection $\pi$ given by

$$\pi(\omega) := \omega \omega^*, \omega \in GL(2, \mathbb{C}).
$$

Here star denotes hermitean conjugation. The base space $D_2(2)$ consists of all not normalized nonsingular $2 \times 2$ density matrices. On $GL(2, \mathbb{C})$ we have a natural metric $g$, which is invariant under right action of $U(2)$

$$g = \Re tr \left( d\omega \otimes d\omega^* \right).
$$

(63)
Therefore, we obtain by an analogous construction as in section 2 a connection \( A_B \) in the bundle (61) and a metric \( g_B \) on its base space. The connection \( A_B \) was proposed by Uhlmann [20] and it governs parallel transport along mixed states, which is related to the concept of purification of density matrices. The metric \( g_B \) is known as the Riemannian metric which comes from the distance of Bures [22] and it is related to the transition probability between mixed states.

It turns out that the connection \( A_B \) is reducible to a connection [19] in the \( SU(2) \) subbundle \( Q_2 \) defined by

\[
Q_2 := \{ w \in GL(2, \mathbb{C}); det(w) \in \mathbb{R}_+ \}.
\]  

(64)

This corresponds to a simple property of the metric (63): If we consider \( GL(2, \mathbb{C}) \) as a \( U(1) \) principal bundle over \( Q_2 \), then every vector tangent to \( Q_2 \) is orthogonal to the direction of the fiber of that bundle. Therefore, the horizontal subspaces of the connection \( A_B \), defined as the orthogonal complements of the vector spaces tangent to the fibers of the bundle \( GL(2, \mathbb{C}) \to D_2(2) \), are tangent to \( Q_2 \). Another consequence of this property is that we can use the restriction of \( g \) to \( Q_2 \) to construct \( g_B \).

In the next step we restrict the base space of \( Q_2 \) to all normalized density matrices,

\[
tr(\rho) = tr(ww^*) = 1.
\]  

(65)

We denote the resulting \( SU(2) \) bundle by \( \hat{Q}_2 \). Obviously, there is a natural embedding \( j : \hat{Q}_2 \to GL(2, \mathbb{C}) \). On \( \hat{Q}_2 \) we have the pull back \( j^* g \) of the metric (63). We will show that there exists a bundle homomorphism \( f : \hat{Q}_2 \to P = SU(2) \times SU(2) \), such that \( j^* g = f^* g_P \), with \( g_P \) being a multiple of the Killing metric on \( SU(2) \times SU(2) \).

Obviously, every matrix \( w \in gl(2, \mathbb{C}) \) can be uniquely represented in the form

\[
w = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} (a + ib),
\]  

(66)

where

\[
a = x_0 1 + x_i i \sigma^i, \\
b = y_0 1 + y_i i \sigma^i, \quad \alpha = 1, 2, 3, \quad x_i, y_i \in \mathbb{R}.
\]

If \( w \in \hat{Q}_2 \) we have \( det(w) > 0 \) and \( tr(ww^*) = 1 \). Hence, we obtain

\[
0 = \Re(det(w)) = \frac{1}{4} \left( \sum_{i=0}^{3} x_i^2 - \sum_{i=0}^{3} y_i^2 \right) ,
\]  

(67)

\[
0 < \Re(det(w)) = \frac{1}{2} \sum_{i=0}^{3} x_i y_i ,
\]  

(68)

\[
1 = tr(ww^*) = \frac{1}{2} \left( \sum_{i=0}^{3} x_i^2 + \sum_{i=0}^{3} y_i^2 \right) .
\]  

(69)
From equations (67) and (69) we find

$$\sum_{i=0}^{3} x_i^2 = \sum_{i=0}^{3} y_i^2 = 1$$  \hspace{1cm} (70)$$

and therefore \(a, b \in \mathbb{P}^1(\mathbb{R})\). So we can define a map \(f : \hat{Q}_2 \rightarrow P\) according to equation (66). One easily checks that this map is an injective bundle homomorphism and because of (68) the set \(f(\hat{Q}_2)\) is an open subset of \(P\). Thus, for every \((a, b) = p \in f(\hat{Q}_2)\) with origin \(w = f^{-1}(p)\) the tangent space \(T_pP\) is isomorphic to the tangent space \(T_{w}\hat{Q}_2\).

With these remarks it is a matter of a simple calculation to show \(j^*g = f^*g_P\). Let \(X_1\) and \(X_2\) be two vectors in \(T_{w}\hat{Q}_2\) and \((A_1, B_1)\) and \((A_2, B_2)\) their images in \(T_pP\), \(p = (a, b) \in SU(2) \times SU(2)\). Taking into account equations (2), (29), (43) as well as

$$\theta(A_i, B_i) = (a^{-1} A_i, b^{-1} B_i) \in su(2) \oplus su(2), \quad i = 1, 2$$

and using

$$-(a^{-1} A) = (a^{-1} A)^* = A^* a, \quad a \in SU(2),$$

we obtain

$$g_P ((A_1, B_1), (A_2, B_2)) = -4 \alpha \left( \text{tr}(a^{-1} A_1 a^{-1} A_2) + \text{tr}(b^{-1} B_1 b^{-1} B_2) \right)$$

$$= 4 \alpha \text{tr}(A_1 A_2^*) + \text{tr}(B_1 B_2^*))$$

$$= 4 \alpha \Re \text{tr}((A_1 + i B_1)(A_2^* - i B_2^*))$$

$$= 16 \alpha \Re \text{tr}(X_1 X_2^*)$$,  \hspace{1cm} (71)$$

showing \(j^*g = f^*g_P\), if \(\alpha = \frac{1}{16}\).

Hence, Uhlnamns connection reduced to a connection in \(\hat{Q}_2\) and the pull back of the Bures metric to the space of nonsingular normalized density matrices coincide with the EYM–system presented in section 5, see equations (45) and (48).

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