LABORATORIO INTERDISCIPLINARE PER
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**Classical Solutions for a Perturbed
N-Body System**

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0. INTRODUCTION

In this review I shall consider the perturbed N -body system, i.e. a system composed of N point bodies of masses $m_1, \dots, m_N$, described in cartesian coordinates

$$x_1, \dots, x_N, \quad x_k \in R^3 \quad x_k \equiv \{x_{k,1}, x_{k,2}, x_{k,3}\}$$

by the system of equations

$$m_k \ddot{x}_k = g \sum_{1 \leq i < k \leq N} \frac{m_i m_k}{|x_i - x_k|^3} (x_k - x_i) + \nabla_k W(x, t) \quad 0.1$$

where

$$\nabla_{k,m} \equiv \frac{\partial}{\partial x_{k,m}}, \quad m = 1, 2, 3$$

The potential W will be sufficiently well behaved at infinity (e.g. $\lim_{|x| \rightarrow \infty} W(x) = 0$), bounded below, of class C^2 away from the coincidence set

$$\Delta \equiv \{x : x_k = x_h \text{ for some } k \neq h\} \quad 0.2$$

and less singular at Δ than the Newtonian potential.

Further assumptions on W will be stated in Sections 4 and 5 where precise statements will be made.

By classical solution of (0.1) I shall mean solution in the strong sense, namely a map $R \ni t \rightarrow x(t) \in (R^3)^N$ which is twice differentiable in t and solves Newton's equations for the given potential.

I will describe recent progress on the following two problems

i) P_T

Existence (and multiplicity) of periodic solutions of (0.1) of preassigned period T

ii) P_E

If W does not depend on time, existence and multiplicity of solutions of (0.1) with prescribed energy E and tracing closed orbits.

As an intermediate step, I shall also consider the same problems but now accepting weak solutions, i.e. solutions for which collisions can occur. A precise definition will be given later.

Under such general assumptions on W very few results about problems P_T or P_E are available through O.D.E. methods (I shall call this approach "local" to distinguish it from the variational approach which will be the main concern of this review), even if one makes very strong regularity assumptions on W and accepts weak solutions.

In **Section 1** I shall briefly recall the variational approach, both for preassigned period and for preassigned energy. The purpose of this section is to establish notation and provide a general setting for the analysis which will be given in the later Sections.

Existence, uniqueness and multiplicity results will be described in Sections 4 and 5.

It should be remarked that, within the variational approach, most of the results available so far on the problems P_T and P_E refer to the two-body case; we shall see that this is due to the particularly simple structure of the $N = 2$ case for $W = 0$.

In **Section 2** I will describe therefore the 2-body problem when $W = 0$. The resulting dynamical system is the Keplerian two-body problem.

All periodic solutions of this system are known since the times of Newton, and therefore a complete answer to problems P_T and P_E is known; in fact it is known that all bounded solutions are periodic, that a solution is periodic precisely if it has negative energy, and the answer to problems P_T and P_E is that there are infinitely many such solutions for any choice of T and of $E < 0$.

There are also weak solutions, i.e. solutions which admit collisions (a precise definition of weak solution will be given later). A subclass of collision solutions can be obtained as limits of classical solutions (this leads to the Levi-Civita regularization of the system).

A complete description of the variational structure of the Kepler problem can therefore be given. This description of the Kepler problem, in a prospective which is coherent with the analysis carried out in the following Sections, should be useful to understand some of the features which will be found in the general case.

In particular, I shall compare in this Section, for the case $W = 0$, the variational structure of (0.1) with the one which is obtained if one adds in (0.1) a (potential) force $\delta \nabla V_S(x)$, $\delta > 0$ where V_S is defined by

$$V_S(x) = \sum_{1 \leq h < k \leq N} \frac{1}{|x_h - x_k|^2} \quad 0.2$$

It corresponds to an *attractive* force which is named "strong force" since it dominates the Newtonian one near the coincidence set.

The new system is then described by the system of equations

$$m_k \ddot{x}_k = g \sum_{i < k} \frac{m_i m_k}{|x_i - x_k|^3} (x_k - x_i) + \nabla_k [W(x, t) + \delta V_S(x)] \quad 0.1_\delta$$

Also for this system the complete description of the flow (the collection of all solutions) is available for $W = 0$; now not all bounded solutions give rise to closed orbits. This is due to the fact that the added potential eliminates a degeneracy of the Kepler system, i.e. the existence of an integral of motion independent of those which are due, via Noether's theorem, to invariance under the euclidean group.

Since a complete description of the flow is available for $\delta \geq 0$ and $W = 0$ one can, in the particular case of the Kepler problem, discuss in detail the limit $\delta \rightarrow 0$. This will provide

hints to what one can expect in the general case; in particular we shall see that only a subset of the collision solutions for the Kepler system can be obtained as limit of solutions of the strong force problem.

Remark that the addition of a strong attractive force leads to collisions for a larger set of initial data (some data with angular momentum greater than zero originate now a collision solution) but, as we shall see, it provides a better structure for the variational problem.

In the variational approach an upper bound on the index of a critical point of a suitable functional can constitute a valuable information on the properties of the solution of the corresponding Euler equations. We shall see in Section 3 examples of this fact. For this reason I shall compare in Section 3, both for Hamilton-type and for Maupertuis-type functionals, the index of the circular solutions with or without a strong force.

Notice that when $W \neq 0$ the O.D.E. methods are ineffective for problems P_T and P_E also for $N = 2$ and even under smoothness assumptions.

One of the main challenges for the variational approach is at present to extend to $N \geq 3$ the results known for $N = 2$; this will require a better understanding of the structure of the flow (the collection of all solutions) and in particular the behaviour of those trajectories which pass very close to the coincidence set.

In view of this in **Section 3** we shall discuss the N -body problem for $N \geq 3$ and $W = 0$, both from the O.D.E. point of view and from the variational one.

As we shall see, this problem is far less understood than the $N = 2$ counterpart; still, a partial O.D.E. answer to problems P_T and P_E is known for $W = 0$ since the times of Euler and Lagrange, and for every T and $E < 0$ infinitely many classical solutions can be explicitly constructed.

Now however these solutions make up only a negligible subset of the flow, so that one does not have an answer to the question of whether there are other initial data which provide a solution to P_T and P_E .

We will describe in some detail the behaviour of the collision solutions, since it leads to the solution of problem P_T and is likely to be of importance also in the solution of P_E . This will lead us into a short analysis of the "central configurations" and of their role in the description of the asymptotic behaviour at collisions. Several questions are still open in this branch of celestial mechanics, most dating back to pioneering work by, among others, Poincaré, Sundman, Van Zeipel and Painlevé. Only recently progress has been made, mostly on the lines of an important paper by Mc Gehee, especially in the solution of the "infinite spin problem" (which we shall discuss in this Section), but much work remains to be done. A modern and complete analysis of the behaviour at collisions can be found in [Mc.1],[M.1]. Central configurations are studied in detail in [M.2].

Also for $N \geq 3$ and $W = 0$ one can obtain a partial description of the flow of (0.1) $_{\delta}$; one can study the convergence for $\delta \rightarrow 0$ of (some) periodic solutions of (0.1) $_{\delta}$ to solutions of (0.1) and the relation between the corresponding indices. This will give clues to the strategy to follow in order to prove corresponding results for the general case.

In **Section 4** I shall provide a brief survey of the results available so far on problems P_T and P_E if weak solutions are accepted. A more extensive treatment can be found in the monography [A,CZ.1] and in the lecture notes [CZ.1].

As explained in that Section the Action functional for the N -body problem is not directly suited for an analysis through variational methods, since it is continuous but not everywhere differentiable in its domain of definition; in particular it is not differentiable at weak solutions of (0.1) which have collisions. These solutions can still be regarded as critical points in a suitably generalized sense [M,W] but then not all critical points need be solutions of

(0.1) in any reasonable way.

In order to avoid this difficulty a family of modified functionals is introduced [B,R]. The modification consists in adding to the potential a term δV_S where V_S is given in (0.3)

The modified functionals are differentiable in their domain of definition, and the critical points are classical solutions of (0.1) $_\delta$.

Solutions of (0.1) are then obtained by taking limits, in a suitable topology, when $\delta \rightarrow 0$, of critical points x_δ^0 of the modified problems. Solutions of (0.1) which can be obtained in this way will be called *weak solutions* [B,R], [A, CZ.1]. Although these weak solutions will in general have collisions, they possess some regularity properties, in particular their collision set has open dense complement in $[0, T]$; moreover, if the system is conservative, their energy takes the same value whenever it is defined. All *O.D.E.* results, in particular continuability past a point in the collision set, refer to weak solutions in this sense.

In favourable cases this allows to prove that some of the weak solutions have no collisions, i.e. are classical solutions of (0.1).

A procedure which is alternative to the introduction of a strong attractive force, and leads to the same regularity results for the limit periodic solution, has been introduced recently by Ambrosetti and Struwe in [A, S]. Here one considers a sequence of "regularized" problems, obtained by smoothing the potential near the coincidence set. The solutions of the modified variational problem are therefore strong solutions, and one can take advantage of the particular regularization chosen to prove convergence to a (possibly weak) solution of P_T and P_E

For both approaches I shall focus on those results which have proved to be important, or I expect to be important, in the search of a solution of P_T or P_E .

In the final **Section 5** I discuss progress and results about problems P_T and P_E . I shall prove existence of strong solutions for every period T under condition on the potential W which are very mild and natural, with the exception of one: that the potential be symmetric with respect to reflections through the origin.

This assumption of symmetry is made in order to restrict the variational problem to the subset of trajectories which satisfy

$$x(t + \frac{T}{2}) = -x(t), \quad \forall t \in R \quad 0.3$$

where T is the period. I shall call them *antiperiodic*.

On these trajectories the sublevel sets of the modified Action functional are compact due to Poincaré's inequality, and this guarantees the existence of an absolute minimum (one of the assumptions on W is that it be bounded below). Local estimates at collisions are then used to prove that at least one of the trajectories on which Φ attains its minimum value must be a strong solution ([D], also [S T] for $N = 3$).

The proof requires a further assumption $(H')_M$, which relates to the structure of the set of central configurations for any subset of M bodies. If for the given choice of masses $(H')_M$ is satisfied, there is a strong solution of P_T which has no M -body collisions.

Condition $(H')_M$ is satisfied for $M \leq 4$ irrespectively of the choice of masses, and is expected to be satisfied for almost all choices of masses when $M \geq 5$. In fact, no collision is known for which $(H')_M$ is not satisfied for every M ; the general validity of condition $(H')_M$ is one of the open problems in celestial mechanics.

As for Problem P_E , a full answer is lacking at present.

For $N = 2$ using a uniform estimate on the index of the approximating solutions of P_E^S and good estimates on the behaviour of these solutions at minimal distance from the collision

set, one can prove existence of a generalized solution which has at most one collision [B][T]. The analysis of the Kepler problem given in Section 1 shows that in this case the limit solution, if it has a collision, has index ≥ 2 and this value exceeds by one the index of the approximating solutions.

It is not known whether one can prove a similar relation between indices under the assumptions we have made on the potential.

In the case $N \geq 3$, under suitable assumptions on W weak solutions can be obtained both for preassigned energy and for preassigned period (without the symmetry requirement), as limits of classical solutions x_δ^0 of $(0.1)_\delta$. The behaviour of x_δ^0 near the collision set has not been controlled for $N \geq 3$, and in fact it is likely to be erratic [M. 3] (reflecting the erratic behaviour at collision for the Newtonian problem, described in Section 3). This has prevented so far to derive information on the index of the limit solution also in cases when one can prove that x_δ^0 , $\delta > 0$ has index ≤ 1 . Therefore even an upper bound on the number of M -body collisions is not known at present for $M \geq 3$.

As a final remark, one should notice that the entire analysis can be carried out if instead of the Kepler potential one were to use a potential of the form

$$\sum_{1 \leq h < k \leq N} \frac{m_h m_k}{|x_h - x_k|^\alpha}, \quad 0 < \alpha < 2 \quad 0.5$$

and a corresponding requirement that W be less singular than this potential at the coincidence points.

For every α in $(0, 2)$ one obtains the same results as in the Newtonian case $\alpha = 1$. Moreover, for $N = 2$ and $\alpha \in (1, 2)$ one can improve the result described above, and prove without symmetry requirement existence of at least one solution of P_T . [T]

In Sections 2 we shall very briefly discuss also the case $\alpha \in (0, 1) \cup (1, 2)$; the flow is here poorly understood, in particular only a very small subset of bounded solutions are periodic. In this respect the flow resembles more to the flow for $\alpha = 1$ in presence of a strong force. The only periodic solutions which can be easily described are the ones which are constructed from planar central configurations (relative equilibria); for them the direct study of the limit of strong-force periodic solutions does not differ in a substantial way, both for the procedure and for the results, from the Kepler case.

Also in the general case treated in Sections 4 and 5 the analysis I give does not differentiate in most instances between the values of $\alpha \in (0, 2)$. I shall restrict myself to the Kepler case, and will mention briefly those stronger results which are specific to other values of α .

1. VARIATIONAL SETTING FOR NEWTON'S EQUATIONS

In this Section we shall review briefly the variational formulation of Newton's equations. We will not discuss here existence but only recall the correspondence between solutions of Newton's equation for a potential force and critical points of suitable functionals. More details and comments on this correspondence can be found in any textbook of Analytic Mechanics.

Consider a newtonian N -body system, with masses $m_1, \dots, m_N$, assume that all the forces are potential, and denote the total potential by

$$V(x, t) \quad x \equiv \{x_1, \dots, x_N\}, \quad R^3 \ni x_k \equiv \{x_{k,1}, x_{k,2}, x_{k,3}\} \quad 1.1$$

Newton's equations for this system are

$$m_k \ddot{x}_{k,\alpha} = \frac{\partial V(x)}{\partial x_{k,\alpha}}, \quad k = 1, \dots, N, \quad \alpha = 1, 2, 3 \quad 1.2$$

which will be written in the abbreviated form

$$M\ddot{x} = \nabla V(x) \quad 1.3$$

The diagonal matrix M will be called *mass matrix*.
Newton's equation can be written in the equivalent form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{k,\alpha}} = \frac{\partial L}{\partial x_{k,\alpha}}$$

or, in abbreviated form

$$\frac{d}{dt} \nabla_2 L(x, y) = \nabla_1 L(x, y), \quad y \equiv \dot{x} \quad 1.4$$

where the suffix 1 (resp 2) indicates that the gradient is taken with respect to the first (resp. second) set of variables. By definition the Lagrangian $L(x, y)$ of the system is

$$L(x, y, t) = \Theta(y) + V(x, t), \quad \Theta \equiv \frac{1}{2} \sum_{k=1}^N m_k y_k^2 = \frac{1}{2} (y, My) \equiv \frac{1}{2} \langle y, y \rangle_M \quad 1.5$$

The function $\Theta(\dot{x})$ will be called kinetic energy.

If the potential V does not depend on time the system is called conservative and the following quantity E , called *energy* (or *total energy*) is a constant of motion

$$E(x, \dot{x}) \equiv \Theta(\dot{x}(t)) - V(x(t)) \quad 1.6$$

The function $-V(x, t)$ is often denoted by the symbol $U(x, t)$ and called *potential energy*.
A function from R (or an interval $I \subset R$) to $(R^3)^N$ which is of class C^2 and for which (1.3) is identically satisfied will be called a *strong* or *classical* solution of Newton's equation (local, if it is defined only for $t \in I$).

We shall be mostly concerned with periodic solutions; for problem P_T the potential is assumed to be T -periodic.

It will be convenient for the following analysis to introduce, following [A,CZ 1], a subclass of generalized solutions, which will be called *weak solutions*.

We shall always assume that the potential V is of class C^2 outside a closed set Δ_V which we call *singular set*. For the Newtonian potential this set is the coincidence set Δ given in (0.2)

If $x(t)$ is a solution of (1.3) we denote by $K_{x(\cdot)}$ the *collision set*

$$K_{x(\cdot)} \equiv \{t \in R : x(t) \in \Delta\} \quad 1.7$$

DEFINITION

The function $x^0(t)$ will be called *weak solution* of (0.1) if the complement of $K_{x^0(\cdot)}$ in $[0, T]$ is open and dense, x^0 is of class C^2 outside K_{x^0} and, if V is time-independent, there is a constant E_0 such that

$$E(x^0(t), \dot{x}^0(t)) = E_0 \quad \forall t \notin K_{x^0(\cdot)} \quad 1.8$$

The constant E_0 will be called *energy* of the weak solution. ///

Classical solutions, in particular periodic ones, have a variational description. We consider first the case of periodic solutions of assigned period T .

Denote by Λ_T^V the following subset of T -periodic functions with values in $(R^3)^N$

$$\Lambda_T^V \equiv \{x(t+T) = x(t), \ x(t) \in H^1((0,T), R^{3N}), \ \int_0^T V(x(t))dt < +\infty\} \quad 1.9$$

and consider on Λ_T^V the function

$$\Phi_T(x(\cdot)) \equiv \int_0^T [\Theta(\dot{x}(t)) + V(x(t))]dt \quad 1.10$$

which we shall call *Action functional*.

Recall that, if Φ_T is of class C^1 , a critical point is one at which the differential vanishes. It is well known, and easy to verify, that if the potential $V(x)$ is of class C^1 then also Φ_T^V is of class C^1 and its critical points are precisely the T -periodic (classical) solutions of (1.2).

Indeed, under the assumption made on V one can verify, through an integration by parts (justified by the smoothness of V) that

$$D\Phi_{x^0} = 0 \Leftrightarrow \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{k,\alpha}} - \frac{\partial L}{\partial x_{k,\alpha}} \right] |_{x(t)=x_0(t)} = 0, \quad \alpha = 1, 2, 3, \quad k = 1, \dots, N$$

The potentials we consider have singular sets, so that in general Φ_T is not of class C^1 everywhere. For example, in the Newtonian N -body problem (in fact, even in the two-body problem) Φ_T is continuous but not differentiable at weak solutions which have a collision.

The definition of critical point has to be suitably extended [W M] if one wants to regard these weak solutions as critical points of Φ . We will see a concrete example of this in the collision solutions for $W \equiv 0$.

In this more general setting, C is a critical value of the functional Φ_T if the topology of the sublevel set $\{x(\cdot) : \Phi_T(x(\cdot)) \leq c\}$ changes at $c = C$ and $x(t) : \Phi_T(x(\cdot)) = C$ is a critical point (at level C) if a change in topology can be "localized" at $x(t)$.

It is important to notice that if $x^0(t)$ is a T -periodic solution there may exist an integer K such that $x^0(t)$ is $\frac{T}{K}$ periodic. If no such K exists we shall say that T is the *minimal period* of x^0 .

We turn now to a brief description of the fixed energy case; here W does not depend on time.

In order to characterize those solutions of (1.3) which have prescribed energy E one introduces a different functional Ψ_E , suggested by the structure of the functional Ψ_E^0 studied by Maupertuis

$$\Psi_E^0(x(\cdot)) \equiv \int_0^1 \sqrt{E + V(x(t))} \sqrt{\Theta(\dot{x}(t))} dt \quad 1.11$$

Notice that this functional is invariant under reparametrization, so the choice of the interval of integration is not relevant.

The functional Φ_E^0 had great historical importance, due to the strong resemblance with the functional "optical lenght" on light rays in geometrical optics, if one interprets $\sqrt{E + V(x)}$ as refraction index and $\sqrt{\Theta(\dot{x}(t))}dt$ as "infinitesimal lenght" of a curve (i.e. as riemaniann structure).

However the functional Ψ_E^0 can be defined only for those trajectories for which

$$E + V(x(t)) \geq 0 \quad \forall t$$

and is therefore defined in a subset which has a boundary; this leads to difficulties in a rigorous existence proof.

A functional which serves the same purpose and presents fewer difficulties is

$$\Psi_E(x(\cdot)) \equiv \int_0^1 \Theta(\dot{x}(t))dt \cdot \int_0^1 (E + V(x(t)))dt \quad 1.12$$

defined on

$$\Lambda_1 \equiv \{x(t), x(t) \in H^1(R, R^{3N}), \quad x(t+1) = x(t), \quad \forall t \quad \int_0^1 V(x(t))dt < +\infty\}$$

Also this functional is reparametrization-invariant.

By Leibnitz's rule it is easy to verify that if the potential V is of class C^1 then Ψ_E is of class C^1 .

We shall recall here the simple proof that, under this condition, there is a one-to one correspondence between classical solutions of (1.2) with energy E and critical points of Ψ_E at which Ψ_E takes a strictly positive value.

The proof follows the same lines of the corresponding proof for the Maupertuis functional Ψ_E^0 which can be found in textbooks of Analytic Mechanics.
Set

$$\omega_0^2 \equiv \frac{\int_0^1 [E + V(x^0(t))]dt}{\int_0^1 \Theta(\dot{x}^0(t))dt} \quad 1.13$$

LEMMA 1.1

Let $x^0(t)$ be a critical point of Ψ_E at which $\Psi_E \in C^1$ and suppose that $\Psi_E(x^0) > 0$. Let ω_0 be a solution of (1.13). Then $x_E^0(t) \equiv x^0(\omega_0 t)$ is a classical periodic solution of (1.2) with energy E and period $\frac{1}{|\omega_0|}$.

Proof

Since Ψ is differentiable and x^0 is a critical point one has, for all $w(t) \in H^1([0, T].R^{3N})$

$$\begin{aligned} 0 &= D\Psi_{x^0} w = \langle \dot{w}, \dot{x}^0 \rangle \cdot \int_0^1 [E + V(t)]dt + \int_0^1 \Theta(\dot{x}^0(t))dt \cdot \int_0^1 (\nabla V(x^0(t)), w(t))dt \\ &= \int_0^1 \Theta(\dot{x}^0(t))dt \left[\omega_0^2 \int_0^1 (\dot{w}(t)\dot{x}^0(t)) + \int_0^1 (\nabla V(x^0(t)), w(t))dt \right] \end{aligned} \quad 1.14$$

From $\Theta(\dot{x}^0(t)) \geq 0$ and $|\omega_0| > 0$ it follows

$$\omega_0^2 \int_0^1 \langle \dot{w}(t), \dot{x}^0(t) \rangle_M + \int_0^1 \nabla V(x^0(t)) dt = 0 \quad 1.15$$

(see (1.5) for the definition of $\langle, \rangle_M$.)
Therefore x^0 is a 1-periodic solution of

$$\omega_0^2 M \ddot{x}(t) + \nabla V(x(t)) = 0 \quad 1.16$$

From (1.16) it follows that the function

$$E_0(t) \equiv \frac{1}{2} \omega_0^2 \langle \dot{x}^0(t), \dot{x}^0(t) \rangle_M - V(x_0(t))$$

has zero derivative at all times for which $\dot{x}_0(t) \neq 0$. Since $V \in C^1$ one has $E_0(t) = E_0 \forall t$. Setting $x_E^0(t) \equiv x^0(\omega_0 t)$ one verifies that indeed $x_E^0(t)$ is a strong periodic solution to (1.3) with energy E and that its period is $\frac{1}{|\omega_0|}$. ///

REMARK 1

Since the system (1.2) we are considering is time-reversal invariant, the two solutions of (1.13) differ only by the sign of ω_0 and are conjugate under time reversal.

REMARK 2

The converse of Lemma 1.1 is also true, as one sees by noticing that if x^0 is a strong periodic solution of (1.2) with period T , then energy is conserved, so that (1.16) holds for the function $x_E^0(t)$ defined by

$$x_E^0 = x^0\left(\frac{t}{\omega_0}\right)$$

where ω is defined as in (1.13).

From this one derives immediately that x_E^0 is a critical point of Ψ_E .

We also notice that $(D\Psi_E)_{x^0}(x^0) = 0$ i.e.

$$\int_0^1 \|\dot{x}^0(t)\|_M^2 dt \cdot \int_0^1 \left[E + V(x^0(t)) + \frac{1}{2} (\nabla V(x^0(t)), x^0(t)) \right] dt = 0 \quad 1.17$$

In particular, for homogeneous potentials $V(x) = \frac{1}{|x|^\alpha}$ one can verify that E must be negative if $\alpha \in (0, 2)$ while it must be positive if $\alpha > 2$. Moreover only if $\alpha \in (0, 2)$ one has $\omega^2 > 0$.

To conclude this Section, it will be useful to consider more in detail the relation between the two variational principles, and in particular compare the variational structures of the two functionals.

We have already remarked that Ψ_E is reparametrization invariant, and in particular invariant under the change of scale $t \rightarrow T_0 t$, $T_0 > 0$.

But ω_0 is homogeneous of order one under this change of scale and it follows that $x_E^0(t)$ is a critical point of the functional

$$\Psi_E^{T_0}(x(\cdot)) \equiv \int_0^{T_0} \Theta(\dot{x}(t)) dt \cdot \int_0^{T_0} (E + V(x(t))) dt$$

where T_0 is the period of $x_0^E(t)$.

On the other hand, $x_E^0(t)$ is a critical point of Φ_{T_0} , and therefore a critical point for both functionals

There is an interesting relation between the two Hessian forms at x_E^0 , which implies a relation between their indices.

Indeed one has

$$i_{\Psi_{T_0}}(x^0) \geq i_{\Phi_{T_0}}(x^0) \quad 1.18$$

We shall give here a proof of (1.18) only in the case in which both functionals are twice continuously differentiable at x^0 , as will be the case for the N-body systems at classical solutions of Newton's equations.

The same relation holds, now comparing generalized indices, if the functionals are less regular at x^0 .

We shall therefore prove (1.18) in the form

$$D^2\Psi_{x^0} \leq \alpha D^2\Phi_{x^0}, \quad \alpha > 0 \quad 1.19$$

where we have omitted the indices E_0 and T_0 .

Define the two functionals

$$\Xi(x(\cdot)) \equiv \frac{1}{2} \int_0^{T_0} (E_0 + V(x(t))) dt, \quad \Sigma(x(\cdot)) \equiv \int_0^{T_0} \Theta(\dot{x}(t)) dt$$

which under our assumption are both of class C^2 at x_0^E .

One has by construction (writing now $\tilde{x}$ for $x_E^0(\cdot)$)

$$\Phi(\tilde{x}) = \Xi(\tilde{x}) + \Sigma(\tilde{x}) - T_0 E, \quad \Psi(\tilde{x}) = \Xi(\tilde{x}) \Sigma(\tilde{x})$$

From

$$D\Xi_{\tilde{x}} + D\Sigma_{\tilde{x}}, \quad \Sigma(\tilde{x}) D\Xi_{\tilde{x}} + \Xi(\tilde{x}) D\Sigma_{\tilde{x}} = 0 \quad 1.20$$

one derives

$$\Sigma(\tilde{x}) = \Xi(\tilde{x}) \quad 1.21$$

since $D\Xi_{\tilde{x}} \neq 0$.

Of course, (1.21) also follows from conservation of energy, since $E + V(x^0(t)) = \Theta(\dot{x}^0(t)) \quad \forall t$.

Under the differentiability assumption stated, one has

$$\begin{aligned} \langle w, D^2\Psi_{\tilde{x}} w \rangle &= \Sigma(\tilde{x}) \langle w, D^2\Xi_{\tilde{x}} w \rangle + \Xi(\tilde{x}) \langle w, D^2\Sigma_{\tilde{x}} w \rangle \\ &\quad + 2 \langle D\Xi_{\tilde{x}}, w \rangle \langle D\Sigma_{\tilde{x}}, w \rangle \quad \forall w \in X \end{aligned}$$

i.e, using (1.20) and (1.21)

$$\begin{aligned} &\langle w, D^2\Psi_{\Xi(\tilde{x})} w \rangle = \\ &\Sigma(\tilde{x}) [\langle w, D^2\Xi_{\tilde{x}} w \rangle + \langle w, D^2(\Sigma_{\tilde{x}} + \Xi_{\tilde{x}}) w \rangle] - 2 \langle D\Sigma(\tilde{x}), w \rangle^2 \quad \forall w \in X \quad 1.22 \end{aligned}$$

This proves (1.15) since by construction the quadratic form Ξ is strictly positive.

One should remark that (1.18) *does not imply* that the inequality (1.18) is strict, nor that if $u_0(t)$ is in the Kernel of $D\Psi_{\tilde{x}}$ then $\langle u, D^2\Phi_{\tilde{x}} u \rangle < 0$.

Therefore even if $D^2\Psi_{\tilde{x}}$ has negative eigenvalues, the corresponding eigenfunctions are not in general related to the Kernel of $D^2\Phi_{\tilde{x}}$.

We shall see an instance of this in Section 2, where we study in detail the two-body problem. In that case, we shall exhibit classical solutions of the (reduced) Kepler problem which are degenerate absolute minima for Φ (in a suitably chosen function space) and mountain-pass solutions for Ψ ; however the kernel of $D^2\Psi$ at this point is also the kernel of $D^2\Phi$.

Before closing this Section some remarks are in order, concerning the role of the minimality of the period.

Let V be independent of time, and let $x^0(t)$ be a weak T -periodic solution of (1.2) with energy E . Since the system is autonomous, $x^0(t)$ is also a nT -periodic solution for any positive integer n , with the same energy E (conservation of energy is a property which holds at each time).

Let x^0 the critical point of Ψ_E one obtains by rescaling time as in Lemma 1.1. By construction then x^0 is T -periodic.

Regard now $x^0(t)$ as a nT -periodic solution of (1.2) and therefore a critical point of the functional Φ_{nT} . Again one can rescale time as in Lemma 1.1 to make it into a critical point x_n^0 of Ψ_E (and therefore into a 1-periodic function); the scaling needed however is now n times bigger, and correspondingly the kinetic energy is scaled by a factor n^2 .

We conclude that

$$\Psi_E(x_n^0) = n^2\Psi_E(x^0)$$

Therefore, while a critical point of Φ_T is also a critical point of Φ_{nT} for every positive integer n , and their energy is the same, the corresponding critical points of Ψ_E form a family on which Ψ_E takes values which increase by a factor n^2 .

This implies in particular that every critical point at level c for Ψ_E has images which are critical points at level n^2c , $n \in \mathbb{Z}$ and represent the same solution of (1.2).

This can lead to an interesting phenomenon, which we will see exemplified in the two-body problem.

One can have a family of functionals Ψ_E^ϵ , $0 < \epsilon < \epsilon_0$ which have critical points x_ϵ^0 such that

$$\lim_{n \rightarrow \infty} \Psi_E^{\epsilon(n)}(x_{\epsilon(n)}^0) = +\infty$$

along a suitable sequence $\epsilon(n)$ with $\lim_{n \rightarrow \infty} \epsilon(n) = 0$, and with the property that the sequence $x_{\epsilon(n)}^0$ converges, in $H_{loc}^1(R, R^{3N})$ and uniformly on compact intervals of time, to a critical point x^0 of the functional Ψ_E^0 obtained by setting $\epsilon = 0$.

In this sense a critical point of Ψ_E^0 can be the limit of a sequence of critical points of the functionals Ψ_E^ϵ in spite of the fact that the value of the "approximate" functionals at the "approximate" critical points increases without bound.

Notice that this degeneracy need not appear in the functional Φ_T . Indeed the sequence considered will be of periodic orbits with minimal period increasing without bound (so that they cannot be critical points of the same fixed-period functional) which converge, in the topology indicated and in particular uniformly over any compact time interval, to a periodic solution of the limit problem with minimal period T .

2. THE KEPLER PROBLEM REVISITED

In this Section we shall review the Kepler problem, pointing out aspects which may be useful in building up an intuition for the more complicated problems ahead.

A complete solution of the Kepler problem was known to Newton. All bounded solutions correspond to negative energy and are periodic, and for every choice of period T or of negative energy $E < 0$ one can find infinitely many solutions.

The fact that all bounded solutions are periodic is due to a degeneracy, and more precisely to the presence of a constant of motion (the Runge-Lenz vector) which is independent of the energy and of the six constant of motion which derive, via Noether's theorem, from the invariance of the lagrangian under the euclidean group of rigid motions in E^3 .

If one restricts itself to relative motion (choosing an inertial reference frame in which the baricentre is at rest) this provides five conserved quantities for the relative motion, which has a phase space of dimension six. Moreover the vector field which describes the flow has no zeroes (there is no rest point). As a consequence, every compact intersection of the level sets of these five conserved quantities must be a closed curve with no rest points, and therefore must be the orbit of a periodic solution.

At a deeper level [M] this degeneracy is due to the fact that, if one considers only non-collision solutions with negative energy, the system is diffeomorphic, as a dynamical system, to the geodesic motion on the Sphere S^3 ; it is therefore invariant under the $O(4)$ group, and so is its completion, which includes now some collision solutions. The new constants of motion reflect this symmetry, via Noether's theorem.

The diffeomorphism requires a rescaling of the time parameter, which becomes singular at the completion orbits. The latter correspond to great circles passing through the south and north pole of S^3 ; in the Kepler problem they correspond to orbits of collision solutions which are periodic, and can be seen as a segment, limit of a sequence of ellipses which have their focus in the origin. The segment has then one of its end points in the origin, and the trajectory runs over it twice, in opposite directions, "in finite time".

This shows that the Kepler problem can be "regularized".

Since the Kepler problem can be described in such great detail, it is not difficult to analyze also its variational structure. We shall do this in this Section, and we shall see that some features emerge which can be of help in understanding the general case.

In Section 4 we shall discuss the role of the addition of a strong attractive force in the general case. It is then wise to see how this affects the variational structure in the Kepler problem. Also, since we shall be interested also in the case of a potential which has a $|x|^{-\alpha}$, $\alpha \in (0, 2)$ singularity at coincidence points, it may be useful to see how the two-body problem for $\alpha \neq 1$ differs from the Kepler case.

We shall see that when a strong force is added or when $\alpha \neq 1$ (or both) the structure of the negative energy solutions is entirely different. Now the $O(4)$ symmetry is absent, and "most" of the bounded solutions are not periodic. Therefore also the variational properties of the Action functional are entirely different.

For $\alpha = 1$ and in presence of the strong force a complete description is still available for any given period T or negative energy E . Therefore one will still be able to describe all critical points of the corresponding Action functional, and in many cases compute their index. The same is true of the fixed (negative) energy solutions.

For $\alpha \neq 1$ the description of the flow is far less complete, and only a simple class of solutions are known, both for preassigned period and for preassigned energy.

We now describe these points in more detail.

We begin with the Kepler problem. The equations of motion are

$$m_1 \ddot{x}_1 = m_1 m_2 g \frac{x_2 - x_1}{|x_1 - x_2|^3}, \quad m_2 \ddot{x}_2 = m_1 m_2 \frac{x_1 - x_2}{|x_1 - x_2|^3}, \quad x_k \in R^3$$

where g is the gravitational constant and m_1, m_2 are the masses of the two bodies. This is a system of second order differential equations which are analytic on the configuration space

$$X \equiv R^6 - \Delta$$

where

$$\Delta \equiv \{x \in X, x_1 = x_2\}$$

Translation invariance of the forces implies that the baricentre moves with constant velocity; without loss of generality one can then take an inertial reference frame in which the baricentre is at rest. We shall do so here and whenever the interaction is invariant under translations.

Introducing the relative coordinates $y \equiv x_1 - x_2$ one has now

$$\ddot{y} = -Mg \frac{y}{|y|^3}, \quad M = m_1 + m_2, \quad y \in R^3 \quad 2.1$$

The system described by (2.1) will be called *reduced system* in accordance with the general theory of reduction of systems with symmetry.

The two sides of (2.1) have different scaling behaviour under the map $y \rightarrow \alpha y$, $\alpha \neq 0$. Therefore, by changing the scale (i.e. the units in which lengths are measured) we can, and will, set $Mg = 1$. The lagrangian of the reduced system is then

$$L(y, \dot{y}) = \frac{1}{2} \dot{y}^2 + \frac{1}{|y|} \quad 2.2$$

Invariance under rotations implies conservation of angular momentum

$$J(y, \dot{y}) \equiv y \wedge \dot{y} = J, \quad J \in R^3$$

and therefore if $J = 0$ all solutions lie on a line through the origin while if $J \neq 0$ they lie in the plane perpendicular to J .

If $J = 0$ the trajectory is of the form

$$y(t) = a(t)\xi, \quad \xi \in R^3, \quad |\xi| = 1$$

where $a(t)$ satisfies

$$\ddot{a}(t) = -\frac{1}{a(t)^2}$$

and this equation has a solution only up to a time $t_0 > 0$ which depends on the initial data and is such that $\lim_{t \rightarrow t_0} a(t) = 0$ (and therefore, by conservation of energy $\lim_{t \rightarrow t_0} |\dot{a}(t)| = +\infty$).

If $J \neq 0$ each trajectory can be described using polar coordinates r, θ in its plane of motion.

Conservation of the component j of angular momentum perpendicular to the plane of motion gives

$$j(r, \dot{\theta}) = r^2 \dot{\theta} = j \in R \quad 2.3$$

where j is a constant (which depends on the trajectory considered) and the equation for r takes then the form

$$\ddot{r} = -\frac{1}{r^2} + \frac{j^2}{r^3} \quad 2.4$$

which is equivalent whenever $\dot{r} \neq 0$ to the equation which describes conservation of energy

$$\frac{1}{2}\dot{r}^2 + \frac{j^2}{2r^2} - \frac{1}{r} = E \quad 2.5$$

If $j \neq 0$ equation (2.4) has a global solution for all initial data and from (2.5) one concludes that, if $j \neq 0$ then $\liminf_{t \in \mathbb{R}} r(t) > 0$.

From the graph of the function $\frac{j^2}{2r^2} - \frac{1}{r}$ one sees that the solution is bounded (and then periodic) only for $E < 0$.

One can use of (2.3) to write (2.5) as

$$\frac{j^2}{2r^2} \left(\frac{dr}{d\theta} \right)^2 + \frac{j^2}{2r^2} - \frac{1}{r} = E \quad 2.6$$

Setting $u \equiv \frac{1}{r} - \frac{1}{j^2}$ equation (2.6) reads

$$\frac{1}{2} \left(\frac{du}{d\theta^2} \right)^2 + \frac{1}{2}u^2 = E + \frac{1}{2j^4} \quad 2.7$$

which is recognized as the law of conservation of energy for an harmonic oscillator with energy $\bar{E} \equiv E + \frac{1}{j^4}$.

Therefore the solution of (2.7) is

$$u(\theta) = u_0 \cos(\theta - \theta_0) \quad 2.8$$

where u_0 and θ_0 depend on the initial conditions.

If one writes (2.8) in terms of $r(\theta)$ one verifies that for $E < 0$ all trajectories span closed orbits which are in fact ellipses with the origin as focus.

Among the solutions of special interest will be the ones which correspond to circular orbits.

If r_0 is the radius, from (2.6) one concludes that r_0 must be a solution of

$$\frac{j^2}{2r^2} - \frac{1}{r} = E$$

and from (2.3) and (2.6) one then derives its period T_{r_0} and its energy E_{r_0} .

In this way one derives

$$r_0 = j^2, \quad T_{r_0} = \frac{2\pi r_0^2}{j} = 2\pi j^3, \quad E_{r_0} = -\frac{1}{2j^2} \quad 2.9$$

One can see from these formulae that for every period T and for every negative energy $E < 0$ (and also for every radius r_0) there is precisely one circular solution.

From (2.9) one can also derive Kepler's laws relating energy to period to radius for circular motions.

One can also prove, by the explicit formulae given above, that for each choice of plane of motion and for each value of T there is a one-parameter family of periodic solutions with (minimal) period T . They can be parametrized by the eccentricity e of their elliptical orbits, with $0 \leq e < 1$. All these solutions have the same energy and same period T of the circular element of the family ($e = 0$).

For each value of T (and of $E < 0$) the family has a limit for $e \rightarrow 1$, pointwise and in $H^1([0, T], \mathbb{R}^2)$. The limit trajectory covers twice (in opposite directions) the limit orbit (a segment with one of its ends in the origin). It is a weak solution and the collision set in $[0, T)$ is a point.

Since all classical periodic solutions are known explicitly and have elliptic orbits, these are the only weak solutions which can be obtained as limits of classical solutions. Other weak solutions of preassigned energy can be constructed for example joining together (in the sense of loops) at collision times two of the solutions described above, with orbits oriented differently in space. Similarly, one can construct a weak solution of minimal period T by joining at collision time two weak solutions of minimal period $\frac{T}{2}$ differently oriented in space. We shall see later that if the two components are oppositely oriented in space the resulting solution can be obtained as limit of classical solutions of the strong force problem. This high multiplicity of periodic orbits for preassigned period or energy is a consequence of the fact that all bounded solutions are periodic; it will not be present, as we shall presently see, if one adds a "strong force" or considers the potential $|x_k - x_h|^{-\alpha}$, $\alpha \in (0, 1) \cup (1, 2)$.

The behaviour near collisions can be more conveniently described if one introduces a new parameter τ defined implicitly by

$$\frac{d\tau}{dt} = \frac{1}{r(t)}, \quad \tau(0) = 0 \quad 2.10$$

Notice that τ is a differentiable monotone function of t on each classical periodic solution. Notice also that, contrary to what happens in the Levi-Civita regularization, the angular motion is not uniform (except for circular solutions). Indeed one has

$$\frac{d\theta}{d\tau} = \frac{j^2}{r(\tau)}$$

However from conservation of energy one can easily verify that, for given $E < 0$ and $j \neq 0$ (i.e. for every classical periodic solution) one has

$$\inf_{t \in \mathbb{R}} r(t) = \frac{1 - \sqrt{1 - 2|E|j^2}}{2|E|}$$

(for classical solutions one has $2|E|j^2 < 1$) so that, for every classical periodic solution

$$\sup_{\tau \in \mathbb{R}} \frac{d\theta}{d\tau} = \frac{2|E|j^2}{1 - \sqrt{1 - 2|E|j^2}}$$

The right-hand side has upper bound 2 and lower bound 1 as function of $|E|j^2$ (the lower bound corresponds to the circular solution, the upper bound is not reached over strong

solutions (it corresponds to the limit in which $|E|j^2 \rightarrow 0$ and therefore, for fixed energy, to collision solutions).

Therefore, for every classical periodic solution one has

$$1 \leq \frac{d\theta}{d\tau} < 2$$

For comparison, remark that when $j \rightarrow 0$, $\sup_t \frac{d\theta}{dt}$ diverges to $+\infty$ and $\inf_t \frac{d\theta}{dt}$ converges to zero; this corresponds to the fact that, at fixed energy and for very small angular momentum the angle varies very fast as a function of time near the two foci of the ellipse and varies very little near the centre.

If the solution has energy $-|E|$ one can also verify that the equation for r takes now the form

$$\frac{d^2 r}{d\tau^2} = 1 + 2Er \quad 2.11$$

which is solved by

$$r(\tau) = \frac{1}{2|E|} \left(1 - e \cos(\sqrt{2|E|}\tau) \right), \quad 0 \leq e < 1$$

where e is the eccentricity. Notice that the limit $e \rightarrow 1$ corresponds to the collision solution and $\tau = 0$ is the collision time.

The collision solution has no singularity in its dependence on τ . In fact, for $e = 1$ and for $\tau \rightarrow 0$ one has

$$r(\tau) \simeq \frac{1}{2}\tau^2 \quad 2.12$$

From (2.10) one derives then the asymptotic behaviour of $r(t)$ at collision time (which is $t = 0$ with the choice of initial data made above in the solution of (2.11)).

$$r(t) = C|t|^{2/3}(1 + O(|t|^\gamma)) \quad 2.13$$

for some $C > 0$ and $\gamma > 0$.

We are now ready to discuss the structure of the critical points of the Action functionals, both of Hamilton and of Maupertuis type, for the Kepler problem.

The Action for the reduced motion, restricted to T -periodic functions, is

$$\Phi_T(y(\cdot)) = \frac{1}{2} \int_0^T \dot{y}^2(t) dt + \int_0^T \frac{1}{|y(t)|} dt \quad y \equiv x_1 - x_2 \quad 2.15$$

defined on

$$\Lambda_T \equiv \{y(t) \in H^1(R, R^3), \int_0^T \frac{1}{|y(t)|} dt < +\infty\} \quad 2.16$$

This functional is positive but not coercive; indeed the sequence

$$y_n(t) \equiv n \xi \quad \forall t, \quad \xi \in R^3 \quad |\xi| = 1 \quad 2.17$$

verifies

$$\lim_{n \rightarrow \infty} \Phi(y_n(\cdot)) = \lim_{n \rightarrow \infty} \frac{T}{n} = 0$$

but has no convergence subsequence.

As a consequence Φ may have no minima; since we know all solutions of Kepler's problem we will verify that indeed Φ has no minima.

Since the lagrangian is symmetric under the map $y \rightarrow -y$ the critical points of Φ which lie in

$$\Lambda_T^a \equiv \{y(t) \in \Lambda_T, \ y(t) = -y(t + \frac{T}{2})\}$$

(we shall call these trajectories "antiperiodic") are precisely the critical points of the restriction of Φ to Λ_T^a .

Indeed, it is immediate to see that

$$y(t) \in \Lambda_T^a, \ h(t) \in \Lambda_T^s \Rightarrow \langle D\Phi_{y(\cdot)}, h(\cdot) \rangle = 0 \quad 2.18$$

where

$$\Lambda_T^s \equiv \{y(t) \in \Lambda_T, \ y(t) = y(t + \frac{T}{2})\}$$

Since every function $g(t) \in \Lambda_T$ can be written uniquely as

$$g(t) = g_s(t) + g_a(t), \quad g_a \in \Lambda_T^a, \quad g_s \in \Lambda_T^s$$

the assertion follows from (2.18).

The same remark applies to minima, even if the functional may not be differentiable there.

Due to Poincaré's inequality, the restriction of Φ to Λ_T^a is coercive and the Palais-Smale condition is satisfied.

The functional has therefore a minimum. We shall indeed verify that it has precisely one minimum, modulo the natural action of time translations (since the system is autonomous) and of rotations and space inversions (since this is a symmetry group of the lagrangian). This minimum corresponds to solutions which are circular and have T as minimal period.

Notice that, once more, since we know all the periodic solutions of the Kepler problem, we also know all the solutions which belong to Λ_T^a . It is easy to verify that among the classical solutions the only ones which satisfy the requirement of being antiperiodic are the circular ones with minimal periods $\frac{T}{2n+1}$, $n \in \mathbb{N}$.

The antiperiodicity condition can be satisfied also by patching together two collision solutions of period $\frac{T}{2}$ which span oppositely oriented segments.

We shall verify that these further solutions (circular with minimal period $\frac{T}{2n+1}$, $n \geq 2$ and antiperiodic collision solutions) are not minima; in particular we shall verify that if $n = 2$ one has a critical point with index 2 and that also the antiperiodic collision solution described above has index two, now in the following generalized sense (Φ is not differentiable at this critical point): there is a two-dimensional space Ξ and a positive constant ϵ_0 such that, denoting by $y^0(t)$ the weak solution considered, one has

$$\Phi(y^0(\cdot) + \epsilon \xi(\cdot)) - \Phi(y^0(\cdot)) < 0 \quad \forall \xi \in \Xi \quad 0 < \epsilon \leq \epsilon_0$$

With little effort one can determine the index of all other critical points in Λ_T^a but we shall not do it here.

We shall now verify these statement; the concrete description we shall achieve of the functional Φ for the Kepler problem will help in the understanding some geometric properties of the Action functional in the more general setting of the Newtonian N -body problem perturbed by the potential W .

From (2.17) one can compute the Hessian of Φ at a critical point $y^0(t)$ at which it is twice differentiable. These are the classical solutions (at weak solutions Φ is not even differentiable). The result is

$$\begin{aligned} \langle w, D^2\Phi_{y^0}w \rangle = & \int_0^T \int_0^T |\dot{w}|^2(t)dt - \int_0^T \frac{1}{|y^0(t)|^3} |w(t)|^2 dt \\ & + 3 \int_0^T \frac{1}{|y^0(t)|^5} (w(t), y^0(t))^2 dt, \quad w(t) \in \Lambda_T \end{aligned} \quad 2.20$$

It is now easy to verify that none of these critical points can be a minimum; indeed if Π is the plane in which the motion takes place, setting

$$w(t) = w^0(t) \equiv \xi \in R^3, \forall t \quad |\xi| = 1, \quad \xi \perp \Pi$$

one has

$$\langle w^0, D^2\Phi_{y^0}(\cdot)w^0 \rangle = - \int_0^T \frac{1}{|y^0(t)|^3} dt < 0$$

Also the weak solutions cannot be minima, since they are critical points which are at the same level as a continuos family of strong solutions, and for the latter the negative eigenvalue of the Hessian is uniformly bounded away from zero; a more direct proof can be given; we shall give it in the more general context of the N -body system for any N in Section 5.

We conclude that indeed the functional Φ has no minima in Λ_T .

We analyze now the restriction of Φ to Λ_T^a . Here we already know from general arguments (which apply for arbitrary N) that at least one minimum exists.

The Hessian is still given by (1.21) but now both y^0 and w must be taken in Λ_T^a .

The only classical periodic solutions in Λ_T^a are a the subset of circular solutions with minimal period $\frac{T}{2n+1}$ for some integer $n \geq 0$ Therefore $|y^0(t)| = r_0$ for some $r_0 > 0$ and one has for every $w \in \Lambda_T$

$$\langle w, D^2\Phi_{y^0}(\cdot)w \rangle = \int_0^T \int_0^T |\dot{w}|^2(t)dt - \frac{1}{r_0^3} \int_0^T |w(t)|^2 dt + 3 \frac{1}{r_0^5} \int_0^T (w(t), y^0(t))^2 dt, \quad 2.21$$

where $y^0 = \cos t \xi_1 + \sin t \xi_2$ and ξ_1 and ξ_2 are two orthogonal vectors in the plane of motion.

By scaling and using Kepler's laws one can restrict attention to the case $T = 2\pi$. By antiperiodicity, only the odd terms will be present in the expansion of $w(t)$ and $y^0(t)$ as Fourier series. In particular, $w(t)$ cannot contain constant terms and the smallest contribution to the Hessian will come from the $\sin t$ and $\cos t$ terms in $w(t)$, since the

term which depends on $\frac{dw}{dt}$ is positive and the *Hessian* is diagonalized by taking Fourier transform.

The last term in (2.21) is positive, and so in order to prove that the Hessian is nonnegative it is sufficient to consider the case

$$\omega(t) \perp y^0(t) \quad \forall t$$

A complete orthonormal basis of functions $w(t) \in \Lambda_T^a$ with this properties is given by

$$\frac{1}{\sqrt{2\pi}} \cos t \xi_3, \quad \frac{1}{\sqrt{2\pi}} \sin t \xi_3, \quad \frac{1}{\sqrt{2\pi}} [\cos t \xi_2 - \sin t \xi_1]$$

where ξ_3 is a unit vector perpendicular to the plane of motion.

Let Σ be the three-dimensional subspace of $L^2((0, 2\pi), \mathbb{R}^3)$ spanned by these functions. If $w(t) \in \Sigma$ one has

$$\langle w, D^2 \Phi_{y^0} w \rangle = \left[1 - \frac{1}{r_0^3} \right] \langle w, w \rangle \quad 2.22$$

Indeed, for any $w \in \Sigma$ one has

$$\int_0^{2\pi} \int_0^{2\pi} |\dot{w}|^2(t) dt = \int_0^{2\pi} \int_0^{2\pi} |w|^2(t) dt$$

We have already determined the radius of the circular orbits on which the motion has period 2π .

From (2.9) one has

$$r_n = \left(\frac{1}{2n+1} \right)^{2/3} \quad 2.23$$

and therefore, for $n = 0$

$$r_0 = 1 \quad 2.24$$

From (2.23) and (2.24) one derives $r_0 = 1$ and therefore, from (2.22)

$$\langle w, D^2 \Phi_{y^0}(\cdot) w \rangle = 0 \quad \forall w \in \Sigma$$

It is worth noticing that the kernel of the Hessian is nontrivial for any autonomous system (it contains always the variation $\dot{x}(t)$, reflecting the invariance of the lagrangian under time translation). A further two-dimensional subspace in the Kernel is due to invariance under rigid rotations.

The kernel of the Action will therefore have dimension at least three in all case in which the system is autonomous and the lagrangian is invariant under rigid rotations, and in particular also when strong forces are added.

It is easy to verify that for the circular solution we are considering, the Hessian is strictly positive in $\Sigma^\perp$, so that Σ coincides with the Kernel of $(D^2 \Phi_T)_{y^0}$.

One can also verify that the value of the functional at the weak solutions in Λ_T^a is greater than the value at the circular orbit with minimal period T . This follows from the degeneracy of the functional on Λ_T and from the fact that in order to be antiperiodic the

weak solution must be made of at least two segments, each of which cannot have minimal period smaller than $\frac{T}{2}$.

From (2.23), (2.24) one also derives that the circular solutions with minimal period

$$\frac{2\pi}{2n+1}, \quad n = 1, \dots$$

have index not smaller than two. A more accurate estimate, using a larger number of test functions with increasing Fourier components, shows that for every $n_0 > 0$ there is a circular solution with index larger than n_0 .

We already know, from general arguments, that a minimum exists; we conclude that the circular solutions with minimal period T are minima of the Action functional on Λ_T^a . This conclusion could have been derived directly by considering the functional Φ on the quotient space modulo rigid rotations and time translations.

We consider next the Maupertuis-type functional which we have introduced in Section 1 to study the problem (P_E) in the general case, and discuss its properties in the particular case of the Kepler system. We shall verify in this particular context that the functional has no minima in Λ_T^a but that the circular solutions with energy $E < 0$ and covering the orbit only once in time T are points of mountain-pass type, and we shall also determine explicitly the eigenvector corresponding to the negative eigenvalue.

There are further critical points, with index ≥ 2 , corresponding to circular solutions of energy E which cover $2n+1$ times the orbit for some integer $n \geq 1$ and further critical points, with generalized index ≥ 2 which correspond to weak solution with energy E .

The Maupertuis-type functional we consider takes here the form

$$\Psi_E(x(\cdot)) \equiv \frac{1}{2} \int_0^1 |\dot{y}|^2 dt \cdot \int_0^1 \left[\left(E + \frac{1}{|y(t)|} \right) dt \right] \quad 2.25$$

Due to the scaling properties of the system there is no loss of generality in considering only the circular solutions of radius one; they correspond, as we have seen above, to angular momentum 1, minimal period 2π and energy $E = -1/2$.

Rescaling time so that the integration is now over the interval $[0, 2\pi]$ we consider the functional

$$\Psi(x(\cdot)) \equiv \frac{1}{2} \int_0^{2\pi} |\dot{y}|^2 dt \cdot \int_0^{2\pi} \left[\left(-\frac{1}{2} + \frac{1}{|y(t)|} \right) dt \right] \quad 2.26$$

defined on $\Lambda_{2\pi}^a$.

Its critical points are solutions of the Kepler problem with energy $-\frac{1}{2}$ in $\Lambda_{2\pi}^a$. We can compute explicitly the Hessian of Ψ at the critical points y_n^0 which correspond to uniform circular motions j_n^0 with minimal periods $\frac{2\pi}{2n+1}$, $n = 0, 1, \dots$. Their radius is $r_n = (2n+1)^{-2/3}$.

The result is (all integrals are over the interval $[0, 2\pi]$)

$$\begin{aligned} \langle w, D^2 \Psi_{y_n} w \rangle &= \int |\dot{w}(t)|^2 dt \int \left(-\frac{1}{2} + \frac{1}{r_n} \right) dt \\ &+ \int |\dot{y}_n^0(t)|^2 dt \int \left(-\frac{|w(t)|^2}{r_n^3} + 3 \frac{(w(t), y_n^0(t))}{r_n^5} \right) dt \end{aligned}$$

$$-2 \int (\dot{w}(t), \dot{y}_n^0(t)) dt \int \frac{(y_n^0(t), w(t))}{r_n^3} \quad w \in \Lambda_{2\pi}^a \quad 2.27$$

We now specialize to the case $n = 0$. We want to show that this critical point is of mountain-pass type and find the eigenfunction corresponding to the negative eigenvalue. Using that $r_0 = 1 = |\dot{y}^0(t)|^2 \forall t$ one can verify that

$$\langle w, D^2 \Psi_{y_0^0} w \rangle = 0 \quad \forall w(t) \in \text{Ker} (D_{y_0^0}^2 \Phi)$$

In fact, due to rotation invariance of Ψ it is sufficient to verify this statement on the function $\xi \cos t$ where ξ is a constant vector perpendicular to the plane of motion. The easy calculation is left to the reader.

Notice that the last term (which is nonpositive) vanishes on this subspace. On the other hand, setting $w(t) = \lambda y_0^0(t)$, $\lambda \neq 0$ so that

$$(w(t), y_0^0(t)) = \lambda |y_0^0(t)|^2 = \lambda, \quad (\dot{w}(t), \dot{y}_0^0(t)) = \lambda |\dot{y}_0^0(t)|^2 = \lambda$$

one derives

$$\frac{2\pi}{\lambda^2} \langle w, D^2 \Psi_{y_0^0} w \rangle = \frac{1}{2} - \frac{1}{2} + \frac{3}{2} - 2 = -\frac{1}{2} < 0$$

An explicit computation shows that the negative eigenspace of $D^2 \Psi_{y_0^0}$ is one-dimensional and therefore spanned by $\lambda y_0^0(t)$, $\lambda \in \mathbb{R}$.

We conclude that y_0^0 is a critical point of mountain-pass type and that y_0^0 generates the negative eigenspace. By Fourier transformation one can also verify that y_0^0 is a degenerate minimum for the restriction of the functional $\Psi_{2\pi}$ to the manifold

$$\mathcal{M}_E \equiv \{y(t) \in \Lambda_{2\pi}^a, \quad g(y(\cdot)) = -E, \quad g(y(\cdot)) \equiv \int_0^1 \frac{1}{2|y(t)|} dt\} \quad 2.28$$

(that all critical points lie on $\mathcal{M}_E$ follows from (1.17)).

This mountain-pass structure, and also the property of being a minimum on $\mathcal{M}_E$ will also be found in the general case of a perturbation of the N-body newtonian system under suitable assumptions on the perturbing potential [A,CZ.1].

We study now the structure of the Kepler problem when one considers an additional *attractive* force inversely proportional to the third power of the distance.

Again one can choose a reference frame in which the baricentre is at rest at the origin. The lagrangian for the reduced system is now

$$L_\epsilon(y, \dot{y}) = \frac{1}{2} |\dot{y}|^2 + \frac{1}{|y|} + \frac{\epsilon}{2} \frac{1}{|y|^2}, \quad y \in \mathbb{R}^3, \quad \epsilon > 0$$

and the corresponding equations are

$$\ddot{y} = -\frac{1}{|y|^3} y - \epsilon \frac{1}{|y|^4} y, \quad y \in \mathbb{R}^3 \quad 2.29$$

For convenience in this Section we shall denote by ϵ the small parameter which multiplies the strong potential, and we shall be interested in the limit $\epsilon \rightarrow 0$.

If $j = 0$ the motion is directional, and leads to a collision in a finite time. A simple scaling argument shows that, if $\epsilon > 0$, the asymptotic behaviour at collision is $y(t) \simeq |t|$ so that $\frac{1}{|y(t)|}$ is not integrable at the origin and periodic trajectories with collisions do not belong to Λ_T . One can in fact also verify that the solutions do not belong to $H^1((0, T), R^3)$. For $j \neq 0$ the motion takes place in a plane, and using polar coordinates conservation of angular momentum reads

$$r^2 \dot{\theta} = j \quad 2.30$$

so that the energy conservation identity takes now the form

$$E = \frac{\dot{r}^2}{2} - \frac{1}{r(t)} + \frac{j^2 - \epsilon}{r^2} \quad f(j, \epsilon) \equiv \frac{\sqrt{j^2 - \epsilon}}{j}$$

and the lagrangian which describes the radial motion is

$$L(r, \dot{r}) = \frac{\dot{r}^2}{2} + \frac{1}{r(t)} - \frac{j^2 - \epsilon}{r^2} \quad 2.31$$

The corresponding motion is asymptotic to the origin if $\epsilon \geq j^2$ (the centrifugal force due to the angular motion is dominated by the added attractive force). Since we will be interested in the limit $\epsilon \rightarrow 0$ we shall consider only the case $\epsilon < j^2$. The limit $\epsilon \rightarrow 0$ will be taken on the manifold of solutions with fixed angular momentum j .

Let then ϵ be smaller than j^2 . If one sets

$$\phi \equiv f(j, \epsilon)\theta, \quad f(j, \epsilon) \equiv 1 - \sqrt{\frac{\epsilon}{j^2}} \quad 2.32$$

one derives from (2.30)

$$r^2 \dot{\phi} = \sqrt{j^2 - \epsilon} \quad 2.33$$

The equations (2.31) and (2.33) describe the Kepler problem in the polar coordinates r and ϕ and here $\sqrt{j^2 - \epsilon}$ is the conserved angular momentum.

Therefore all solutions of (2.31) (2.33) with negative energy are periodic and all bounded orbits are closed and can be described by writing r as a periodic function of ϕ , of period 2π .

As a consequence, in view of (2.32), if $r(\phi)$ is not a constant function, r can be written as a periodic function of θ only if $f(j, \epsilon)$ is a rational number, and the period will be π only if $f(j, \epsilon) = 1$.

But by construction $f(j, \epsilon) < 1$ if $\epsilon > 0$ and $f(j, \epsilon)$ is a continuous function of ϵ .

We conclude that "for most initial data" the solution is not periodic.

Still, all solutions are known since for each initial datum one has explicit expressions for $r(t)$ and $\psi(t)$ (as in the Kepler problem) and therefore for $\theta(t)$.

Since $\frac{d\phi}{d\theta} \equiv f(j, \epsilon)$ is constant along each orbit (for which $j^2 > \epsilon$), but may take different values on different orbits, one describes the motion as a Keplerian motion with precession.

This will allow also in this case to give a rather complete description of the variational problem and in particular of the index of the periodic solutions.

We have seen that at a fixed value of the parameter ϵ the motion on orbits different from the circular ones will be periodic only if

$$1 - \frac{\epsilon}{j^2} = \frac{P}{Q}, \quad P, Q \in \mathbb{Z}^+, \quad Q > P \quad 2.34$$

and the period will be QT if T is the period of the periodic solution for $\epsilon = 0$ which has the same initial data.

From (2.34) one derives

$$j^2 = \frac{\epsilon}{\frac{Q^2}{P^2} - 1} \quad 2.35$$

so that, if Q and P are kept fixed, or even if one places an upper bound on Q (i.e. on the period of the sequence of periodic trajectories one is considering) one has

$$\lim_{\epsilon \rightarrow 0} j(\epsilon) = 0 \quad 2.36$$

This means that, if an upper bound is given to the periods, every subsequence will consist of orbits which have a larger and larger eccentricity, and any limit solution is a collision. On the other hand, if one lets $Q(\epsilon)$ increase, and still wants a T -periodic solution (rather than a QT -periodic one) one must consider a solution of the Kepler problem which has period $\frac{T}{Q}$ and precedes in such a way as to give rise to a closed trajectory after Q Kepler periods.

We know that the diameter of a Kepler orbit is proportional to the period (for fixed angular momentum), and that its energy is inversely proportional to the period. Therefore the sequence of trajectories considered in this case have smaller and smaller diameter, and both kinetic energy and absolute value of energy increase without bound (the energy of the orbits which precedes is approximately equal to the energy of the Keplerian orbit).

It is interesting to describe what happens to the corresponding critical points of the variational functionals.

Consider first the Action, which for $\epsilon > 0$ has the form

$$\Phi_T^\epsilon \equiv \int_0^T \left[\frac{1}{2} \dot{y}^2 + \frac{1}{|y(t)|} + \frac{\epsilon}{2} \frac{1}{|y(t)|^2} \right] dt \quad 2.37$$

As we have already remarked, the behaviour at collisions is now $|y(t)| \simeq t$ and the functional does not take a finite value on these solutions.

The kinetic energy of any sequence of T -periodic solutions y_ϵ^0 of (2.29) for $\epsilon \rightarrow 0$ grows without bound. Since all terms in (2.37) are positive, one has

$$\lim_{\epsilon \rightarrow 0} \Phi^\epsilon(y_\epsilon^0) = +\infty$$

On the other hand, if the period is not fixed but the average kinetic energy is required to go to a limit, the sequence of solutions corresponds to an increasing eccentricity tending to one. Therefore these orbits tend to pass closer and closer to the origin, and it is not a-priori obvious whether the contribution of the last term in (2.37) has a limit when $\epsilon \rightarrow 0$ along this set of solutions, since the integral diverges when $\epsilon \rightarrow 0$.

We shall prove now that if the sequence y_ϵ^0 is made of critical points of Ψ_{MT}^ϵ for some fixed positive integer $M \geq 2$, then the contribution from this term converges to zero, so that the function $\Psi^\epsilon(y_\epsilon^0)$ is continuous at zero.

To prove this assertion, we must provide an accurate description of the sequence we consider.

Consider first the case $M = 1$. For all $\epsilon > 0$ the y_ϵ^0 are circular orbits; and denoting by r_0^ϵ the corresponding radii (which we have determined for any given T in the first part of this Section) it is easy to verify that $\lim_{\epsilon \rightarrow 0} r_0^\epsilon = r_0^0 > 0$. Therefore the integral

$$\int_0^T \frac{1}{(r_0^\epsilon)^2} dt$$

converges to a finite limit when $\epsilon \rightarrow 0$, and due to the proportionality factor ϵ the last term of (2.37) converges to zero.

We have already seen that circular orbits are minima for the action functional for $\epsilon = 0$. A similar proof can be given directly for the case $\epsilon > 0$ since the explicit dependence of the radius on T is known. One could also derive this conclusion from the fact that the index is zero for $\epsilon = 0$ and that the kernel has dimension three and is entirely due to the fact that the lagrangian is invariant under time translation and rigid rotations. Since also the modified lagrangian has these invariances the kernel is the same and the continuity described above implies that there can be no negative eigenvalues (at least for ϵ small enough).

We consider next the case $M \geq 2$. We give details only for $M = 2$; the other cases can be treated similarly. Also, without loss of generality, we consider only the case $T = 2\pi$ so that for $\epsilon = 0$ the energy is $E = -\frac{1}{2}$.

We have already noted that, if the orbit is not circular, for $\epsilon > 0$ the period of the precessing orbits will be $2\pi Q$, where Q is the least integer positive for which

$$1 - \frac{\epsilon}{j^2} = \frac{P}{Q}, \quad P, Q \in \mathbb{Z}^+, \quad P < Q \quad 2.38$$

Since we take $M = 2$ we must have $Q = 2$ and therefore $P = 1$.

This leads to the condition on j

$$j^2(\epsilon) = \epsilon$$

The orbits we are considering have therefore, as mentioned above, an eccentricity which increases to one when $\epsilon \rightarrow 0$ and therefore they pass closer and closer to the origin.

For these orbits, which we denote again by y_ϵ^0 we want to compute

$$c(\epsilon) \equiv \epsilon \int_0^{2\pi} \frac{1}{|y_\epsilon^0(t)|^2} dt$$

It will again be convenient to make use of the time parameter τ . It is easy to verify that one obtains

$$r_\epsilon^0(\tau) = 1 - \sqrt{1 - j^2(\epsilon)} \cos \tau = 1 - \sqrt{1 - \epsilon} \cos \tau \quad 2.40$$

Using the Taylor expansion of $\cos \tau$ near $\tau = 0$ and the rescaling $\tau \equiv \sqrt{\epsilon} \sigma$ one verifies that the integral diverges, when $\epsilon \rightarrow 0$ as $c_0 \epsilon^{-1/2}$ so that

$$\limsup_{\epsilon \rightarrow 0} c(\epsilon) \leq \lim_{\epsilon \rightarrow 0} c_0 \epsilon^{1/2} = 0 \quad 2.41$$

proving our assertion.

Using the same estimates one concludes that

$$(f^\epsilon(t), y_\epsilon^0(t)) \equiv 0 \Rightarrow \lim_{\epsilon \rightarrow 0} (\Xi^\epsilon(y_\epsilon^0 + f^\epsilon) - \Xi^\epsilon(y_\epsilon^0)) = 0 \quad 2.42$$

where we have set

$$\Xi^\epsilon \equiv \Phi^\epsilon - \Phi$$

In particular, if $(f(t), y_\epsilon^0(t)) \equiv 0$ and $\Phi(y^0 + \delta f) < \Phi(y^0)$ for δ sufficiently small, then the index of y^ϵ as a critical point of Φ^ϵ is not smaller than one.

One may also notice that the sequence y_ϵ^0 we are considering has a limit when $\epsilon \rightarrow 0$ in the $H^1([0, T], R^3)$ topology.

In order to describe the limit, it is convenient to remember that a 2π -periodic collision solution of the Kepler problem, limit of strong 2π -periodic strong solutions with orbits of increasing eccentricity, is described in polar coordinates, choosing as reference axis the one along which the collision takes place, by

$$\theta(t) = 0 \quad t \in (0, \pi), \quad \theta(t) = \pi \quad t \in (\pi, 2\pi) \quad 2.43$$

$$r^2 = 1 + \frac{2}{r}$$

The limit trajectory y_0^0 , which is 4π periodic, is then described in polar coordinates by (2.43) and

$$\theta(t) = 0 \quad t \in (0, \pi) \cup (3\pi, 4\pi), \quad \theta(t) = \pi, \quad t \in (\pi, 3\pi) \quad 2.44$$

Notice that this is indeed a weak solution of the Kepler problem, it is antiperiodic and is a generalized critical point of $\Phi_{4\pi}^0$. In fact, the entire sequence y_0^ϵ is made of functions which have minimal period 4π and are antiperiodic.

Therefore they are critical points of the corresponding functionals restricted to antiperiodic functions.

Using the estimates given above, it can be proved that the norm of the differentials of $\Phi_{4\pi}^\epsilon$ at y_ϵ^0 grows without bound (in fact its asymptotic behaviour is $\epsilon^{-1/2}$) and that $\Phi_{4\pi}^0$ is not differentiable at y_0^0 .

One can obtain uniform estimates on the Holder norm of order β for every $\beta < \frac{1}{2}$ and one can also prove that $\Phi_{4\pi}^0$ is Holder continuous of every order less than $\frac{1}{2}$ and that one has convergence of the local Holder norm of $\Phi_{4\pi}^\epsilon$ at y_ϵ^0 to the corresponding local norm of $\Phi_{4\pi}^0$ at y_0^0 .

Convergence in a sequence of neighborhoods is more difficult to obtain although some directions can be controlled, as seen above for vector-valued functions which are at all times perpendicular to $y^0(t)$.

The knowledge of the generalized index can be used to prove that the limit trajectory is in fact a weak solution, and that there are at most two times of collision in $[0, 4\pi)$ (and in fact precisely two, which are $t = 0$ and $t = 2\pi$ in the notations given above).

Of course these statements can be easily verified here from the explicit description of the solution; we have presented them in a more general form since in this setting some of them can be proved in the case of the perturbed 2-body system.

We now consider the functional

$$\Psi_E^\epsilon \equiv \int_0^1 \Theta(\dot{y}(t)) dt \cdot \int_0^T \left(E + \frac{1}{|y(t)|} + \frac{\epsilon}{2|y(t)|^2} \right) dt \quad 2.45$$

which is a variational functional for problem $(P)_E$.

Circular periodic solutions of $(1.1)_\epsilon$ with energy E are again (after rescaling time) critical points of Ψ_E^ϵ .

If one denotes by y_ϵ^0 these solutions and by r_E^ϵ their radius, it is easy to verify that the radius is a continuous function (in fact analytic) of ϵ for every E , and that $y_\epsilon^0(t)$ is continuous in ϵ in the topology of $H^1((0, T), \mathbb{R}^3)$.

From this one derives that the functions Ψ_E^ϵ converge in a neighborhood of y_0^0 .

All these critical points are of mountain-pass type for their functionals, as can be verified directly in the same way as we have done before for the case $\epsilon = 0$. They are also minima for the restriction of the functional $\Psi_{2\pi}$ to the manifold $\mathcal{M}$ defined (see 1.17)

$$\mathcal{M}_E \equiv \{y(t) \in \Lambda_{2\pi}^a, \quad g(y(\cdot)) = -E, \quad g(y) \equiv \int_0^1 \left[V(y) + \frac{1}{2} y \cdot \nabla V(y) \right] dt$$

Notice that $\mathcal{M}_E$ does not depend on ϵ since the strong force potential is homogeneous of degree -2 .

Consider now the critical points of Ψ_E^ϵ which correspond, after rescaling, to periodic solutions of (2.29) which have energy E but are not circular.

From the analysis given above it follows that each of them traces an orbit which is obtained by considering a solution of the Kepler problem with energy E (this orbit is an ellipse with focus in the origin) and making it precede in its plane around the origin with a period which is rational with respect to the period T of the Kepler orbit considered.

If $\frac{P}{Q}$, $P < Q$ is the minimal description of this rational number, QT is the period of the corresponding solution, which we shall denote by $y_{Q,P}^0$.

To compare $\Psi_E^\epsilon(y_{Q,P}^0)$ with $\Psi_E(y^0)$ one must rescale time so that the minimal period of y^0 becomes QT . This gives a contraction of the kinetic energy and leads to

$$\Psi_E(y_{P,Q}^0) = Q^2 \Psi_E(y^0) + O(\epsilon)$$

where y^0 is the periodic circular solution which in its period covers only once its orbit.

We end this Section by a remark on the case $V(y) = |y|^{-\alpha}$, $\alpha \in (0, 2)$, $\alpha \neq 1$.

For this potential the degeneracy of the case $\alpha = 1$ is no longer present, and in this respect our knowledge of the structure of periodic solutions does not change appreciably if one adds a strong force. Again a necessary condition for periodicity of a solution is that its energy be negative, but now this condition is no longer sufficient, and periodic solutions correspond to initial data which have zero Lebesgue measure in phase space.

Therefore one has a much poorer description of the structure of the critical points of the variational functionals; for circular solutions, however, the results (in particular convergence when $\epsilon \rightarrow 0$ in the strong-force case), are the same as for $\alpha = 1$.

3. THE N-BODY PROBLEM

The N -body problem is described in cartesian coordinates by the system of equations

$$m_k \ddot{x}_k = \nabla_k V_K, \quad V_K \equiv \sum_{h < k} \frac{m_h m_k}{|x_k - x_h|} \quad 3.1$$

or, in compact notation

$$M \ddot{x} = \nabla V_K, \quad M \in L(R^{3N}, R^{3N}) \quad 3.2$$

where we have set equal to one the gravitational constant (the suffix K in the potential stands for Kepler; I could not use N for Newton because N is used here to denote the number of bodies in the system).

This system is defined and smooth in the configuration space

$$X \equiv R^{3N} - \Delta$$

where as before the coincidence set Δ is defined by

$$\Delta \equiv \{x : \exists h \neq k, x_k = x_h\} \quad 3.3$$

The system (3.2) is variational, and the corresponding lagrangian is

$$L \equiv \frac{1}{2} \sum_k m_k \dot{x}_k^2 + V(x) \equiv \frac{1}{2} \|\dot{x}\|_M^2 + V(x) \quad 3.4$$

We recall that $\langle, \rangle_M$ is defined by

$$\|x\|_M^2 \equiv \langle x, x \rangle_M, \quad \langle x, y \rangle_M \equiv \sum_k m_k (x_k, y_k) \quad 3.5$$

Since the system is invariant under translations, one can find an inertial reference frame in which the baricentre is at rest at the origin. We shall always make this choice, so that at each time the coordinates will satisfy the identity

$$\sum_k m_k x_k(t) = 0 \quad \forall t$$

For future reference we introduce two second order monomials in the positions and momenta of the bodies; they will play an important role in what follows. These are the moment of inertia I of the system and the angular momentum J , both relative to the baricentre (i.e. relative to the origin of the coordinates).

$$I \equiv \frac{1}{2} \|x\|^2, \quad J \equiv x \wedge M \dot{x} \quad 3.6$$

Standard theorems of ordinary differential equations provide existence of a solution of (3.2), for every initial datum in X , in a maximal interval $[0, t^*]$.

Painlevé [Pa] proved that $x(t)$ approaches Δ for $t \rightarrow t^*$. If the limit of $x(t)$ is a point $x^* \in \Delta$ one says that one has a *collision* at time t^* at the point x^* . If the limit set contains more than one point, one speaks of *pseudocollisions*.

Painlevé [Pa] showed also that for $N = 3$ only collisions can occur; for $N \geq 5$ pseudocollisions are known to occur [Ma Mc], [X.2]. They are related to motions in which at least two clusters, each composed of two or more bodies, collapse to a total collision at time $t = t^*$, the separation among any two of these clusters goes to infinity for $t \rightarrow t^*$ and other particles (or clusters) oscillate indefinitely between two of the separating clusters. The oscillating bodies acquire a kinetic energy which grows without bound for $t \rightarrow t^*$ and this is compensated by a corresponding loss of potential energy in collapsing clusters [X.1].

One can prove [Su], [W] that when $t \rightarrow t^*$ the moment of inertia either diverges or goes to a finite limit. If $I(t)$ is uniformly bounded for $t \rightarrow t^*$ then the singularity is a collision [v.Z]. In this case the N bodies group themselves in one or more clusters; if a cluster contains more than one body, then all the bodies in the cluster collide simultaneously for $t \rightarrow t^*$.

We shall only be concerned with singularities which are collisions. In fact, if $I(t)$ diverges when $t \rightarrow t^* < +\infty$ then the kinetic energy of the system diverges at t^* and the estimates which we shall give below imply that the trajectory is not in $H^1([0, t^*], (R^3)^N)$ and therefore will not appear as critical point of the variational functionals we will consider.

Collision solutions, on the contrary, are in $H^1([0, t^*], (R^3)^N)$ and moreover the potential is integrable along these solutions; the functionals we consider take then a finite value at collision solutions.

It should be remarked that by construction the collision solutions we consider in this Section will be such that the collision set (in time) has open dense complement. More precisely, since one can have $t^* < T$ where T is the prefixed period, T -periodic collision solutions will be obtained by composing (in the sense of paths) a countable number of the maximal solutions described above. For $N \geq 3$ the collision set of a weak solution can have accumulation points; for example, a N -body collision can be the limit of M -body collisions for $M < N$.

We have remarked above that at the maximal time t^* the bodies group themselves in several clusters, each of them exhibiting total collision. If Ω is a partition of the N bodies in K clusters, then the singularity set of the partition Ω is the subset of Δ_Ω defined by the condition that in each element of the partition (i.e. in each cluster) all the bodies are at the same point in space.

We shall only discuss in some detail the case in which the partition is trivial (there is only one cluster). In this case one speaks of a *simultaneous collision*.

In the general case, one can treat separately the behaviour at collision within each cluster [El]; the presence of the bodies in the other clusters plays a marginal role in the structure of the collision, and indeed can be seen as the addition of a nonsingular perturbation to the Newtonian problem relative to the bodies in the cluster considered. We shall make brief comments on this later on; a thorough treatment is given in [El].

For simultaneous collisions, in the reference frame we have chosen the collision takes place at the origin and we shall agree to choose the time parameter so that $t^* = 0$. There is no loss of generality in this, since the system is autonomous.

When several clusters are present, in each of which a simultaneous collision takes place at $t = 0$, there is in general no reference frame in which all the baricentres of the clusters are at rest. One proves that the motion of each baricentre is differentiable so that on the new time scale it becomes very slow; through the implicit function theorem one can then reduce the problem to the case of a simultaneous collision.

The properties of the solution "near $t=0$ " for a simultaneous collision are obtained by "zooming in" at the origin and "slowing up time" so that in the new time parameter it takes infinitely long for the bodies to reach the origin. This will be the purpose of the Mc

Gehee coordinates, which we shall discuss shortly.

From the scaling properties of the equation (under scaling of space and time) one concludes that $I(t) \simeq t^{4/3}$ for $t \rightarrow 0_+$. This suggests defining

$$\xi_k(t) \equiv t^{-2/3} x_k(t), \quad k = 1, \dots, N \quad 3.7$$

One can prove (Lemma 3.2) that there is a constant $C > 1$ (which depends on the mass of the bodies) such that

$$\frac{1}{C} \leq \min_{h < k} \liminf_{t \rightarrow 0} |x_k(t) - x_h(t)| \leq \max_{h < k} \limsup_{t \rightarrow 0} |x_k(t) - x_h(t)| < C \quad 3.7_1$$

This will allow to "blow up" the collision point to a manifold $\mathcal{M}$ (the collision manifold) to which the flow is asymptotic in the rescaled time.

This procedure defines a flow on $\mathcal{M}$; the properties of this flow will provide information on the behaviour at (isolated) collisions and also about the flow of (3.2) *near the collision set* Δ (i.e. information on the behaviour of classical solutions when they come close to a collision). In turn, if x^0 has a collision, this provides an estimate on the index of the approximating x_g^0 .

If the initial value problem is well posed for the flow extended to $X \cup \mathcal{M}$ (i.e. if one has for this extended flow continuity of the solutions with respect to initial data) one says that the system (3.2) *can be regularized* [E].

Only in this case does the behaviour of the flow near collisions bear *a simple relation* to the flow on the collision manifold (and in general to the properties of collision solutions).

An equivalent way of defining a regular system (or rather a regular extension of a system) is to state that one can add to the strong solutions a minimal number of collision solutions, suitably extended beyond the critical time t^* , in such a way that for the the resulting system one has continuous dependence on initial data.

Notice that not all collision solutions (nor all weak solutions) are added to obtain a regular extension, if this exists. As an example, the two-body problem is regularized by adding only bounce collisions (i.e. solutions which trace back their path after collision). Other collision solutions in which the two bodies change direction after collision are not included in the regularized system (although, as we have seen, some of them can be obtained as limit from a strong-force system).

If the extended flow on $X \cup \mathcal{M}$ is not regular, it is much more difficult to derive from properties of the flow on $\mathcal{M}$ informations on the behaviour of non-collision solutions which pass "very near" the singular set Δ . One can for instance consult [M.2] where information on the behaviour near collision is obtained from a careful study of the flow on the collision manifold.

We have seen in Section 2 that the two-body problem can be regularized. This is not the case for $N \geq 3$. One has

STATEMENT 3.1

For every $N \geq 3$ the N -body problem cannot be regularized.

A detailed proof is given in [Mc.1] for $N = 3$ and "most" value of the masses; the proof makes use of the theory of isolating blocks ([C],[C E]). The approach is very instructive because it gives evidence of the irregular behaviour of solutions "near a collision" if one considers a sequence of initial data $x^n(0)$ for trajectories $x^n(t)$ such that

$$\inf_n \inf_{t \in R} \text{dist}(x^n(t), \Delta) = 0$$

This should be compared with the very regular behaviour of the extended system in the case of the two-body problem. This regular behaviour was at the heart of the estimates we gave about the minimal distance from collision of the approximating solutions and also of Tanaka's estimates [T] on the number of collisions for suitable weak solutions.

Similar estimates are lacking at present for motion near M -body collisions, $M \geq 3$. This makes it very difficult to compare the behaviour "near collision" of classical solutions with and without strong force and therefore to compare topological properties of critical points of the Action functionals (or the Maupertuis functionals) with and without strong force. From this point of view, the approach in [A,S] could provide a simpler scheme.

A very interesting description of the chaotic behaviour of the flow near collisions is given for the three-body problem in [M.2] in terms of orbits which are homoclinic or heteroclinic to a suitable family of periodic solutions; later we will briefly describe this behaviour.

The irregular behaviour *near collisions* does not reflect itself in an irregular behaviour *at collisions*. In fact, as we shall see, the simultaneous collision of three bodies takes place "directionally" in the sense that all the unit vectors $x_k(t)|x_k(t)|^{-1}$ have a limit when $t \rightarrow t^*$ (in general, when this happens, we shall say that *the collision is directional*). Moreover, for $N = 3$ all such limits are known.

Much less is known on simultaneous collisions of M bodies, for $M \geq 4$; we will discuss these points in greater detail, since they are crucial for the proof we shall give in Section 5 of the existence, for each positive T of at least one T -periodic solution of the perturbed N -body problem which is free of three and four bodies collisions and also of N -body collisions, $N \geq 5$, for "most" values of the masses.

It should be pointed out that two-body collisions can be regularized (keeping also here only bounce trajectories) also for the N -body system.

We shall prove that the flow on the collision manifold is of gradient type i.e. there is a continuous function F which is bounded above and increases along the flow at every point which is not a rest point. For $N \geq 3$ this defines a partial order among rest points, given by the value of F there. This partial order is responsible for the fact that the N -body system cannot be regularized; one can in fact exhibit sequences of strong solutions which come arbitrarily close to a collision but such that, if the limit to a collision were continuous, the limit orbit on the collision manifold would violate the partial order. The problem is most easily understood within the frame of Conley's theory of isolating blocks [C] [C,E]. We refer to [Mc1], [M.2] and [E] for a detailed discussion, especially for the three-body problem.

REMARK 1

It is an interesting and unsolved problem to characterize variationally those weak solutions which can be added to a flow to make it regular, when this is possible. The difficulty lies in the fact that a regularized T -periodic weak solution could be the limit, uniformly over compact time intervals, of strong solutions which are bounded but not periodic. It is not known whether the requirement of continuity with respect to initial data for T -periodic solutions gives a weaker restriction in the definition of regularizability.

REMARK 2

A different definition of regular collision solution can be found in the literature [S,M]. Here the collision solution $x(t)$ is said to be regularizable (at collision time t^) if it can be continued for $t \geq t^*$ to a collision solution (at $t = t^*$ the function $x(t)$ is not even C^1). With this definition every three-body collision can be made regular, for example by continuing it in the solution one obtains by space inversion. Still, the three-body system*

cannot be regularized in the sense described above.

Before introducing the Mc Gehee coordinates and the collision manifold with its induced flow, we discuss the construction of a family of T -periodic classical solutions (or classical solutions with prescribed negative energy E) for the N -body problem. This will allow us to give some information on the variational structure of the Action functional and on the Maupertuis functional also in the N -body case; there will be some interesting similarities with the two-body case discussed at length in Section 2, but the resulting description is far less complete. As we shall see, these constructions (due in part to Euler [Eu] and Lagrange [Lg]) will also provide the basis for the study of the behaviour of collision solutions.

Euler was the first to give explicit (weak) periodic solutions of the N -body problem, $N > 2$. He generalized to $N > 3$ the collision solutions of the Kepler problem. He did so seeking particular solutions of the 3-body problem for which the particles collapse homotetically to triple collisions remaining on a given straight line through the origin. That the motion is homotetic means then that the configuration of the bodies changes in time only by a common scale factor.

In general, one can consider the same problem for $N \geq 3$; the particles now collapse towards the origin along straight lines, without requiring colinearity. These solutions, if they exist, have the form

$$x_k(t) = c(t)x_k^0, \quad k = 1, \dots, N, \quad x_k^0 \in R^3$$

Substituting in Newton's equations one derives

$$\ddot{c}(t)Mx^0 = \nabla V(c(t)x^0) = \frac{1}{c(t)^2} \nabla V(x^0) \quad 3.8$$

where we have used the compact notation of (3.2) and have exploited the homogenous character of the potential.

Taking scalar product with x^0 (in R^{3N}) gives

$$\ddot{c}(t) = -\frac{\lambda}{c^3(t)}, \quad \lambda \equiv \frac{V(x^0)}{\langle x^0, x^0 \rangle_M} \quad 3.9$$

$$\nabla V(x^0) = \lambda Mx^0, \quad 3.10$$

Equation (3.9) is recognized to be Newton's equation of a body with total mass λ attracted to a fixed centre by a gravitational force, when $J = 0$. Its flow was described in Section 2. Equation (3.10) is more difficult to solve; in fact a complete set of solutions is known only for $N = 3$. In this case, one can prove that there are only two types of solutions: either $x_k^0, k = 1, 2, 3$ are the coordinates of the vertices of an equilateral triangle (these solutions were found by Lagrange [Lg]) or they lie on a straight line, and the value of the x_k^0 depends on the masses of the three bodies (these solutions were found by Euler [Eu]).

It is important to notice, for future reference, that the equation (3.10) is covariant under rigid rotations around the origin. The set of solutions is therefore invariant under the action of the rotation group, and under this action the solutions are partitioned into equivalence classes.

A larger collection of solutions for the N -body problem was found by Lagrange; these will be classical periodic solutions. The problem he posed was to find solutions which are

homotetic but for which the motion does not necessarily take place on a straight line. The most general form of these solutions, if they exist, is

$$x_k(t) = c(t)R(t)x_k^0 \quad 3.11$$

where $R(t)$ is a family of orthogonal matrices and $c(t)$ is a common scale factor (to be precise, Lagrange considered only the case $c(t) \equiv 1$). Again, substituting (3.11) in (3.2) one derives

$$c^2 \left(\ddot{c} I + 2\dot{c} A + c R^{-1} \ddot{R} \right) x^0 = M^{-1} \nabla V(x^0), \quad A(t) \equiv R^{-1}(t) \dot{R}(t) \quad 3.12$$

where $A(t)$ is the angular velocity of the motion described by $R(t)$. One now makes use of the fact that $A(t)$ is antisymmetric, and that

$$R^{-1} \ddot{R}(t) = \frac{dA(t)}{dt} + A^2(t)$$

to prove [M.1] that (3.11) can have only two types of solutions:

a) Solutions for which $R(t) \equiv I \quad \forall t$. In this case x^0 must be a solution of (3.10) and for each k , $c(t)x_k^0$ is a collision solution of the Kepler problem.

b) Solutions for which x^0 are planar and satisfy (3.10); in this case, if Π is the plane defined by the x_k , equation (3.2) is solved precisely if $R(t)$ are rotations in Π around the origin, and denoting by $\theta(t)$ the angle of rotation, the following identities are satisfied

$$c^2(t)\dot{\theta}(t) = \text{const}, \quad \ddot{c}(t) = c(t)\dot{\theta}^2(t) - \frac{\lambda}{c^3(t)} \quad 3.13$$

where λ is defined in (3.9).

These are the equations for the motion of a particle of mass λ attracted by the origin by a Kepler force (with gravitational constant one). They are also the equations for the relative coordinates in the two-body problem.

Let Π be the plane which contains all x_k^0 ; the solutions of type *b*) are then parametrized by two parameters, E and j and are obtained by letting $x_k(t)$ be the trajectory in the plane Π that a body of mass λ would follow if it were attracted by the origin through a Kepler force, its initial position being x_k^0 , its energy E and its angular momentum j .

If $E < 0$ these solutions of the N -body problem are therefore periodic, and classical solutions if $j \neq 0$. Notice that, according to (3.9), the "mass" λ depends on x_k^0 .

Among these solutions, the only ones which satisfy

$$x_k(t + \frac{T}{2}) = -x_k(t) \quad \forall t$$

are the circular ones, for which one can find a reference frame, rotating uniformly with respect to the fixed inertial frame, in which all bodies appear at rest. For this reason, planar solutions of (3.10) are called *relative equilibria*.

Solutions of (3.10), whether planar or not, are called *central configurations*; the reason for this terminology is that, as we have seen, if the N particles are initially at x_k^0 with zero

initial velocity, they collapse along straight lines to a simultaneous collision at the origin, and the law of collapse is a homotetic motion described by a collision solution of the Kepler problem.

It should also be remarked that a) and b) are not mutually exclusive; a solution which is of both types is obtained if the x_k^0 are coplanar and the angular momentum j of the Kepler motion is zero.

REMARK

It can be verified [M.1] that those solutions of (3.10) which are planar are critical configurations in the following sense: a configuration x^0 is called critical if there exists at least one point (x^0, p^0) in phase space such that at (x^0, p^0) the energy and the three components of angular momentum are not all independent. In particular one can prove that the circular solutions described above are entirely made of critical configurations. Therefore attention has to be paid in the determination of the dimension of the null space of the Hessian of the Action functionals at these solutions.

We shall see that central configurations play another important role; in fact ([Su],[Mc.1],[M.1])
if the solution $x(t)$ has a simultaneous collision at the origin at time $t^* = 0$ and if for some $\epsilon > 0$

$$x(t) \notin \Delta \quad \forall 0 < t < \epsilon \quad 3.14$$

(for the sake of simplicity in the formulae, we consider the limit $t \rightarrow 0_+$ instead of the limit $t \rightarrow 0_-$) then for each k and uniformly in $[0, \epsilon]$ the vectors $t^{-1/3}x_k(t)$ are bounded and bounded away from zero and the ω -limit set of $t^{-1/3}x(t)$ are precisely the central configurations. They will also be the rest points of the flow on the collision manifold.

Define the central collapses to be those collision solutions for which the bodies collapse to a N -body collision moving along straight lines, i.e. the solutions $\tilde{x}(t)$ which correspond to initial data for which all velocities are zero and the configuration $\tilde{x}(0)$ is central.

We say that the collision solution is *directional* if

$$\lim_{t \rightarrow 0} \frac{x_k(t)}{|x_k|(t)}$$

exists for all k .

We say that $x(t)$ is asymptotic (at collision time $t = 0$) to a central collapse $\tilde{x}(t)$ if

$$\lim_{t \rightarrow 0} \frac{x_k(t)}{|x_k|(t)} = \frac{\tilde{x}(0)}{|\tilde{x}(0)|}$$

Notice that a solution can be asymptotic to two different central collapses for $t \rightarrow 0^\pm$.

One can prove [El] that a collision solution is directional if its ω -limit set is contained in the equivalence class of a single central configuration, i.e if it is isolated as equivalence class. We shall come back to this point later in this Section.

We shall prove in Section 5 that if a solution $x(t)$ is directional at collisions, it cannot be a minimum of the Action functional Φ_T over antiperiodic functions; in fact estimates on the index of the solution can be obtained from knowing that it is directional at collisions. We have seen an instance of this in the two-body case (for this system all collisions are directional). This gives further reason to analyze the structure of the central configurations.

We shall always work in the inertial reference frame in which the baricentre is at rest at the origin, and restrict attention to the solutions of (3.7) which lie on the "carved sphere"

$$S_0 \equiv \{x : x \in R^{3N} - \Delta, \quad \|x\|_M = 1, \quad \sum_{k=1}^N m_k x_k = 0\}$$

where as before Δ is the coincidence set

$$\Delta \equiv \{x \in R^{3N} : \exists k \neq h, x_k = x_h\}$$

There is no loss in generality in doing so, since (3.10) is invariant under scaling if one also changes the value of λ and the condition that the baricentre be at the origin is automatically satisfied by any solution of (3.10).

On the carved sphere S_0 the equation (3.10) becomes

$$M^{-1} \nabla V(x) + V(x) x = 0 \quad 3.15$$

The left hand side can be recognized to be the gradient of the restriction of V to S_0 (the vector field is easily seen to be tangent to S_0 and the second term plays the role of a Lagrange multiplier).

Writing $\tilde{V}$ for the restriction of V to S_0 we can then look at the solutions of (3.15) as the critical points of the function $\tilde{V}$ or also as the equilibrium points for the flow of the gradient of $\tilde{V}$ on S_0 . Notice that $\tilde{V}$ is continuous on S_0 but S_0 is open (since Δ is closed) and therefore not compact. To obtain a compact set in which one can apply minimax techniques or Morse theory to derive existence and multiplicity of critical points of $\tilde{V}$ one notices [M.1] that the positive function $\tilde{V}$ increases without bound along anyh sequence which converges to ∂S and for every $K > 0$ there is a neighborhood $\mathcal{N}_K$ of Δ on which $\tilde{V} > K$. From this one derives

STATEMENT 3.2 [Sh]

There is a neighborhood $\mathcal{N}_0$ of Δ in the closure of S_0 which contains no central configurations (the size of this neighborhood depends on the value of the masses). Moreover $V(x)$ is uniformly bounded in $S_0 - \mathcal{M}$

All critical points lie then a-priori in the set $S_0 - \mathcal{N}_0$ which is closed and therefore compact. We have already remarked that (3.10) and then (3.15) are invariant under rigid rotations. Therefore, if x^0 is a critical point of $\tilde{V}$ then for every $R \in \mathbf{O}(3)$ and for the reflection $\sigma : x \rightarrow -x$ also Rx^0 and σx^0 are critical points. This may suggest studying the reduced functional in the quotient space with respect to the $\mathbf{O}(3)$ action, but one must pay attention to the fact that $\{\cup_{R \in \mathbf{O}(3)} Rx^0\}$ has dimension two if x^0 is colinear (i.e. all x_k^0 are colinear) and dimension three if x^0 is not colinear. Indeed if x^0 is colinear, the stabilizer of all x_k^0 in $\mathbf{O}(3)$ is not trivial. This difficulty will be overcome because we will show now that the colinear critical points are always isolated as equivalence class.

To see this, notice that the set of colinear configurations is obtained through rigid rotations from the colinear configurations along a preassigned axis

$$x_k = y_k \zeta, \quad y_k \in R \quad 3.16$$

for a given unit vector ζ . Substituting (3.16) in (3.15) one obtains

$$\frac{\partial W}{\partial y_k} + m_k y_k W(y) = 0, \quad W(y) \equiv \sum_{h < k} \frac{m_k m_h}{|y_k - y_h|} \quad y_k \in R, \quad k = 1, \dots, N \quad 3.17$$

The function W is the restriction of V to colinear configurations along the vector ζ and the solutions of (3.18) are the critical points of W on

$$\Gamma_1 \equiv \{y_k, k = 1, \dots, N, \sum m_k y_k = 0, y_k \neq y_h \forall h \neq k, \|y\|_M = 1, y_k \in R\} \quad 3.18$$

There are $N!$ connected components in Γ (corresponding to the ways one can order N points on a line); they are pairwise equivalent under reflection about the origin, and only one member must be retained for each equivalence class, since reflections about the origin on a line can be achieved by rotation of an angle π around an axis perpendicular to the line. Since $W(y)$ diverges to $+\infty$ when y tends to the coincidence set, and as we have stated before there is an open neighborhood of this set which contains no critical point, by category theory W has at least $N!$ critical points (in fact minima) and $\frac{N!}{2}$ of them are independent modulo reflection through the origin. [Mou]

To prove that these minima are the only critical points, one proves that W is strictly convex in each of the $N!$ disconnected regions [P],[M.1].

This shows that colinear central configurations are isolated *from each other* as equivalence classes. To prove that they are isolated, still as equivalence classes, from any other central configuration as well, one estimates the Hessian of $\tilde{V}$ at these critical points; one can prove [M.1] that its kernel has dimension two and is entirely due to the symmetry of the problem. Let y^0 be a colinear critical point; we already know that the Hessian has $N - 1$ positive eigenvalues, corresponding to the strict convexity described above. The remaining eigenvalue, which exist if $N > 2$, are strictly negative [M.1], so that, for $N > 2$, y^0 is a saddle point if considered as critical point of $\tilde{V}$. It follows also that each colinear central configurations is isolated as equivalence class.

Since $\tilde{V}$ is positive and diverges at the boundary of S_0 , $\tilde{V}$ has a minimum, and this critical point corresponds to a central configuration which is not colinear.

With a similar procedure one proves that, if $N > 3$ there is at least one central configuration which is not planar (for $N = 3$ every solution of (3.15) is planar since any three vectors which sum to zero define a plane if they are not colinear).

For this purpose one considers the restriction Z of $\tilde{V}$ to configurations which lie in a fixed plane Π . Denote by z_k the coordinates of the bodies in this plane. Then the central configurations which lie in this plane are the critical points of

$$Z(z) \equiv \sum_{h < k} \frac{m_k m_h}{|z_k - z_h|} \quad z_k \in R^2, \quad k = 1, \dots, N \quad 3.19$$

on

$$\Gamma_2 \equiv \{z_k, k = 1, \dots, N, \sum m_k z_k = 0, y_k \neq y_h \forall h \neq k, \|z\|_M = 1, z_k \in R^2\} \quad 3.20$$

The function Z is positive and its critical points lie a-priori in a compact. Therefore it attains its minimum value, and this cannot be a colinear central configuration since we have seen that they are not minima for $N \geq 3$.

Let $z^0 \in R^{2N}$ be a minimum of Z . The study of the hessian of $\tilde{V}$ at y^0 shows that, if $N > 3$ there is a negative eigenvalue. Therefore z^0 is not a minimum for $\tilde{V}$. But such minimum exists because $\tilde{V}$ is positive and its critical points lie in a compact set. Therefore, for $N > 3$ there is a central configuration which is not planar.

Many important problems about the structure of the set of central configurations are still open. These configuration form a compact set [Sh] but it is not known whether for some choice of $N \geq 4$ and of the masses $m_1, \dots, m_N$ there are noncolinear central configurations which are not isolated as equivalence classes. One can pass to the quotient modulo the rotation group, since away from colinear configurations the stability group is trivial and the quotient provides a smooth manifold. One looks now for nonisolated points of the image of V on this manifold. It is known [W] that for $N = 4$ there is only one central configuration which is not planar, i.e. the regular tetrahedron (independently of the value of the masses). This central configuration is therefore isolated as equivalence class.

For arbitrary N , under the assumptions of nondegeneracy, one can use Morse inequalities to provide a lower bound on the number of equivalence classes of central configurations. In this way one arrives [X.1] to the following estimate for the number $Q(N)$ of classes of *noncolinear* central configurations for the N -body problem

$$Q(N) \geq \frac{(2N-4)(N-1)!}{2}$$

This estimate can be improved if one takes into account also the invariance of the system under reflection through the origin. This leads, still under the assumption of nondegeneracy, to the sharper estimate for the number $Q_1(N)$ of classes of *planar non colinear* central configurations for $N \geq 3$

$$Q_1(N) \geq N! \left(\sum_{k=3}^N \frac{1}{k} \right)$$

That these numbers provide only lower bounds but the system may have in fact many more central configurations is seen from a recent result [X.1]. If one considers an N -body system in which the masses are distributed according to a suitable hierarchy (with one mass being by far the biggest, and the rest of the masses being distributed in subsets of vastly different weight), one can give a lower bound (in the notations employed above) valid for $N \geq 3$

$$Q(N) \geq (N-2)! (2^{N-1}(N-2) + 1), \quad Q_1(N) \geq \frac{(N-2)!}{16} (3^{N-1}(2N+3) + 2N^2 - 6N + 1) \quad \blacksquare$$

Here no assumption is made a-priori of nondegeneracy, but these numbers refer to isolated equivalence classes.

We now review briefly the Mc Gehee change of variables [Mc.1], [El], which allows a simple analysis of the behaviour at collisions; it introduces also the collision manifold (the blow-up of the collision point) and the induced flow, which are important tools for the proof that the N -body system cannot be regularized for $N \geq 3$.

We consider the limit $t \rightarrow 0_-$ so that $t \leq 0$ in what follows. Identical results hold for $t \rightarrow 0_+$.

Let $r \equiv I^{1/2}$ (where I is the moment of inertia introduced in (3.6)) and $\eta \equiv \frac{z}{r}$.

These can be seen as polar coordinates in $X \equiv R^{3N} - \Delta$ (with the scalar product defined by the moment of inertia).

As we mentioned before, from the homogeneity property of the Kepler potential it follows that at simultaneous collisions $r(t) \simeq |t|^{2/3}$.

We expect that $|x_k|$ has the same asymptotic behaviour for all k and introduce therefore "homogeneous" coordinates. We also break the momentum $p \equiv M\dot{x}$ in a radial part and a tangential one (with respect to the scalar product $\langle, \rangle_M$).

Let

$$S \equiv \{x \in X : r = 1\} \quad 3.21$$

We set

$$s \equiv \frac{x}{r}, \quad y = \frac{(p, x)}{r}, \quad \eta \equiv p - yMs, \quad x, \eta \in R^{3N}, \quad y \in R \quad 3.22$$

By construction $s \in S$ and $\langle \eta, s \rangle_M = 0$. Therefore

$$T \equiv \{s, \eta : s \in S, \langle \eta, s \rangle_M = 0\} \quad 3.23$$

can be regarded as the tangent bundle of S with s and η as coordinates.

The map (3.22) can be inverted to give

$$x = r, \quad \eta = x + yMs \quad 3.24$$

and (3.24) defines a real analytic diffeomorphism

$$\phi : R^+ \otimes R^1 \otimes T \ni (r, y; s, \eta) \Rightarrow (x, p) \in X \otimes R^{3N}$$

In these coordinates the system of equations (3.2) is transformed in a system which has the property that the equation for r is regular at $r = 0$ (the point of simultaneous collisions) while the others have coefficients which are singular.

This corresponds to the fact that at simultaneous collisions at least some of the velocities must diverge, by conservation of energy. The scaling we shall perform is suggested by our expectation that the velocities of the N bodies diverge at the same rate, so that only the "radial" part of the velocity may need rescaling. Comparing the scaling properties of the kinetic and the potential energies under dilations, we expect that the asymptotic behaviour at collision for the radial velocity (or equivalently for the kinetic energy) be $r^{-1/2}$.

We make therefore a further change of variables

$$u \equiv r^{1/2}\eta, \quad v \equiv r^{1/2}y$$

In the coordinates u, s, r, v the equations (3.2) become

$$\begin{aligned} r^{-1/2} \dot{r} &= v, & r^{-1/2} \dot{v} &= \frac{v^2}{2} + (u, M^{-1}u) - V(s) \\ r^{-1/2} \dot{s} &= M^{-1}u, & r^{-1/2} \dot{u} &= -\frac{vu}{2} - (u, M^{-1}u)Ms + U(s)Ms + \nabla U(s) \end{aligned} \quad 3.25$$

and the conservation of energy becomes, if E is the energy of the solution considered

$$\frac{1}{2}(u, M^{-1}u) + \frac{1}{2}v^2 - V(s) = Er \quad 3.26$$

It is now clear that one can remove the singularity of these equations at $r = 0$ by performing the time transformation

$$t \rightarrow \tau; \quad \frac{dt}{d\tau} = r^{1/2}(t) \quad 3.27$$

Notice that this transformation of the time parameter depends on the trajectory. Notice that $t < 0$ is equivalent to $\tau < 0$ and $\lim_{t \rightarrow 0^-}$ is equivalent to $\tau \rightarrow -\infty$.

Denoting by a prime the differentiation with respect to τ one obtains now the following expression for Newton's equations for the N -body system in the Mc Gehee space-time coordinates.

$$\begin{aligned} r' &= rv, & v' &= \frac{v^2}{2} + (u, M^{-1}u) - V(s) \\ s' &= M^{-1}u, & u' &= -\frac{vu}{2} - (u, M^{-1}u)Ms + U(s)Ms + \nabla U(s) \end{aligned} \quad 3.28$$

The conservation of energy has still the form (3.26) The manifold $\mathcal{M}_N$ in $R^+ \otimes R \otimes T$ defined by

$$\mathcal{M}_N \equiv \{r, v, s, u : r = 0, \frac{1}{2}(u, M^{-1}u) + \frac{1}{2}v^2 = V(s)\} \quad 3.29$$

is invariant under the flow described by (3.28) (since energy is conserved). Notice that this manifold is independent of E .

We shall call $\mathcal{M}_N$ defined in (3.29) the *N-body collision manifold* (since it corresponds to $r = 0$ i.e. to a simultaneous collision). It can be parametrized by the coordinates s and u and is therefore diffeomorphic to T defined in (3.23) (i.e to the unit tangent bundle of S .) The equations for s and u in (3.28) define a flow ϕ on $\mathcal{M}_N$ which we shall call *collision flow*.

It is defined in terms of the coordinates s, u by

$$s' = M^{-1}u, \quad u' = -\frac{vu}{2} - (u, M^{-1}u)Ms + V(s)Ms - \nabla V(s) \quad 3.30$$

where $v(\tau)$ is defined by the condition that the flow lies on $\text{cal}M$ i.e.

$$\frac{1}{2}(u, M^{-1}u) + \frac{1}{2}v^2 = V(s)$$

so that

$$v' = \frac{v^2}{2} + (u, M^{-1}u) - V(s) \quad 3.31$$

Its rest points are the central configurations; in fact, setting to zero in (3.31) all the derivatives with respect to τ and keeping into account (3.31) one obtains at the limit point s^0, v^0, u^0

$$u^0 = 0, \quad \lambda Ms^0 = \nabla V(s^0), \quad \lambda \equiv \frac{1}{2}\|v^0\|_M^2$$

i.e. s^0 is a central configuration.

In this notations it is not difficult to prove [Mc.1] the following results, first obtained by Sundman ([S],[W]) by an accurate use of tauberian estimates.

LEMMA 3.3

Let $x(t) \in R^{3N}$ be a solution of the N -body problem which has an isolated simultaneous collision at the origin at $t = 0$. Then (recall that $\lim_{\tau \rightarrow -\infty}$ is equivalent to $\lim_{t \rightarrow 0_-}$)

i)

The total angular momentum J of the system along $x(t)$ is zero (recall that J is a constant of motion).

ii)

There is a constant $C > 1$ such that (3.7₁) holds, i.e.

$$\begin{aligned} \frac{1}{C} &\leq \min_{h < k} \liminf_{t \rightarrow 0_-} |t|^{-2/3} |x_k(t) - x_h(t)| \\ &\leq \max_{h < k} \limsup_{t \rightarrow 0_-} |t|^{-2/3} |x_k(t) - x_h(t)| < C, \quad k, h = 1, \dots, N \end{aligned} \quad 3.32$$

Moreover

iii) The flow on the collision manifold is of gradient type, the increasing function being v .

Analogous statements hold for the limit $t \rightarrow 0_+$.

Proof

Notice first that the variables v and $\|u\|$ are uniformly bounded, since $V(s)$ is uniformly bounded on S .

In the Mc Gehee variables the angular momentum takes the form

$$J = r^{1/2} ((Ms) \wedge u)$$

Since $\lim_{\tau \rightarrow \infty} r(\tau) = 0$ and J is a constant of motion, it follows $J \equiv 0$.

We next prove iii). This follows directly from (3.31); indeed on the collision manifold one has

$$v' = \frac{1}{2}(u, M^{-1}u) + \frac{v^2}{2} + \frac{1}{2}(u, M^{-1}u) - V(s) = \frac{1}{2}(u, M^{-1}u) \quad 3.33$$

so that indeed $v(\tau)$ is strictly increasing outside the equilibrium points. This shows that the collision flow is of gradient type.

To prove ii), notice that v^2 is uniformly bounded and $v(\tau)$ is nondecreasing so that it has a limit which we denote by v_∞ .

Moreover $v_\infty > 0$. In fact, one cannot have $v_\infty = 0$ since in view of (3.31) one would have $V(s^0) = 0$ which is impossible since V is bounded away from zero on the rest points. One cannot have $v_\infty < 0$ because from (3.28) one would derive $\lim_{\tau \rightarrow -\infty} r(\tau) \neq 0$ contrary to the fact that in the limit one must have $r = 0$.

From $v_\infty > 0$ and (3.28) one derives that $r(\tau)$ converges to zero exponentially fast when $\tau \rightarrow -\infty$. When this is expressed in the variables x_k and time t one obtains (3.7)₁ if one recalls that, in view of statement 3.1, the differences $|s_k - s_h|$, $k \neq h$ have a common lower bound and a common upper bound (the upper bound coming from $\|s\| = 1$) ///

It should be remarked that the flow ϕ is not regular over the entire manifold $\mathcal{M}_N$ since there could be partial collisions (i.e. collisions involving a subset of the bodies). Double collisions can be regularized, and a new manifold can be constructed, with a new flow on it, which is regular in correspondence to the double collisions blown up now in a suitable

way; a very detailed description of the new manifold is given in [Mc.1] for the case $N = 3$. In this case, the collision flow has singularities only at triple collisions.

REMARK 1

If one adds to the N -body problem a potential force with potential W which satisfies for some $\beta > 0$

$$\lim_{x \rightarrow \beta} \max_{h < k} [|x_k - x_h|^{1-\beta} W(x), + |x_k - x_h|^{2-\beta} |\nabla W(x)|] = 0 \quad 3.34$$

$$\frac{(x, \nabla W(x))}{|x|^3} + W(x) \geq 0 \quad 3.35$$

the results in Lemma 3.2 still hold, with the exception of the second which becomes now i)'

The total angular momentum J with respect to the origin is of class C^1 at $t = 0$ (J is not a constant of motion if W is not invariant under rotations) and moreover $\lim_{t \rightarrow 0+} J(t) = 0$. In fact, it is easy to see that, under the assumption (3.34) the system of equations (3.28) is replaced by

$$\begin{aligned} r' &= rv, & v' &= \frac{v^2}{2} + (u, M^{-1}u) - V(s) - \frac{(s, \nabla W(rs))}{r} & s' &= M^{-1}u, \\ u' &= -\frac{vu}{2} - (u, M^{-1}u)Ms + U(s)Ms + \nabla U(s) - \frac{1}{r}(s, \nabla W(rs))Ms + r\nabla W(rs) \end{aligned}$$

The manifold $\mathcal{M}_N$ in $R^+ \otimes R \otimes T$ is still defined by

$$\mathcal{M}_N \equiv \{r, v, s, u : r = 0, \frac{1}{2}(u, M^{-1}u) + \frac{1}{2}v^2 = V(s)\} \quad 3.36$$

On $\mathcal{M}_N$ one still has $v' \geq 0$ and the proof of i) and iii) is the same as before.

Note that condition (3.35) is not necessary. It has been introduced here to give the simple proof using only the equation for v . If one takes into account also the equation for u , or follows the lines of the more complicated proof in [Su], [W], one can substitute the assumption (3.35) with the more natural one: for some $\delta > 0$

$$\lim_{x \rightarrow \Delta} \max_{h < k} |x_k - x_h|^{3-\epsilon} \|D^2 W(x)\| = 0 \quad 3.37$$

REMARK 2

One must stress that the coordinates we have introduced above are adapted to the study of the behaviour near a simultaneous collision of all bodies, since only in this case we can expect that at collision all velocities have the same (divergent) asymptotic behaviour.

On the manifold $\mathcal{M}$ at a collision in which the bodies separate into clusters each of which undergoes simultaneous collision (for clusters made of only one body the last statement must be understood in the sense that the body is not colliding at the time considered) other systems of coordinates may be preferable, in which for each cluster one introduces the moment of the inertia relative to the baricentre and decomposes the momentum in a radial part and a tangential part relative now to the scalar product defined by the mass matrix of the cluster.

More attention must be paid now, since the total momentum or the total angular momentum of each cluster are not constants of motion. A more detailed analysis shows that these quantities vary smoothly in time at $t = 0$ so that after rescaling they become for each cluster "almost" constants. The analysis of the asymptotic behaviour at collision can then be performed in this new system of coordinates precisely as in the case of a simultaneous collisions, with the same conclusions. Of course one must control the change of variables in the regions of $\mathcal{M}_N$ where all systems of coordinates can be used (i.e. away from collisions). When considering one cluster K and its collapse, the interaction with the other bodies outside the cluster act as a small perturbation the effects of which become negligible at collision; in fact a detailed estimate shows [El] that all these contributions are exponentially small in the time parameter τ_K adapted to the cluster under consideration.

For a detailed treatment of the multi-cluster case one should consult [El]. The conclusion is that the second and third statement still hold (with respect to the coordinates of the bodies in the cluster) while the first has to be modified as follows:

i)'

The total angular momentum J_K of the cluster K with respect to its baricentre is of class C^1 at $t = 0$ (J_K is not a constant of motion) and moreover $\lim_{t \rightarrow 0+} J_K(t) = 0$.

We have already remarked that the structure of the flow on the collision manifold is also important in the study of the properties of the flow of (0.1) near collisions.

In a very interesting paper [M.2] Moeckel studies for the planar three-body problem the behaviour of solutions of (0.1) which enter a sufficiently small neighborhood of the coincidence set Δ . To describe his results, we shall use the following notation. A solution $x(t)$ of (0.1) for the three-body problem has a *near collision* at time t^0 in the bounded time interval I_0 if for ϵ small enough one has

$$\min_{t \in I_0} I_{x(\cdot)}(t) = I_{x(\cdot)}(t^0) \leq \epsilon \quad I_{x(\cdot)}^2(t) \equiv \sum_{k=1}^3 m_k x_k^2(t)$$

where $t^0 \in I^0$ is uniquely defined, and moreover $I_{x(\cdot)}(t)(t - t_0)$ is strictly increasing in $I_0 - \{t^0\}$.

We shall call "simple solutions" of eccentricity e and period T the T -periodic solutions of (0.1) which have the form $c(t)R(t)x^0$, $R(t) \in \mathbf{O}(3)$ where e is the eccentricity of the corresponding ellipses and x^0 is a relative equilibrium.

It is easy to verify that the natural order given by the flow on the collision manifold places the Lagrange equilibria above the Euler ones.

If the relative equilibrium is denoted by Y we shall denote by $Y_{e,T}$ the corresponding simple solution with eccentricity e and minimal period T .

We say that the solution $x(t)$ of (0.1) which has at t^0 a near collision, "shadows" in I_0 a simple solution $Y(t)$ if

$$\sup_{t \in I_0} \left| \frac{x(t)}{|x(t)|} - \frac{Y(t)}{|Y(t)|} \right| \leq \delta(\epsilon)$$

where δ is infinitesimal in ϵ .

This definition can be suitably generalized to include also the case when the interval I_0 is unbounded on one or both sides, and correspondingly $t_0 = \pm\infty$. In this case the solution $x(t)$ is *asymptotic* to one of the exact solutions.

For the three-body system with almost equal masses it is proved in [M.2] that, if the eccentricity is small enough, for each simple solution $Y_{e,T}$ of eccentricity e and period T one can find a solution of (0.1) which is homoclinic to $Y_{e,T}$ (asymptotic to $Y_{e,T}$ both in the past and in the future), and that for each pair of simple solutions $Y_{e,T}^1, Y_{e',T'}^2$ one can find a solution $x(t)$ which is eteroclinic from $Y_{e,T}^1$ to $Y_{e',T'}^2$.

He also proves that for any finite sequence of simple solutions $\{Y^k\}$, $k = 1, \dots, K$ one can find a solution $x(t)$ of (0.1) for which there is a disjoint sequence of time intervals I_k such that $x(t)$ shadows Y^k in I_k .

This shows in particular that the three-body system cannot be regularized; otherwise one would obtain, for any pair of central configurations Y^1 and Y^2 , a solution of the collision equation on the collision manifold which connects Y^1 to Y^2 asymptotically (in time τ) as limit when ϵ, T and $(1 - e)$ converge to zero, of the portion between t_1 and t_2 of a trajectory which shadows $Y_{e,T}^1$ near t_1 and $Y_{e,T}^2$ near t_2 . This solution does not exist if Y^1 follows Y^2 in the natural order induced by the collision flow.

It is not known whether this shadowing can be obtained using a *periodic* solution of (0.1). This problem is very hard in the *O.D.E.* approach, since one does not have uniform continuity with respect to initial data, so that it is difficult to use an implicit function theorem without losing the shadowing property. It would be very interesting to reobtain Moeckel's results from a variational point of view.

The proof of these "shadowing" properties is obtained in [M.2] exploiting the fact that when the masses are almost equal, all relative equilibria are hyperbolic, and so are all the simple periodic solutions, at least if e is small enough. The proof of the shadowing property relies then on a study of the intersection properties of the corresponding stable and unstable manifolds, and should therefore be amenable to a variational study, much in the spirit of the multi-bump analysis of lagrangian systems.

Coming back to the description of the flow at or near collision, it may be instructive to see how a central collapse is described in Mc Gehee's coordinates. This will help in understanding the role of these coordinates.

A central collapse (starting from a central configuration x^0) is described by

$$x_k(t) = c(t)x_k^0, \quad \dot{x}_k(t) = \dot{c}(t)x_k^0 \quad 3.38$$

where $c(t)$ solves $\ddot{c} = -\frac{\lambda}{c^3}$ for a suitable $\lambda > 0$. The asymptotic behaviour at collision is $x_k(t) \simeq t^{2/3}x_k^0$

From the definition it follows

$$\eta(t) = \eta(0) = 0, \quad s(t) = s(0), \quad r(t) \simeq r(0)|t|^{2/3}, \quad v \simeq c|t|^{2/3} \quad 3.39$$

(the factor $r^{-1/2}$ in the definition of v has the purpose to give v the same asymptotic behaviour at collision as $r(t)$).

Moreover near collision one has

$$\frac{dt}{d\tau} \simeq \frac{1}{t}, \quad t \simeq e^\tau$$

so that

$$|x(\tau)| \simeq e^{\beta\tau}, \quad \left| \frac{dx}{d\tau} \right| \simeq e^{\beta\tau} \quad 3.40$$

for some positive constant β .

It could be expected that a similar behaviour (in particular exponential convergence when $\tau \rightarrow -\infty$ to a unique rest point) be present in all collisions, and that the asymptotic study of the solutions of (3.2) near a collision can be correspondingly made in the framework of stable and unstable manifolds for dynamical systems, using the Mc Gehee coordinates and time scale.

Notice that by definition, for a trajectory $x(t)$ which has a simultaneous collision at $t = 0$, this collision is *directional* if the image of $x(t)$ on the collision manifold (obtained by associating to $x(t)$ the trajectory on $\mathcal{M}_N$ defined by the coordinates $u(t)$, $s(t)$) converges, for $\tau \rightarrow \infty$, to a central configuration. Remark that a-priori one only knows that its ω -limit set is contained in the set of central configurations).

There is no general proof available that every weak solution of (3.2) has only directional collisions (the direction can be different for $t \rightarrow 0_+$ and $t \rightarrow 0_-$), although this is known for two-body and three-body collisions. However, no collision has been constructed which is not directional.

The following important result has been proved recently ([El], see also [SH]).

LEMMA 3.4 [El]

A collision is directional if its ω -limit set is the equivalence class of a single central configuration.

REMARK

The condition in Lemma 3.4 is equivalent to the requirement that the set of equivalence classes of noncolinear central configurations does not contain connected components which are not singletons (this allows the presence of accumulation points).

We shall not reproduce here in detail the proof of Lemma 3.4, for which we refer to [El]. We only remark that, if the ω -limit set is as in the assumption of the Lemma, the only obstruction to being directional comes from the possibility of an endless rigid rotation; this motion is called in Celestial Mechanics *an infinite spin and its existence is the Painlevé-Wintner problem*. [Pn],[W].

In [El] (see also [S,H]) it is proved that no collapsing cluster can enter an infinite spin. This result holds also if the limit set does not belong to an isolated equivalence class.

To prove Lemma 3.4 one notices that for any trajectory $\tau \rightarrow s(\tau)$, at each time τ the vector $\dot{x}(t)$ can be uniquely decomposed in the sum of the angular velocity associated to the rigid rotation and its orthogonal complement. In the Mc Gehee coordinates and time parameter the decomposition of $\dot{x}$ corresponds to the fact that the vector $s' \in R^{3n}$ can be written *uniquely* as

$$s'(\tau)_k = \zeta(\tau) \wedge s_k(\tau) + Y_k(\tau) = \zeta \wedge s + Y \quad \langle Y, (\zeta \wedge s) \rangle = 0$$

where $(\zeta(\tau))$ is the angular velocity of the system.

It is also convenient to write each s_k uniquely as $s_k = \beta_k \zeta + z_k$.

In [El] using simple estimates derived from the equations which describe the flow on the collision manifold, it is proved that for some $\gamma > 0$

$$\lim_{\tau \rightarrow \infty} e^{\gamma \tau} |\zeta| \|z\|^2 = 0 \quad 3.35$$

while, if ζ is bounded or $\|z\|$ is bounded away from zero

$$\lim_{\tau \rightarrow \infty} e^{\gamma \tau} \|\zeta \wedge s\| = 0 \quad 3.36$$

This proves that an infinite spin is impossible at least if the limit configuration is not colinear (in this case at least one of the two conditions which lead to (3.36) could be violated). A separate proof is then given for the case in which the limit central configuration is colinear. ///

We end this Section with some remarks on the index of circular solutions. For many purposes it is useful to have estimates on the (Morse) index of critical point of the Action functional which corresponds to a given periodic solution.

We shall sketch the proof of a relation between the (Morse) index of a critical point y^0 of $\tilde{V}$ which corresponds to a planar central configuration and the Morse index of Φ_T at the periodic solution which one obtains by letting y^0 rotate rigidly with a suitable angular velocity.

Denote by $\tilde{V}$ the restriction of Newton's potential to the carved sphere S_0 . Its critical points can also be characterized as the critical points of

$$F_V(y) \equiv \frac{1}{2} \left(\frac{2\pi}{T} \right)^2 \sum m_k |y_k|^2 + \sum_{i < k} \frac{m_i m_k}{|y_i - y_k|} \quad 3.45$$

conditioned to $\|y\|_M^2 = 1$, $\sum_k m_k y_k = 0$.

If y_k lie in the plane Π a solution of the equations of motion with potential V is given by

$$x(t; y) \equiv R_\Pi^T(t) y$$

where $R_\Pi^T(t)$ denotes uniform rotation around the origin in Π with period T . Therefore $x(t; y)$ is a critical point of Φ_V

$$\Phi_V \equiv \frac{1}{2} \int_0^T \sum_k m_k |\dot{x}_k(t)|^2 dt + \int_0^T V(x(t)) dt$$

By construction $x(t; y)$ belongs to Λ_T^a and has no collisions.

We shall study the relation between the index of $x(t; y)$ as critical point of Φ_V defined over Λ_T^a and the index of y as a critical point of F_V .

Recall that $|x_i(t; y) - x_k(t; y)| = |y_i - y_k| \forall t$. Therefore one has

$$\begin{aligned} D^2 F^V(x(\cdot; y))(w) &= \int_0^T \sum_k m_k |\dot{w}_k(t)|^2 dt + \sum_{i, k} \rho_{k, i}^{-1} V'(\rho_{k, i}) |w_k(t) - w_i(t)|^2 \\ &\quad + \sum_{i, k} \left[\rho_{k, i}^{-2} V''(\rho_{k, i}) - \rho_{k, i}^{-3} V'(\rho_{k, i}) \right] (y_i - y_k, \zeta_i(t) - \zeta_k(t))^2. \end{aligned}$$

where $\zeta_k(t) \equiv R_\Pi^T(-t) w(t)$ and $\rho_{k, i} \equiv |y_i - y_k|$.

Defining $\zeta_k(t)$ by

$$\zeta_k(t) \equiv R_\Pi^T(-t) w_k(t)$$

one has

$$\zeta_k(t) = \sum_{n \geq 0} c_n^k \cos\left(\frac{\pi t}{T}\right) + \sum_{n \geq 1} s_n^k \sin\left(\frac{\pi t}{T}\right)$$

It is straightforward to verify that

$$D^2\Phi_V(x(\cdot; y)(R_\Pi^T(\cdot)\zeta)) = D^2F_V(y)(\zeta), \quad \zeta \in R^{3N} \quad 3.46$$

so that one has

STATEMENT 3.5

The index of Φ_V at the critical point $x(t; y)$ is not smaller than the index of F_V at the critical point y .

We specialize now our analysis to the Kepler problem.

The function F_V in (3.45) takes the form

$$F(y) = \frac{1}{2} \left(\frac{2\pi}{T} \right)^2 \sum_k m_k y_k^2 + \sum_{i < k} m_i m_k |y_i - y_k|^{-1}$$

and the Action is

$$\Phi = \int_0^T \frac{1}{2} \sum_k m_k \dot{x}_k^2 + \int_0^T \sum_{i < k} m_i m_k |x_i(t) - x_k(t)|^{-1}$$

By rescaling, there is no loss of generality in considering only the case $T = 2\pi$.

One has then, if y is a planar critical point of F

$$D^2F(y)(w) = F_1(w) - F_2(w) + F_3(w) \quad w \equiv \{w_k\}, \quad w_k \in R^3$$

where

$$\begin{aligned} F_1 &\equiv \sum_k m_k |w_k|^2 & F_2 &\equiv \sum_{i \neq k} m_i m_k |w_i - w_k|^2 |y_i - y_k|^{-3} \\ F_3 &\equiv 3 \sum_{i \neq k} m_i m_k (w_i - w_k, y_i - y_k)^2 |y_i - y_k|^{-5} \end{aligned}$$

Also, using (3.46) one verifies that $D^2\Phi_V$ is diagonalized by the Fourier series.

Let P^n be the orthogonal projection onto the n^{th} Fourier component and define

$$G_k^n(\zeta) \equiv G_k(P^n \zeta)$$

One has then

$$D^2\Phi(P^n [R_\Pi^{2\pi}(t)(\zeta(t))]) = G_1^n(\zeta) - G_2^n(\zeta) + G_3^n(\zeta)$$

where

$$G_1^n = (2n+1)^2 G_1^0, \quad G_2^n = G_2^0, \quad G_3^n = \frac{(2n+1)^2}{2} G_3^0$$

Moreover

$$G_m^0(R(t)w) = F_m(w), \quad \forall w \in R^{3N}, \quad m = 1, 2, 3 \quad 3.48$$

where, in complex notations

$$R(t)(x_1 + ix_2, x_3) = (e^{it}(x_1 + ix_2), x_3), \quad x \in R^3$$

(we have chosen for Π the plane (1,2)).

The functions F_m and G_m are positive, while F_3 and G_3 vanish when the argument is in each point perpendicular to the plane Π . Therefore, if the critical point y of F has index zero, one must have

$$F_1(w) - F_2(w) \geq 0 \quad \forall w \quad 3, 49$$

It follows from (3.48), (3.49) and the positivity of G_3 that, for all w

$$G_1(w(t)) - G_2(w(t)) + G_3(w(t)) \geq 0 \quad \forall w(t)$$

so that the critical point $x(t, y)$ of Φ_T has index zero. This proves

STATEMENT 3.6

For the case of the Kepler potential, if the planar central configuration $\{y_k\}$ as a critical point of $\tilde{V}$ has index zero, then the index of $x(t; y)$ as critical point of Φ is also zero.

With the same arguments, one can prove that if the index of y is N , then also the index of $x(t, y)$ is N . If the index of y is $N_1 \geq N$ one can in general only conclude that the index of $x(t, y)$ is not smaller than N_1 .

More detailed conclusions can be drawn if one knows explicitly the configuration y .

4. RESULTS FROM CRITICAL POINT THEORY

In this Section we shall review briefly some recent results on critical point theory for singular lagrangian, with special reference to those results which have found applications to the N-body problem. For a general exposition of critical point theory one can consult [A], [E], [M,W] [R]. Results specially adapted to the N-body problem are described in [A,CZ.1] and in [CZ.1].

Generally speaking, we shall consider functionals defined on a Hilbert space $\mathcal{H}$ (typically, $\mathcal{H} \equiv H^1(R^{3N}, R)$ or one of its closed subspaces) or a Hilbert manifold (typically, $\{u \in \mathcal{H}, g(u) = 0\}$ where g is of class C^1). Critical points will be found by minimax arguments. Category theory in the sense of Ljusternik and Schnirelmann is often employed to define minimax classes.

In applying variational methods to a functional f on a Banach space X one requires in general

$$f \in C^1(\mathcal{B}_f), \quad \mathcal{B}_f \equiv \{u \in \mathcal{H}, |f(u)| < +\infty\} \quad (A)$$

If $u^0 \in \mathcal{B}_f$ and $Df(u^0) = 0$ then u^0 satisfies weakly the corresponding Euler-Lagrange equations; for the functionals we shall introduce (see Section 1) these variational equations are equivalent to Newton's equations.

Under the assumptions we make on the potentials, in particular to be regular outside the coincidence set Δ and to have a suitable scaling behaviour at Δ , the variational functionals will have the property to be of class C^2 at all points at which they are differentiable, so that an index can be defined there.

The variational functionals for the N-body problem do not satisfy condition (A) in absence of strong forces; in particular they are continuous but not differentiable at (weak) solutions of Newton's equations in which collisions occur. We have seen examples of these "collision solutions" in Sections 2 and 3.

A collision solution x^0 can be regarded as a critical point in a generalized sense [M,W] (roughly speaking, the topology of the level sets changes in a neighborhood of x^0) and its

(generalized) index can be defined as the maximal dimension of a subspace $\mathcal{K}$ for which one can find a positive constant ϵ_0 for which

$$\forall g \in \mathcal{K}, \quad f(x^0 + \epsilon g) < f(x^0) \quad \forall \epsilon < \epsilon_0$$

This definition coincides with the usual one if f is twice differentiable at x^0 .

On the other hand, not all critical points in this more general sense correspond to solutions of Newton's equations. Therefore, while this extended definition of index can be of use to establish properties of the collision solutions (and then to give an upper bound on the number of collisions) the general theory of critical points cannot be applied to obtain existence and multiplicity of classical or weak solutions of (0.1).

This difficulty is overcome in [B,R] by constructing a sequence of functionals f_δ which take value $+\infty$ on collision solutions and satisfy (A) for every $\delta > 0$. Critical points x_δ^0 of these modified functionals are obtained by minimax arguments, and suitable estimates are then used to prove that the x_δ^0 converge along subsequences, weakly in $\mathcal{H}$ and pointwise, to a (periodic) weak solution $x^0(t)$ of Newton's equations. We recall that in Section 1, a periodic solution was called weak if its collision set has open dense complement (and, if the system is autonomous, the energy takes the same value whenever it is defined).

It is important to notice that, since an index is defined for every x_δ^0 and a generalized index is defined for x^0 , properties of $x^0(t)$ at times close to collisions and/or uniform properties of x_δ^0 at times at which the distance from Δ is minimal can be used to give upper bounds on the number of collisions which take place in a period. Results in this direction will be discussed in Section 5.

In this context an interesting open problem is the relation between the indices of the x_δ^0 (as critical points of f_δ) and the (generalized) index of x^0 as (generalized) critical point of f ; we shall comment briefly on this point in Section 5, using as an example the two-body Kepler problem (in which the analysis can be carried out in detail).

Critical points for the modified functionals are obtained by minimax methods; this requires an amount of compactness, which is usually obtained through the Palais-Smale condition. For the convenience of the reader we shall recall here some definitions and results. Details can be found in [A][A,CZ.1][E][M,W].

To select minimax sequences u_n with suitable properties (in particular to control $\|Df(u_n)\|$) is important a result by I. Ekeland [E], which we shall state in the following form (the result by Ekeland is in fact more general)

PROPOSITION 4.1 (Ekeland)

Let $\mathcal{E}$ be a Banach space and f be a semicontinuous function bounded from below on $\mathcal{E}$. Then for every $\epsilon > 0$ and $u \in \mathcal{E}$ such that

$$f(u) \leq \inf_{\mathcal{E}} f(v) + \epsilon$$

one can find $v \in \mathcal{E}$ such that

$$f(v) \leq f(u), \quad \|u - v\| \leq 1, \quad \forall w \quad f(w) \geq f(v) - \epsilon \|v - w\|$$

Recall that if $\mathcal{E}$ is a Banach space (or a Banach manifold) a function f on $\mathcal{E}$ is *coercive* if $y_n \in \mathcal{E}$, $\|y_n\| \rightarrow \infty$ implies $f(y_n) \rightarrow \infty$ and is *lower semicontinuous* if $y_n \rightarrow^w y$ implies $f(y) \leq \liminf_{n \rightarrow \infty} f(y_n)$. Here the superscript w denotes weak convergence.

With these notations one has [E], [M,W]:

PROPOSITION 4.2

If $\mathcal{E}$ is a reflexive Banach space (in particular a Hilbert space) and f is coercive and lower semicontinuous, then f has a minimum on $\mathcal{E}$.

Notice that reflexivity is used to guarantee that the weak and weak * topologies coincide, so that from any bounded sequence one can extract one which converges weakly.

Combining the two previous Propositions one obtains

PROPOSITION 4.3

Let $f \in C^1(\mathcal{H}, R)$ be bounded below. For any minimizing sequence $\{u_n\}$, one can find another minimizing sequence $\{v_n\}$ such that $\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0$ and $\lim_{n \rightarrow \infty} \|Df(v_n)\| = 0$.

It is then natural to find a condition which will imply that the v_n in Proposition 4.3 converge (and then to a critical point). This will be therefore a compactness condition, and a very important tool in the direct method in variational calculus.

For a continuous function f on a Banach space (or manifold) $\mathcal{E}$ we shall denote by f^c and f_c respectively the sublevel sets and the level sets

$$f^c \equiv \{u \in \mathcal{E}, f(u) \leq c\}; \quad f_c \equiv \{u \in \mathcal{E}, f(u) = c\}$$

DEFINITION

Let $\mathcal{E}$ be a Banach space, and f be a real valued differentiable function. By definition, f satisfies the Palais-Smale (abbreviated P.S) condition at level c (i.e. on f_c) if from every sequence v_n with $f(v_n) \rightarrow c$, $\|Df(v_n)\| \rightarrow 0$, one can extract a convergence subsequence.

If $\bar{u}$ is the limit, one has then

$$f(\bar{u}) = c, \quad Df(\bar{u}) = 0$$

i.e. $\bar{u}$ is a critical point and $\{u : f(u) = f(\bar{u})\}$ is a critical level.

From this definition and Proposition 4.3 it follows in particular that f has a minimum if it is bounded below and satisfies the P.S. condition at level f_{c_0} where

$$c_0 \equiv \inf_{u \in \mathcal{E}} f(u)$$

Another tool which is of importance for the problemes treated here is a minimax result known as Mountain-pass theorem [A,R].

DEFINITION

Let $\mathcal{E}$ be a Banach space, and $f \in C^1(\mathcal{E})$. One says that f has a mountain pass geometry if

- i) $f(0) = 0$ and one can find positive constants a and r such that $\|u\| = r \Rightarrow f(u) \geq a$.
- ii) There exists $v \in \mathcal{E}$ such that $\|v\| > r$, $f(v) < a$.

Consider now the class of paths

$$\Gamma \equiv \{\gamma \in C^1([0, 1], \mathcal{E}) : \gamma(0) = 0, \gamma(1) = v\}$$

and set

$$c \equiv \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} f(\gamma(s))$$

One has then

PROPOSITION 4.4 (Mountain-pass Theorem)

If f has a mountain-pass geometry and satisfies the Palais-Smale condition at level c , then $c \geq a$ and c is a critical level for f , i.e. there is a critical point of f on level c .

The paths considered in the previous Proposition form a *minimax class* at level c for f .

Other minimax classes can be used to find critical points of the functional f , and are particularly important if one is looking for multiplicity.

To define in general a minimax class for the function f we need to introduce a Lemma (deformation lemma) which roughly speaking states that one can construct a flow along which f decreases if one is away from the critical points (this can be seen as a deformation of the level sets which puts the critical points in evidence). This construction allows to concentrate near the critical points the change in topology which occurs at critical levels.

PROPOSITION 4.5 (deformation lemma)

Let $\mathcal{H}$ be a Hilbert space and let $f \in C^{1,1}(\mathcal{H}, R)$ satisfy the Palais-Smale condition at level c . Then, given any $\epsilon_0 > 0$, for every neighborhood $\mathcal{U}$ of the set of critical points at level c one can find $\epsilon \in (0, \epsilon_0)$ and a homotopy $\eta \in C([0, 1] \times \mathcal{H}; \mathcal{H})$ such that

- i) $\eta(s, u) = u$ for all s and all $u \notin f^{c+\epsilon_0} - f^{c-\epsilon_0}$.
- ii) $f(\eta(s, u))$ is for all u a nonincreasing function of s .
- iii) $f(\eta(1, u)) \leq c - \epsilon$ for all $u \in f^{c+\epsilon_0} - \mathcal{U}$

We define now the minimax classes (for the function f) as follows:

DEFINITION

A minimax class Γ at level c for the function f on a Hilbert space $\mathcal{H}$ is a collection of subsets of $\mathcal{H}$ which for some ϵ_0 is invariant for the deformations described in Proposition 4.5 and is moreover such that

$$c \equiv \inf_{A \in \Gamma} \sup_{u \in A} f(u) \quad 4.1$$

It is easy to verify that with this definition the paths considered in Proposition 4.4 form a minimax class.

One has then the following generalization of Proposition 4.4.

PROPOSITION 4.6

Assume that $f \in C^{1,1}(\mathcal{H}; R)$, that Γ is a minimax class at level c and that P.S. holds at level c .

Then c is a critical level for f .

Several topological constructions have been introduced in the context of the N -body problem to find minimax classes (and therefore existence and multiplicity of solutions of the problem modified by the introduction of a strong force). We will briefly mention here three constructions. Not all of them are equally suited if one wants to obtain also information on the index of the critical points one finds; we have already remarked that an upper bound on the index can provide an upper bound on the number of collisions.

A construction which works well for existence and multiplicity, but has not been developed so far to provide estimates on the index, is related to category theory in the sense of Lusternik-Schnirelman.

Recall that if $\mathcal{M}$ is a topological space a subset $\mathcal{O} \in \mathcal{M}$ is called *contractible* (in $\mathcal{M}$) if one can find a point $m_0 \in \mathcal{M}$ and a deformation ϕ of $\mathcal{O}$ such that $\phi(1, m) = m_0$ for all $m \in \mathcal{O}$.

The Lusternik-Schnirelman (L.S.) category of $\mathcal{O}$ relative to $\mathcal{M}$, denoted by $\text{cat}(\mathcal{O}, \mathcal{M})$ is the smallest integer K such that $\mathcal{O}$ can be written as the union of K contractible sets. If no such K exists, $\text{cat}(\mathcal{O}, \mathcal{M}) \equiv +\infty$.

The category so defined has important properties with respect to inclusions and unions, which we shall not repeat here [M,W]. What is especially important is that it is upper semicontinuous under deformations, i.e.

$$\mathcal{O} \text{ closed, } \phi \text{ a deformation} \Rightarrow \text{cat}(\mathcal{O}, \mathcal{M}) \leq \text{cat}(\phi(1, \mathcal{O}), \mathcal{M})$$

This property allows to construct minimax classes for the N-body problem with strong force. Indeed, the singularity of the strong force potential allows the decompositions of the sublevel sets f^c into subsets which are not contractible (in f^c) and are invariant under the deformation described in Proposition 4.5 (the invariance does not hold in absence of strong force). For every integer k the class of subsets $\mathcal{O}$ for which $\text{cat}(\mathcal{O}, \mathcal{M}) \geq k$ is therefore a minimax class.

If some compactness is available, one derives existence and multiplicity of critical points. Indeed, for each integer k define

$$c_k \equiv \inf_{\gamma \in \mathcal{O}_k} \sup_{u \in \gamma} f(u)$$

Then

PROPOSITION 4.7

Let $\mathcal{M}$ be a complete $C^{1,1}$ manifold and let $f \in C^1(\mathcal{M}, \mathbb{R})$. If f satisfies P.S. at each level c one has

- i) every finite c_k is a critical level
 - ii) if $c_k = +\infty$ for some k , then f is not bounded above
 - iii) if for some k one has $c_k = c_{k+1} = \dots c_{k+m}$ then $\text{cat}(\mathcal{Z}_c, \mathcal{M}) \geq m+1$
- where $\mathcal{Z}_c$ is the collection of all critical points at level c .

In particular if f is bounded below, using the fact that one of the properties of category is $\text{cat}(\emptyset, \mathcal{M}) = 0$, one concludes that the number of critical points of f is at least $\text{cat}(\mathcal{M}, \mathcal{M}) \equiv \text{cat } \mathcal{M}$.

Category theory has been used to prove existence and multiplicity for the N-body problem (with the added strong force term) in [B,R] and in [M.T.1][M.T.2]; the starting point is a result of E.Faddell and S.Husseini [F,H.1][F,H.2] which describes the category for maps from S^1 to S^{n-1} , $n \geq 2$.

Denoting by Λ^1 these maps and by Λ_0^1 the collection of elements in Λ^1 which satisfy $x(t+1/2) = -x(t)$ one has

PROPOSITION 4.8 (Faddell-Husseini)

i)

$$\text{cat } \Lambda^1 = \text{cat } \Lambda_0^1 = +\infty$$

ii) for every integer $k \geq 1$ there exist compact sets $\mathcal{X} \in \Lambda^1$ (resp. $\mathcal{X}_0 \in \Lambda_0^1$) such that $\text{cat}(\mathcal{X}, \Lambda^1) = k$ (resp. $\text{cat}(\mathcal{X}_0, \Lambda_0^1) = k$).

This Proposition is used in the N-body problem considering as variational class the set of collections $\{\eta(x, t)\}$ of closed curves in $\mathbb{R}^{3N}$ parametrized by $x \in S^{n-1}$ and such that $\eta(x, t) \notin \Delta$ for all x, t .

A geometrical index is defined by considering the map

$$(x, t) \rightarrow \frac{\eta(x, t)}{|\eta(x, t)|}$$

and a corresponding homotopical index is defined by considering deformations of these sets. Notice that the functional one considers diverges to $+\infty$ if $x(t) \in H^1([0, T], R^{3N})$ approaches the coincidence set Δ and this allows to use (4.1)) in the minimax analysis. In fact, in presence of a strong force, the deformations of the level sets of the (action) function preserve this category structure.

Some of the contractible sets given by the Liusternik-Schirelmann category do not contain trajectories in which the bodies are very far apart from each other and have very small velocities; this provides an amount of compactness and then existence of critical points.

For the N -body problem, depending on the assumptions on the potential (in particular if it is not invariant under reflections, see below) some generalizations of Proposition 4.7 are needed (in particular to cover the case of critical points for the Maupertuis-type functional) in order to include [A,CZ.2] cases in which the P.S. compactness condition does not hold at all level sets c but there is a constant c_0 such that P.S. holds when $c \geq c_0$ (notice that for a functional of Maupertuis type compactness cannot hold at level zero, since this level contains all constant functions).

Other compactness results have been obtained [M,T.1][M,T.2] by restricting the variational functional f to sublevel sets $g \leq c_0$ of a suitable C^1 function g , and making use of properties of the gradient flow of f (or some approximated flow) at the boundary $g = c_0$.

These approaches do not always provide good upper bounds on the index of the critical points for which existence is proved.

A related approach, in which Morse theory is used to find non-contractible invariant sets makes use of a *linking structure* [Be, R][Be] (a generalization of the mountain pass structure described above). One considers as minimax classes a set of paths γ in a fibered structure over $\mathcal{H}$, with fibers the spheres of radius one and dimension $n - 1$, chosen so that $\gamma(0) \equiv \{x_0 \in \mathcal{H}, S^0\}$ where x^0 is a critical point and S^0 is a maximal sphere of radius one in the positive eigenspace of $D^2 f(x^0)$. A critical value of f is then defined as usual by minimax, and if compactness can be proved one concludes that there is a critical point at this level. In this case, the index of the critical point x^0 can be estimated by $i(x^0) \leq n$ (notice that the mountain pass geometry corresponds to the case $n = 1$).

Also in this case, in applications to the N -body problem, one must make use of a family of modified functionals f_δ to which the linking argument can be applied, and then prove that one can find a sequence of critical points x_n^0 for the modified functionals, which converge along subsequences $\delta_n \rightarrow 0$ to a weak solution of Newton's equations and are such that their index is bounded above by n .

The importance of a good uniform *upper bound* on the index of the sequence of approximating critical points x_δ^0 will be apparent in the next Section, where we describe results about existence of classical solutions of (0.1). Indeed, these results will be obtained by providing a *lower bound* on the index of x_δ^0 for δ sufficiently small (or on the generalized index of the limit weak solution x^0) *under the assumption that the collision set of x^0 be non empty*. The proof of existence of a classical solution (or an upper bound on the number of collisions) is then given by comparing the bound from above (obtained by variational techniques) with the bound from below.

In what follows, we shall mostly be interested in bounds which can be derived from assumptions on the potential which refer only to its behaviour near the coincidence set.

We now describe briefly some results on existence of weak solutions for the N -body problem through the variational techniques described above, selecting in particular those results which will be used in Section 5 to give bounds on the number of collisions of the weak solutions obtained.

More details, and an extensive list of references can be found in the monography [A,CZ.1] and in the lecture notes [CZ.1].

It is convenient to describe separately the cases of preassigned period T (problem P_T) and preassigned energy E (problem P_E). In the former case we assume that the potential is T -periodic, in the latter we assume that the potential is time-independent.

In what follows we denote by V_K the newtonian N -body potential

$$V_K \equiv \sum_{1 \leq k < h \leq N} V_{h,k}(|x_k - x_h|), \quad V_{h,k}(|y|) \equiv m_k m_h |y|^{-1} \quad x_k \in R^3 \quad 4.1$$

where $\{m_k, k = 1, \dots, N\}$ are the masses of the N bodies and we have chosen units in which the gravitational constant takes value one.

Notice that V_K is invariant under the map $x_k \Rightarrow -x_k \forall k$. Most of the results which we shall describe hold also if

$$V_{k,h} = \frac{C_{h,k}}{|y|^\alpha}, \quad 0 < \alpha < 2$$

or even if in a small neighborhood of the coincidence set

$$\frac{D_{h,k}}{|y|^\alpha} \leq V_{k,h} \leq \frac{C_{h,k}}{|y|^\beta}, \quad 0 < \alpha \leq \beta < 1$$

We shall point out, when appropriate, special features corresponding to further restrictions on the parameters α and β .

As we shall see shortly, there is a major difference both in the approach and in the results depending on whether or not $V_{h,k}(x) = V_{h,k}(-x)$.

We discuss first results which bear on problem P_T . With this notation, for problem P_T the functional considered is, as described in Section 1,

$$\Phi_T(x(\cdot)) \equiv \int_0^T \left[\frac{1}{2} \|\dot{x}(t)\|_M^2 + V_K(x(t)) + W(x(t), t) \right] dt \quad 4.2$$

defined on

$$\Lambda_T \equiv \{x(t) : x(t) \in H^1([R^{3N} - \Delta] \times [0, T], R), \int_0^T [V_K(x(t)) + W(x(t), t)] dt < +\infty\}$$

Precise assumptions on $W(x, t)$ will be described later, since they are different in the different cases described below. Roughly speaking, the main assumption is that W be bounded below, T -periodic, of class C^2 in x away from the coincidence set Δ , uniformly in t , and that near Δ it be less singular than V_K .

As remarked previously, this functional is not everywhere differentiable in its domain of definition. To apply the general results and procedures which we have briefly described above one considers then the sequence of functionals

$$\Phi_T^\delta(x(\cdot)) \equiv \int_0^T \left[\frac{1}{2} \|\dot{x}\|_M^2 + V_K(x(t)) + W(x(t), t) + \delta V_S(x(t)) \right] dt. \quad \delta > 0 \quad 4.3$$

where the additional potential V_S represent an attractive force, which is chosen so singular that Φ_T^δ is differentiable on its entire domain of definition. A common choice is

$$V_S(x) \equiv \sum_{h < k} \frac{C}{|x_h - x_k|^2}, \quad C > 0 \quad 4.4$$

but other choices are possible (with a stronger singularity at the coincidence set) and have been made [T].

An alternative procedure, which leads to similar results and could be more versatile for use of local estimates, is to "mollify" the newtonian potential near the coincidence set [A,S], making it regular there (for example by convoluting it with a smooth positive function of total mass one, supported in a ball of radius δ at the origin), and then to consider sequences of solutions for the modified problem when $\delta \Rightarrow 0$.

We shall not discuss any further here this regularizing scheme; we remark however that it may be more suitable to provide a relation between the indices of the approximating sequence and the (generalized) index the limit weak solution.

REMARK

It is perhaps worth remarking that all the assumptions which we shall discuss are satisfied by the newtonian potential, and therefore also if one adds perturbations which are "sufficiently small".

Within the scheme of modification through a strong attractive force, the strongest results have been obtained under the assumption that the potential be invariant under reflection about some point in R^{3N} (e.g. the baricentre of the system if it is isolated).

This case will be treated first, both for P_T and for P_E .

It is difficult to say at present whether this condition reflects an inadequacy of the variational techniques which have been employed so far, or whether it is really needed to obtain the stronger results.

Without loss of generality we shall consider the case in which the symmetry is required with respect to the origin; if the potential is T -periodic the condition reads

$$V_{i,k}(x) = V_{i,k}(-x), \quad W(x, t + \frac{T}{2}) = W(-x, t) \quad \forall x \in R^{3N}, \quad t \in R \quad (H)_2$$

We treat first the case in which $(H)_2$ is satisfied.

A_T : FIXED PERIOD PROBLEM; $(H)_2$ SATISFIED

If $(H)_2$ is satisfied, one can restrict the variational formulation to "T-antiperiodic" functions, i.e. functions which satisfy $x(t + T/2) = -x(t)$; in particular the domain of definition becomes now

$$\Lambda_T^a \equiv \{x \in \Lambda_T, \quad x(t + \frac{T}{2}) = -x(t) \quad \forall t\}$$

Indeed, it is easy to verify that if $x(t) \in \Lambda_T^a$ is a critical point for the restriction of Φ_T^δ to Λ_T^a , then $x(t)$ is also a critical point of Φ_T^δ .

By Poincaré's lemma, every bounded set in Λ_T^a is compact in the topology of continuous functions; this leads to a simpler and more general application of the minimax procedure to obtain critical points.

One can prove [CZ.1]

PROPOSITION 4.9

Assume that the potential W satisfies $(H)_2$ and moreover for all $x \in R^{3N} - \Delta$, $t \in R$

- i) $W \in C^2((R^{3N} - \Delta) \times R, R)$
- ii) $W(x, t) \leq 0 \quad W(x, t + T) = W(x, t)$
- iii)

$$\frac{|\partial_t W(x, t)|}{V_k} + \frac{W(x, t)}{V_K(x, t)} + \frac{|\nabla_x W(x, t)|}{|\nabla V_K(x)|} + \frac{\|D_x^2 W(x, t)\|}{\|D^2 V_K(x)\|} < C d(x, \Delta)^\gamma$$

for some $\gamma > 0$, where d denotes euclidean distance.

Then for each $\delta > 0$ Φ_T^δ attains its minimum value in Λ_T^a . Moreover, denoting by x_δ^0 a collection of minima, there is a positive constant C such that

$$\Phi_T^\delta(x_\delta^0) \leq C \quad \forall \delta > 0 \tag{4.5}$$

From (4.5) and the assumption $W \geq 0$ one concludes that

$$\int_0^T \|\dot{x}_\delta^0\|_M^2 dt \leq C \tag{4.6}$$

Since bounded sets in Λ_T^a are weakly compact, one can find a sequence of x_δ^0 which converge, weakly in H^1 and pointwise, to a function x^0 which can be proved [A,CZ.1][B,R] to be a weak solution of (0.1).

The property of x_δ^0 to be an absolute minimum will be exploited in the next section to prove that if W is time-independent there exists a classical periodic solution of P_T provided a condition $(H')_M$ is satisfied; this condition is related to the structure of the set of central configurations for the N -body problem.

B_T : FIXED PERIOD PROBLEM; $(H)_2$ NOT SATISFIED

If assumption $(H)_2$ is dropped, still in the fixed period problem, weaker results can be obtained, under stronger assumptions on $W(x, t)$. Now one has to rely on Morse theory or on the theory of category for the free loop space to prove existence of critical points x_δ^0 for the functionals Φ_T^δ and convergence, weakly in H^1 and pointwise, to a weak solution of (0.1).

For this one has to impose various conditions on the behaviour of the potential outside a neighborhood of the collision set.

Consider for example the case in which the modified N -body potential can be written as

$$V_K + W \equiv V \equiv \frac{1}{2} \sum_{h \neq k} V_{h,k}(x_h - x_k, t)$$

with

$$V_{h,k} \in C^1((R^{3N} - \Delta) \times S_T; R), \quad V_{h,k}(x, t) = V_{h,k}(x, t + T)$$

Existence of critical points (in fact, of infinitely many geometrically distinct critical points) for the system described by the potential

$$V(x, t) + \delta V_S(x), \quad V_S(x) \equiv \sum_{h < k} \frac{1}{|x_h - x_k|^2}$$

have been obtained under further assumptions on the functions $V_{h,k}(y, t)$.

We quote here two types of conditions; both lead to existence and also to estimates of the type (4.6) from which one derives existence, as limit of subsequences, of a weak solution of (0.1). Remark that, since weak limits are taken, and the variational classes which are used are not closed under weak convergence, no multiplicity result are given for these weak solutions.

The first results in this directions were obtained by A. Bahri and P. Rabinowitz. In addition to the assumptions on $V_{h,k}$ given above, they assume

$$V_{h,k}(x, t) \geq 0 \quad \lim_{|x| \rightarrow 0} \sup_t V_{h,k}(x, t) = 0 \quad 4.7$$

For $N \geq 3$ one has lack of compactness; in particular the failure of the *P.S.* condition is due to the fact that one can have subsequences along which the system splits in two or more widely separated clusters, each of which is an approximate solution of Lagrange's equations.

This difficulty has been overcome in [B,R],[Ri] by exploiting the fact that the topology of the configuration space for N bodies is much richer than the topology of the product of the configuration spaces of the composing clusters (topology "at infinity").

Along these lines, one can prove [B,R],[Ri]

PROPOSITION 4.10

Suppose that $V_{h,k}$ satisfies, in addition to the requirements described above (including (4.7))

$$\forall M > 0 \quad \exists R(M) > 0 : y \cdot \nabla V_{h,k}(y, t) > M |\nabla V_{h,k}(y, t)| \quad \forall |y| > R(M) \quad 4.8$$

Then for every $\delta > 0$ the functional Φ_T^δ has infinitely many critical points which are geometrically distinct and satisfy (4.6) (for an increasing sequence of positive constants C).

Notice that condition (4.8) is essentially a requirement that the function $V_{h,k}$ be, as a function of y , "essentially radial" at very large distances.

Results of the same type have been obtained also by Majer and Terracini [Mj,T.1], under conditions on the functions $f_{h,k}(y, t)$ which are somewhat different but still related to the angle between the gradient and the ray through the origin. They prove

PROPOSITION 4.11

Suppose that instead of (4.8) the functions $V_{h,k}(y, t)$ satisfy the following assumption

$$\exists \rho > 0, \quad \theta \in [0, \pi/2), \quad \text{s. t. } \text{ang}(y, \nabla_y V_{h,k}(y, t)) \leq \theta \quad \forall |y| > \rho \quad 4.9$$

where $\text{ang}(\xi, \eta)$ is the angle between the two vectors ξ and η .

Then the functional Φ_T^δ has infinitely many critical points which are geometrically distinct and satisfy (4.6) (for an increasing sequence of positive constants C).

The idea of the proof is rather different from that of Proposition 4.10. Here one makes use of an auxiliary function g on Λ_T such that the *P.S.* condition is satisfied in its the sublevel sets and that the intersections of the sublevel sets of g and of Φ_T^δ have a rich topological structure.

The topological arguments used in the proof of Propositions 4.10 and 4.11 do not lead to sharp estimates on the index of the critical points whose existence is stated. As a result, no explicit upper bound is known for the number of collisions in the limit weak solution $x^0(t)$.

An exception is the case $N = 2$ or which the following result is known

PROPOSITION 4.12 [B,L][L,S],

For $N = 2$ the critical points x_δ^0 obtained by the minimax procedure described in Proposition 4.10 are of mountain-pass type, i.e. $i_\delta(x_\delta^0) \leq 1$ where $i_\delta(x_\delta^0)$ is the index of x_δ^0 as critical point of Φ_T^δ .

We discuss next results which bear on problem $(P)_E$.

Now the potential is independent of time, and the variational functional is, as indicated in Section 1,

$$\Psi_E(x(\cdot)) \equiv \frac{1}{2} \|x(\cdot)\|_M^2 \cdot \int_0^1 (E + V(x(t))) dt, \quad \|x(\cdot)\|_M^2 \equiv \frac{1}{2} \int_0^T \sum_{k=1, \dots, N} m_k \dot{x}_k(t)^2 dt$$

defined on Λ_1^a .

If x^0 is a critical point of Ψ_E at a positive level and Ψ_E is differentiable at x^0 , and one sets

$$\omega_0^2 \equiv \frac{2 \int_0^1 (E + V(x(t))) dt}{\|x(\cdot)\|_M^2}$$

then $\tilde{x}^0(t) \equiv x^0(\omega_0 t)$ is a classical periodic solution of (0.1) with energy E .

As before, this functional is not everywhere differentiable in its domain of definition and one is lead to consider the family of functionals Ψ_E^δ obtained through the addition of a strong force potential, as in the fixed period case

$$\Psi_E^\delta(x(\cdot)) \equiv \frac{1}{2} \|x(\cdot)\|^2 \cdot \int_0^1 (E + V(x(t)) + \delta V_S(x(t))) dt, \quad V_S(x) \equiv \sum_{h < k} |x_k - x_h|^{-2}$$

Also here we shall consider first the case in which condition $(H)_2$ is satisfied.

A_T: FIXED ENERGY PROBLEM, $(H)_2$ SATISFIED

The best results so far for $N \geq 3$ are in [A,CZ.3]

The potential $V(x) \equiv V_K(x) + W(x)$, $x \in R^{3N}$ has again the form

$$V(x) = \sum_{h \neq k} V_{h,k}(x_k - x_h), \quad V_{h,k} \in C^1((R^3 - \{0\}), R) \quad 4.10$$

and satisfies

- i) $V_{h,k}(y) = V_{h,k}(-y)$
- ii) $\exists \alpha \in [1, 2[$ such that $y \cdot \nabla V_{h,k}(y) \leq -\alpha V_{h,k}(y) \quad \forall y \neq 0$
- iii) $\exists \beta \in]0, 2[$ and $r > 0$ such that $y \cdot \nabla V_{h,k}(y) \geq -\beta V_{h,k}(y) \quad \forall 0 < |y| \leq r$
- iv) $V_{h,k}(y) \rightarrow 0$ as $|y| \rightarrow \infty$, $V_{h,k}(y) \leq 0$

Notice that, as in the case of prescribed period without symmetry, these conditions are non-local; in fact they impose restrictions on W on the structure of the potential on the entire configuration space (for comparison the conditions for $\mathbf{A}_T$, with the exclusion of symmetry, only required information on the behaviour of W at the coincidence set Δ).

It is proved in [A,CZ.3] that if V satisfies i) to iv) the functional Ψ_E^δ has for all $\delta > 0$ a mountain pass structure (in a somewhat more general sense than given before). Indeed it can be verified that under assumption i) to iv) one can find positive constants a and $\rho' > \rho$ such that

- a) $\Phi_E^\delta \geq a \quad \forall x \in \Lambda^a, \quad \|x(\cdot)\|_1 = \rho'$
- b) $\exists \delta_0 > 0, x_0, x_1 \in \Lambda^a, \|x_0(\cdot)\|_1 = \rho, \|x_1(\cdot)\|_1 = \rho'$ such that $\Phi_E^\delta(x_0(\cdot)), \Phi_E^\delta(x_1(\cdot)) < a$
- c) $(x, \nabla \Phi_E^\delta(x)) > 0$ for all $\|x\|_1 = \rho$.

and it can also be proved that the *P.S.* condition is satisfied at the minimax level.

Assumptions i) ... iv) are also sufficient to prove that the mountain-pass critical points one obtains satisfy (4.8) so that a weak solution x^0 of (0.1) can be obtained as limit of a subsequence of these critical points.

One has therefore [A,CZ.1]

PROPOSITION 4.13

Suppose that V is the form given in (4.9) and satisfies i), ... iv). Then for every $E < 0$ and $\delta > 0$ the functional Φ_E^δ has a critical point x_δ^0 of mountain-pass type. Moreover x_δ^0 converges, for $\delta \rightarrow 0$, along subsequences, to a function $x^0(t)$ such that $\tilde{x}^0(t) \equiv x^0(\omega_0 t)$ is a weak solution of (0.1), where

$$\omega_0 = \lim_{\delta \rightarrow 0} \omega_\delta, \quad \omega_\delta \equiv \frac{2 \int_0^1 [E + V(x_\delta^0(t)) + \delta V_\delta(x_\delta^0(t))] dt}{\|x_\delta^0(\cdot)\|^2}$$

A sharper statement is proved if one makes stronger assumptions on the functions $V_{h,k}$.

If one assumes, in addition to i) ... iv)

$$v) \quad 3(x, \nabla V_{h,k}(x)) + (x, D^2 V_{h,k}(x)x) \neq 0 \quad (x, \nabla V_{h,k}(x) < 0 \quad \forall x \in R^{3N} - \Delta,$$

then the functional $\Phi_E^\delta(x_0)$ restricted to a suitable manifold of codimension one attains there its minimum. This is a sharper result since it may be possible by a local analysis to prove that this cannot be true if x^0 has a collision.

To understand the role of assumption v) recall that all critical points must lie, for every $\delta > 0$, on the set

$$\mathcal{M} \equiv \{x(t) \in \Lambda_1^a, \quad g(x(\cdot)) = 0\}, \quad g(x(\cdot)) \equiv \int_0^1 [2V(x(t)) + x(t) \cdot \nabla V(x(t))] dt + 2E \quad 4.12$$

Assumption v) implies that if $E < 0$ the half line $\lambda x(t)$, $\lambda > 0$ intersects $\mathcal{M}$ transversally so that $\mathcal{M}$ is a smooth manifold.

Moreover for every $\delta > 0$ critical points x_δ^0 of the restriction of Φ_E^δ to $\mathcal{M}$ are also critical points of Φ_E^δ . In fact, they satisfy

$$(D\Phi_E^\delta)_{x_\delta^0} + \lambda_0 Dg_{x_\delta^0} = 0$$

and evaluating this function on x_δ^0 one derives $\lambda_0 = 0$.

Moreover assumption v) implies that the restriction of Φ_E^δ to $\mathcal{M}$ is bounded below and that the intersection of the sublevel sets of Φ_E^δ with $\mathcal{M}$ is closed and bounded in Λ_1^a .

One concludes

PROPOSITION 4.14

Suppose that V is the form given in (4.10) and satisfies $i), \dots, v$). Then for every $E < 0$ and $\delta > 0$ the functional Φ_E^δ restricted to the manifold $\mathcal{M}$ defined in (4.12) attains its minimum. Let x_δ^0 be a minimum point. Then x_δ^0 is a critical point of Φ_E^δ of mountain pass type with index one. Moreover as in Proposition 4.13 x_δ^0 converges, for $\delta \rightarrow 0$, to a function $x^0(t)$ such that $\tilde{x}^0(t) \equiv x^0(\omega_0 t)$ is a weak solution of (0.1), where ω_0 is given in Proposition 4.13.

As last case, we indicate some results which have been obtained for the fixed energy problem when condition $(H)_2$ is not satisfied

B_E : FIXED ENERGY PROBLEM, $(H)_2$ NOT SATISFIED

Results for this case are in [A,T,V]. Related results for the case of strong force potentials (and positive energies) are in [Mj,T.2] where some of the techniques used in [A,T,V] are introduced, in particular critical point theory with boundary conditions, through the introduction of an auxiliary functional to provide subsets in which the lack of the Palais-Smale condition can be overcome. This procedure was not needed in case A_E since bounded closed sets in Λ_1^a are compact in the topology of continuous functions.

One restricts now the functional to the manifold

$$\mathcal{M}' \equiv \{X(t) \in \Lambda, \quad \sum_k m_k \int_0^1 \dot{x}_k^2(t) dt = 2\}$$

and proves that critical points of the restriction of Ψ_E^δ to $\mathcal{M}'$ are also critical points of Ψ_E^δ . One then relates the critical points of the restriction of Ψ_E^δ to $\mathcal{M}'$ to the critical points of a suitable functional Ψ' on $\mathcal{M}'$. The functional Ψ' is bounded below, and critical points are found using critical point theory with boundary conditions. Notice that now the Palais-Smale condition must be verified, since one is working in Λ .

While this procedure is effective to find critical points, their index cannot be easily controlled.

We refer to [A,T,V] for the detailed analysis. Here we only quote the results

The following assumptions are made on the potential

As in case A_E the potential $V(x) \equiv V_K(x) + W(x)$, $x \in R^{3N}$ has the form

$$V(x) = \sum_{h < k} V_{h,k}(x_k - x_h), \quad V_{h,k} \in C(R^3 - \{0\})$$

and satisfies

- i) $\exists \alpha \in [1, 2[$ such that $y \cdot \nabla V_{h,k}(y) \geq -\alpha V_{h,k}(y) \quad \forall y \neq 0$
- ii) $\exists \beta \in]0, 2[$ and $r > 0$ such that $y \cdot \nabla V_{h,k}(y) \leq -\beta V_{h,k}(y) \quad \forall 0 < |y| \leq r$
- iii) $V_{h,k} \rightarrow 0$ as $|y| \rightarrow \infty$
- iv) $3y \cdot \nabla V_{h,k}(y) + (D^2 V_{h,k} y, y) \neq 0 \quad \forall y \in \mathbb{R}^3$
- v) $2V_{h,k}(y) + y \cdot \nabla V_{h,k}(y) \rightarrow 0 \quad \text{as } |y| \rightarrow \infty$
- vi) There exists $\theta \in [0, \frac{\pi}{2})$ and $r_2 > 0$ such that

$$\text{ang}(\nabla V_{h,k}(y), y) \leq \theta \quad \forall |y| \geq r_2$$

where $\text{ang}(\xi, \eta)$ is the angle between the vectors ξ and η if both are nonzero, and is zero otherwise.

Under these assumptions one has

PROPOSITION 4.15

Suppose i) to vi) hold. Then for all $E < 0$ there is a critical point x_δ^0 of Φ_E^δ . Moreover one can choose these critical points so that they converge, weakly in H^1 and pointwise, to a function $x^0(t)$ such that for a suitable constant ω_0 the function $\tilde{x}^0(t) \equiv x^0(\omega_0 t)$ is a weak solution of (0.1) with energy E .

Since the critical points described in this theorem for $\delta \neq 0$ are obtained by using category theory, little information is obtained on their index.

For this reason one cannot draw useful conclusions on the maximum number of collisions in $\tilde{x}^0(t)$.

REMARK

The solution $x^0(t)$ of (0.1) whose existence is asserted in Propositions 4.11 to 4.14 has a collision set with open and dense complement. For $N = 2$ it has further regularity properties [A.CZ.1]. In particular the number of collisions is finite. The proof is based on conservation of energy (or similar estimates in the time-dependent case) which, for $N = 2$ provides a lower bound on the kinetic energy of the relative motion; this, in turn, implies a lower bound on the time which must elapse between successive collisions. This conclusion cannot be drawn immediately when $N \geq 3$ since the lower bound applies to the total kinetic energy but not to the kinetic energy of the relative motion of bodies in a cluster. In fact, it is known that a triple collision can be an accumulation point of two-body collisions. From local analysis it is known that total collisions cannot accumulate; it would be very interesting to have sharper results from variational estimates.

5. CLASSICAL PERIODIC SOLUTIONS FOR THE WEAKLY PERTURBED N-BODY SYSTEM

In the previous Section we have outlined results about existence of weak solutions of the system (0.1) under suitable conditions on the perturbing potential W . We have distinguished four cases (denoted by $\mathbf{A}_T$, $\mathbf{A}_E$, $\mathbf{B}_T$, $\mathbf{B}_E$); $\mathbf{A}$ refers to potentials which satisfy the symmetry property while in case $\mathbf{B}$ this condition is not satisfied. The subscript T and E refer respectively to the case of preassigned period and preassigned (negative) energy.

In particular in case $\mathbf{A}_T$ (i.e. preassigned period and condition $(H)_2$ holds) condition $(H)_1$ is sufficient to guarantee the existence of a periodic weak solution which is the limit, in $H^1(\mathbb{R}, \mathbb{R}^{3N})$ and pointwise, of a sequence of critical points $x_{\delta_n}^0$ of the Φ_T^δ . Moreover these approximating critical points are minima. In case $\mathbf{A}_E$ (preassigned energy and $(H)_2$

holds) further assumptions on W are required, and one concludes that x_δ^0 are mountain pass critical points (so that their index is bounded above by one).

In this Section we shall describe results about existence of classical solutions of (0.1), i.e. answers to problems P_T and P_E .

As remarked in Section 4, the general strategy is to prove that, if the limit critical point x^0 described in Section 4 has a collision, then one can give *lower bounds* on the generalized index of x^0 or on the index of the approximating solutions x_δ^0 . If this lower bound is sufficiently sharp, and if the upper bounds one finds by variational arguments are sufficiently sharp, a contradiction will follow, and this establishes the existence of a classical periodic solution.

In cases in which the two bounds do not contradict each other, one may at least set an upper bound on the number of collisions.

For $N = 2$ the behaviour of x^0 at collisions and of x_δ^0 at the minimal distance between the two bodies are comparatively simple, and in fact they do not differ in any essential way from those described in Section 2. One can therefore give sharp lower bounds on the indices of x_δ^0 for $\delta > 0$ and on the generalized index of x^0 . This allows to derive upper bounds on the number of collisions and in favourable cases to conclude that x^0 has no collisions, and is therefore a classical solution of (0.1). We shall briefly discuss these results, which are due to K. Tanaka [T].

For $N \geq 3$ lower bounds are more difficult to obtain since much less is known about the behaviour of the approximating functions x_δ^0 "at minimal distance between any two of the bodies". For this reason, for $N \geq 3$ the results obtained so far are limited to the case $\mathbf{A}_T$ [D],[S,T] and require a further assumption $(H')_M$ which we will introduce shortly.

In this Section we shall describe results for $\mathbf{A}_T$ when $N \geq 2$ and for $\mathbf{B}_T$ and $\mathbf{B}_E$ when $N = 2$.

THE CASE $\mathbf{A}_T$

We shall make use of Proposition 4.9; for convenience of the reader we recall here the setting and the assumptions.

We consider the perturbed N-body problem described by the system of equations (0.1)

$$m_i \ddot{x}_i = \nabla_i(V_K + W) \quad x_i \in R^3, \quad i = 1, \dots, N \quad (0.1)$$

$$V_K \equiv - \sum_{k \neq j} \frac{m_i m_k}{|x_i - x_k|}$$

where the gravitational constant has been set equal to one.

We shall assume that the "perturbing potential" W is *time independent* and has the following "regularity" property $(H)_1$,

$$W \in C^2((R^{3N} - \Delta), R), \quad W \geq 0,$$

$$\frac{W(x)}{V_K(x)} + \frac{|\nabla_x W(x)|}{|\nabla V_K(x)|} + \frac{\|\frac{\partial^2 W(x)}{\partial x_i \partial x_j}\|}{\|D^2 V_K(x)\|} < C d(x, \Omega)^\gamma \quad (H)_1$$

for all t and for $0 < d(x, \Omega) < \delta_0$, where $C, \delta_0, \gamma > 0$ and $d(x, \Omega)$ is the (euclidean) distance of the point of coordinates x from the collision set Δ defined by

$$\Delta \equiv \{x \in R^{3N}, : x_i = x_k \text{ some } k \neq i\}.$$

In addition to $(H)_1$ we assume the "symmetry property" $(H)_2$:

$$W(x) = W(-x) \quad (H)_2$$

REMARK 1

We recall that condition $(H)_2$ is introduced in order to be able to find critical points by restricting the functionals Φ to a subspace of $H^1(R, R^{3N})$ in which Poicaré inequality provides compactness (here the subspace of T -antiperiodic functions). Condition $(H)_2$ does not play other roles in what follows, and does not seem to be a "natural" condition; one may conjecture that existence of at least one strong periodic solution for every preassigned period T can be proved using only assumption $(H)_1$.

REMARK 2

The condition that W be time-independent enters crucially in our method of proof, because we need to compare the value of Φ_T at different trajectories, constructed by deleting parts of the orbit and tracing the remaining part with the same speed but at different times. This will be needed in order to construct new trajectories which have only collisions which are strongly regular (the definition will be given presently). The case $N = 2$ is exceptional from this point of view, since all collisions are strongly regular. Only for this type of collisions can one derive easily estimates on the "generalized index" (or on the indices of an approximating sequence). Proving the same results in the case in which the potential W is time-dependent seems a difficult task.

REMARK 3

It is easy to verify that the estimates we give for the variation of Ψ_T along special directions (which can be constructed if the collision is strongly regular) apply also to the functional Φ_E if all collisions of the solution x^0 itself are strongly regular, and this is a very strong assumption which cannot be verified in general.

We shall prove that for every preassigned T the system (0.1), with W time-independent, has a strong (classical) T -periodic solution under assumptions $(H)_1$, $(H)_2$ and under the further condition $(H')_M$ which we state now

For the given value of the masses, every non planar central configuration for any subset of $M \leq N$ bodies is isolated modulo rotations $(H')_M$

Central configurations and their role were discussed in Section 3. We recall that $(H')_2$, $(H')_3$, $(H')_4$ are satisfied for all value of the masses, while it is expected that $(H')_M$, $M \geq 5$ be satisfied for all value of the masses, with the possible exception of an algebraic set (i.e. a set of masses which satisfies an algebraic equation). We have also remarked there that no collision is known for which $(H')_M$ is not satisfied for some M .

As remarked in the previous Sections, to obtain a classical solution one considers a family of modified action functionals Φ_T^δ defined by

$$\Phi_T^\delta((x(\cdot))) \equiv \Phi_T((x(\cdot))) + \delta \int_0^T V_S(x(t)) dt, \quad V_S(x) \equiv \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^2}$$

defined on

$$\Lambda_T^a \equiv \{x(t) : x(t) \in \Lambda_T, \quad x_i(t + \frac{T}{2}) = -x_i(t)\}$$

Under $(H)_2$ critical points of Φ_T^δ are classical solutions of the system of equations

$$m_i \ddot{x}_i = - \sum_{k \neq j} \frac{m_i m_k}{|x_i - x_k|^3} (x_i - x_k) + \nabla_i (V_K + W + \delta V_S)$$

We recall from Section 4, Proposition 4.9 that under assumptions $(H)_1$ and $(H)_2$ there exists a collection of minima x_δ^0 of Φ_T^δ which converge along subsequences when $\delta \rightarrow 0$, weakly in $H^1((0, T), R^{3N})$ and uniformly in $[0, T]$, to a function $x^0(t)$ which is a weak solution of (0.1).

Notice that all x_δ^0 are antiperiodic, so that they cannot correspond to critical points of $V + W + \delta V_s$ (since this potential is singular at the origin, which is the only point invariant under reflections).

As a first step in the proof of existence of a classical solution of (0.1) we shall prove the following simple Lemma

LEMMA 5.1

Let x^0 be the limit weak solution described in Proposition 4.9. Then one cannot find a T -periodic function $\tilde{x}(t)$ with values in R^{3N} and a vector $\eta \in R^{3N}$ such that

$$i) \quad \Phi_T(\tilde{x}^0(\cdot) + \eta) < \Phi_T(\tilde{x}^0(\cdot)) \leq \Phi_T(x^0(\cdot))$$

$$ii) \quad \inf_t d[\tilde{x}^0(t) + \eta, \Delta] > 0$$

where d denotes euclidean distance.

Proof

Since x_δ^0 are minima one has

$$\Phi_T^\delta(\tilde{x}^0(\cdot) + \eta) \geq \Phi_T^\delta(x_\delta^0(\cdot)) > \Phi_T(\tilde{x}_\delta^0(t)) \quad 5.2$$

Condition *ii*) implies that $\tilde{x}^0(t) + \eta$ has no collisions and therefore

$$\lim_{\delta \rightarrow 0} \Phi_T^\delta(\tilde{x}^0(\cdot) + \eta) = \Phi_T(\tilde{x}^0(\cdot) + \eta) \quad 5.3$$

Since $\Phi_T^\delta(x(\cdot)) - \Phi_T(x(\cdot)) > 0$ for every $x(t)$ taking the limit $\delta \rightarrow 0$ in (5.2) and using (5.3) one derives

$$\Phi_T(\tilde{x}^0(\cdot) + \eta) > \Phi_T(x^0(\cdot))$$

contradicting *i*). ///

We shall now prove that, if x^0 in Proposition 4.9 has a collision, one can construct $\tilde{x}^0$ and η with the properties *i*) and *ii*) in Lemma 5.1. This leads to a contradiction, proving that x^0 is a classical solution.

It will be convenient to introduce some more notation.

DEFINITION

We shall say that a trajectory $x(t) \in C^0(R, R^{3N})$ has a *left-isolated M -collision* at time 0 if there is a set I_M of M indices for which

$$x_i(t) \in C^2((-\epsilon, 0), R^3) \quad \forall i \in I_M,$$

$$i, k \in I_M \Rightarrow x_i(0) = x_k(0) \quad i \in I_M, k \notin I_M \Rightarrow x_i(0) \neq x_k(0)$$

In a similar way, considering now right neighborhoods of 0 one defines *right-isolated collisions*.

A collision which is both left- and right-isolated is called *isolated*. Without loss of generality we shall assume that the cluster of colliding bodies we consider have indices $1, \dots, M$. Other clusters may or may not have a collision at $t = 0$. If $M = N$, we shall speak of a *simultaneous collision*.

Without loss of generality we take an inertial reference frame for which $x_k(0) = 0$, $k = 1, \dots, M$.

Let $\Omega_M^\pm$ be the limit set for $t \rightarrow 0^\pm$ of

$$\xi(t) \equiv \{\xi_k(t)\} \quad \xi_k(t) \equiv |t|^{-2/3} x_k(t), \quad k = 1 \dots M \quad 5.4$$

Under our assumptions on V one can prove that the baricenter of the M points we are considering is of class C^1 in a neighborhood of $t = 0$.

Therefore $\sum_1^M m_k x_k(t) = 0(t)$ and we conclude that

$$\xi \in \Omega_M^\pm \Rightarrow \sum_1^M m_k \xi_k = 0$$

DEFINITION

We shall say that the isolated M -collision at $t = 0$ is

i) *Right-directional (resp. left-directional)* if the set Ω_+ (resp. Ω_-) reduces to a point; in this case we use the notation

$$\lim_{t \rightarrow 0^\pm} \xi_k(t) \equiv \xi_k^\pm, \quad k = 1, \dots, M \quad 5.5$$

with

$$\sum_1^M m_k \xi_k^\pm = 0$$

ii) *Directional* if it is both left and right directional. Notice that ξ^+ may differ from ξ^- . If these two configurations coincide we shall say that the collision is *strongly directional*.

iii) *Planar to the right (resp. to the left)* if one can find vectors $\zeta^\pm$ such that

$$\lim_{t \rightarrow 0^\pm} (\zeta^\pm, \xi_k(t)) = 0 \quad 5.6$$

iv) *Planar* if it is planar on both sides, and *strongly planar* if $\zeta^+ = \zeta^-$.

v) *Regular* if on each side it is either directional or planar (or both), and *strongly regular* if it is either strongly directional or strongly planar.

REMARK

Two-body collisions are directional (and therefore regular). Isolated two- and three-body collisions are directional. We have recalled the proof for $W = 0$ [Mc.1], [Su], [S M] in Section 3 for simultaneous collisions. We have also sketched the extension of the proof for $W \neq 0$ and for collision in a cluster (the latter is treated in detail in [El]).

Condition $(H')_M$, $2 \leq M \leq N$ implies that every isolated collision of any subset of the N bodies is regular.

To see this, recall that we have seen in Section 3 that there is no infinite spin, so that a collision is directional if the ω -limit set Ω of $\xi(t) \equiv x(t)\|x(t)\|^{-1}$ is contained in the equivalence class of a central configuration, and this is certainly the case if the equivalence class (under rigid rotations) of Ω is a singleton (notice that this is a weaker condition than being isolated, since it allows for accumulation points).

Recall now that Ω is entirely made of central configurations. The analysis we have given in Section 3 leads then to

PROPOSITION 5.2

- i) If $[\Omega]$ is a singleton, the collision is directional.
- ii) If Ω is entirely made of planar configurations, then the collision is planar.

It is known [Pl] that if ξ is a colinear central configuration then $[\xi]$ is a singleton (in fact, it is an isolated point).

From Proposition 3.3 one derives therefore

PROPOSITION 5.3

Every isolated M -body collision for a weak solution of (1.1) is regular if for the given masses condition $(H')_M$ is satisfied.

REMARK

$(H')_2$ and $(H')_3$ are satisfied for all choices of masses because there are no nonplanar central configurations (in fact, for $M = 3$ one has the stronger result that all collisions are directional, since modulo $O(3)$ there is only one noncolinear central configuration, an equilateral triangle).

$(H')_4$ holds for all choices of masses because modulo $O(3)$ there is only one nonplanar central configuration (a tetrahedron).

It is expected that $(H')_M$ $M \geq 5$ holds for almost all values of the masses. In fact, no example is known in which $(H')_M$ fails for some M ; the natural place where to look for counterexamples would be at bifurcation points of planar into nonplanar central configurations (see e.g. [M S]).

We can now prove

PROPOSITION 5.4

If V satisfies $(H)_1$ and $(H)_2$ and if for the given value of the masses condition $(H')_M$ is satisfied for every $M \leq N$, then $x^0(t)$ in Proposition 4.9 is a classical solution of (0.1).

REMARK 1

Proposition 5.4 extends to the case in which the Newtonian potential is substituted by

$$V_K^\alpha(x) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|^\alpha} \quad 0 < \alpha < 2$$

The proofs require only minor modification and will not be given here. Details can be found in [D].

REMARK 2

The case $N=3$ has been discussed previously by E. Serra and S. Terracini [S T] under somewhat stronger conditions and using estimates on the behaviour of the Action functional at an isolated collision which are less sharp than the ones used here.

Proof of Proposition 5.4

The proof will be given in two steps. First we prove (step A) that if the collision set of x^0 is not empty, and all isolated collisions are regular, one can construct $\tilde{x}^0(t)$ in such a way that it is antiperiodic, has only one collision in $[0, T/2)$ and this collision is strongly regular. As second step (step B) we prove that for such $\tilde{x}^0$ one can find η such that $i)$ and $ii)$ in Lemma 5.1 hold. But this leads to a contradiction, according to Lemma 5.1. One concludes that the assumption that the collision set of x^0 is nonempty is false, so that x^0 is indeed a classical solution.

Step A) (parts of this construction can be found in [CZ.1] and in [S,T]).

We first construct a trajectory $\hat{x}^0(t)$ which is antiperiodic, satisfies $i)$ and has only one collision in $[0, T/2)$, and this collision is directional. As a second step, we shall construct $\tilde{x}^0$ with all the desired properties.

Let $t_0 \in [0, T/2)$ be a time at which $V(x^0(t))$ attains its minimum value. Such t_0 exists since $x^0(t)$ is antiperiodic and continuous outside the collision set $K_{x^0(\cdot)}$, and $V \equiv V_K + W$ is continuous outside the coincidence set (where it diverges to $+\infty$).

Let

$$t_1 \equiv \max\{t \in [0, T/2], t < t_0, t \in K_{x^0(\cdot)}\}, \quad t_2 \equiv \min\{t \in [0, T/2], t > t_0, t \in K_{x^0(\cdot)}\}$$

They exist because the collision set is nonempty (by assumption) and closed (in fact its complement is open and dense, since x^0 is a weak solution of (0.1)).

Let now $\hat{x}^0$ be the trajectory constructed as follows:

$$\begin{aligned} \hat{x}^0(t) &= x^0(t + t_1), \quad t \in [0, t_0 - t_1], & \hat{x}^0(t) &= x^0(t_0) \quad t \in [t_0 - t_1, T/2 - t_2], \\ \hat{x}^0(t) &= x^0(t + t_2), \quad t \in [T/2 - t_2, T/2] \end{aligned}$$

and by antisymmetry in $[T/2, T]$.

Since V is time independent, and by assumption $V(x^0(t_0)) \leq V(x^0(t)) \quad \forall t$, the contribution to Φ_T coming both from the kinetic energy and from the potential energy are not larger for $\hat{x}^0$ than for x^0 . Therefore $i)$ is satisfied. Moreover $\hat{x}^0(t)$ has only one collision (at $t = 0$) in $[0, T/2)$ and this collision is directional in view of Assumption $(H')_M$.

Notice that time independence of W is essential for this construction, since $x^0(t)$ and $\hat{x}^0(t)$ have common parts of the orbit, but these common parts are covered at different times.

We now use $\hat{x}^0(t)$ to construct a new trajectory $\tilde{x}^0(t)$ for which the only collision in $[0, T/2)$ is strongly directional. Let

$$\Phi_1(\hat{x}^0(\cdot)) \equiv \int_0^{T/4} [\Theta(\hat{x}^0(t)) + V(\hat{x}^0(t))] dt, \quad \Phi_2(\hat{x}^0(\cdot)) \equiv \int_{T/4}^{T/2} [\Theta(\hat{x}^0(t)) + V(\hat{x}^0(t))] dt \quad 5.7$$

where $V \equiv V_K + W$ and Θ is kinetic energy.

Since the sum is $\frac{1}{2}\Phi_T(\hat{x}^0(\cdot))$ at least one of them is not larger than $\frac{1}{4}\Phi(\hat{x}^0(\cdot))$. Assume for definiteness that this is the case for Φ_1 . Then

$$\Phi_T(\hat{x}^0(\cdot)) \geq \Phi_T(\tilde{x}^0(\cdot)) \quad 5.8$$

where $\tilde{x}^0(t)$ coincides with $\hat{x}^0(t)$ in $[0, T/4]$ and is obtained in $[T/4, T/2]$ tracing $\hat{x}^0(t)$ backwards. and in $[T/2, T]$ by antiperiodicity.

Also here (5.8) holds because W is time-independent, so that the contributions to $\Phi_T(\tilde{x}^0(\cdot))$ coming from the time intervals $[0, T/4]$ and $[T/4, T/2]$ coincide.

By construction, $\tilde{x}^0(t)$ satisfies $i)$, has $\{0, T/2\}$ as collision set, and the collision is strongly regular at each of these points.

We can now take *step B*).

We construct a function $f \in \Lambda_T^a$ and positive constants ϵ_1, δ_1 such that $i)$ and $ii)$ of Lemma 5.1 are satisfied for $\eta = \epsilon_1 f$.

We exploit the fact, proved in Section 3, that for an isolated M-collision there exist positive constants $C > 1$ and ϵ_1 such that for every pair of distinct indices $i, k = 1, \dots, M$ one has

$$\frac{1}{C} \leq |t|^{-2/3} |\tilde{x}_k^0(t) - \tilde{x}_i^0(t)| \leq C \quad \forall |t| \leq \epsilon_1 \quad 5.9$$

As in Section 3, set $|t|^{-2/3} \tilde{x}_k^0(t) \equiv \xi_k(t)$ and define

$$f_i(t) \equiv \left(w_0(t) - w_0\left(t + \frac{T}{2}\right) \right) \zeta_i \quad 5.10$$

where $w_0(t)$ is of class $H^1((-\frac{T}{2}, \frac{T}{2}), R)$, equal to one in $[0, \epsilon_0]$ and with support in $[0, 2\epsilon_0]$ and such that

$$\sup_t |\dot{w}_{\epsilon_0}| = \epsilon_0^{-1}, \quad \|\dot{w}_{\epsilon_0}(\cdot)\|_M \leq \epsilon_0^{-\frac{1}{2}} \quad 5.11$$

The constant ϵ_0 (which gives the size of the support of w_0) will be chosen later.

The vector $\zeta \equiv \{\zeta_i\}$ in (5.10) is chosen so that

$$\|\zeta_i - \zeta_k\|^2 > 0 \quad 5.12_1$$

$$\lim_{t \rightarrow 0^\pm} (\zeta_i - \zeta_k, \xi^i(t) - \xi^k(t)) = 0 \quad \forall i \neq k = 1, \dots, M \quad 5.12_2$$

Such choice of ζ is possible since the collision is strongly regular. Indeed, if it is strongly planar, one takes all the ζ^k perpendicular to the plane of asymptotic motion, and such that (5.12₁) holds. If the collision is strongly directional but not planar, one chooses a plane Π and lets ζ_k be the projection of ξ_k^+ onto Π , rotated by ninety degrees. This gives (5.12₂). By an appropriate choice of the plane Π one can guarantee that (5.12₁) holds as well.

We prove now that ϵ_0 can be chosen so that $i)$ and $ii)$ in Lemma 5.1 hold.

We prove first that $ii)$ holds.

Consider the function $\tilde{x}^0(t) + \epsilon f(t)$. By antisymmetry it is sufficient to consider the interval $[-T/2, T/2]$. In this interval the collision set coincides with $t = 0$. Choose $\delta > 0$ and small. Then

$$\inf_{T/2 \geq |t| \geq \delta} \min_{i < k} |\tilde{x}_k^0(t) - \tilde{x}_h^0(t)| \equiv C(\delta) > 0$$

so that, choosing ϵ small enough

$$\inf_{T/2 \geq |t| \geq \delta} \min_{i < k} |\tilde{x}_k^0(t) + \epsilon f_k(t) - \tilde{x}_h^0(t) - \epsilon f_h(t)| > C/2 > 0$$

On the other hand, for δ sufficiently small, in view of (5.12)₂ one has for $|t| \leq \delta$

$$2|\tilde{x}_k^0(t) + \epsilon f_k(t) - \tilde{x}_h^0(t) - \epsilon f_h(t)| \geq |\tilde{x}_k^0(t) - \tilde{x}_h^0(t)| + \epsilon |f_h(t) - f_k(t)| \geq \epsilon |f_h(t) - f_k(t)|$$

Combining these estimates, for ϵ_0 small enough and $f(t)$ antiperiodic and supported in $[-2\epsilon_0, 2\epsilon_0] \cup [T/2 - 2\epsilon_0, T/2 + 2\epsilon_0]$

$$\inf_{i < k} |\tilde{x}_k^0(t) + \epsilon f_k(t) - \tilde{x}_h^0(t) - \epsilon f_h(t)| > C_1 > 0$$

so that *ii*) holds as claimed.

We prove next that *i*) holds.

We must show that

$$\Phi_T(\tilde{x}^0(\cdot) + \epsilon f(\cdot)) - \Phi_T(\tilde{x}^0(\cdot)) < 0 \quad \forall 0 < |\epsilon| < \epsilon^0, \quad 5.15$$

for some $\epsilon^0 \neq 0$.

To simplify notations, from now on we shall write $x(t)$ for $\tilde{x}(t)$ and shall omit the index T in the functional.

Consider first the case of simultaneous collision and the support of $W(x) = 0$ is disjoint from the coincidence set Δ ; the result will be then extended along the same line to the case of M -collisions and for a generic $W(x)$ satisfying $(H)_1$.

The support of the function f will be contained in an arbitrarily small (but finite) neighborhood of the origin. Therefore if the support of W is disjoint from Δ in proving (5.15) we can assume $W = 0$ so that the Action is

$$\Phi(x(\cdot)) \equiv \frac{1}{2} \sum_i m_i \int_{-\frac{T}{2}}^{\frac{T}{2}} \dot{x}_i^2 dt + \sum_{i < k} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{m_i m_k}{|x_i - x_k|} dt$$

We shall use the identity

$$\Phi(x(\cdot) + \epsilon f(\cdot)) - \Phi(x(\cdot)) = \int_0^1 \frac{d}{ds} [\Phi(x(\cdot) + \epsilon s f(\cdot))] ds$$

which holds if $\frac{d}{ds} [\Phi(x(\cdot) + \epsilon s f(\cdot))] ds$ is integrable in $[0, 1]$; we shall verify this for our choice of $f(t)$.

In our case one has

$$\begin{aligned} & \Phi(x(\cdot) + \epsilon f(\cdot)) - \Phi(x(\cdot)) = \\ & \int_0^1 \frac{d}{ds} \sum_i m_i \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\dot{x}_i(t) + s \epsilon \dot{f}_i(t), \epsilon \dot{f}_i(t) \right) dt + Q_\epsilon(x(\cdot), f(\cdot)) \end{aligned} \quad 5.16$$

where

$$Q_\epsilon \equiv - \int_0^1 ds \sum_{i < k} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \frac{m_i m_k (x_i(t) - x_k(t) + s \epsilon f_i(t) - s \epsilon f_k(t), \epsilon f_i(t) - \epsilon f_k(t))}{|x_i(t) - x_k(t) + s \epsilon f_i(t) - s \epsilon f_k(t)|^3} \quad 5.17$$

If ϵ_1 is chosen small enough the support of $f(t)$ is made of two disconnected components (separated by $\frac{T}{2} - 4\epsilon_0$), which give equal contribution to the integrals we compute. In addition, by strong regularity, the ratio of the contributions from $[-2\epsilon_0, 0]$ and from $[0, 2\epsilon_0]$

tends to one when $\epsilon_0 \rightarrow 0$. Therefore we shall restrict our attention to the integral over $[0, 2\epsilon_0]$.

Making use of the support property of w_{ϵ_0} , of the bounds on the behaviour of $x_k(t)$ near collisions and of Schwartz's inequality it is easy to verify that the contribution from $[0, 2\epsilon_0]$ to the first integral on the right hand side of (5.17) can be bounded by

$$\left| \int_0^1 ds \sum_i m_i \int_0^{2\epsilon_0} dt \left((\dot{x}_i(t) + s\epsilon \dot{f}_i(t), \dot{\epsilon} f_i(t)) \right) \right| < C_1 \epsilon \epsilon_0^{-1} + C_2 \epsilon^2 \epsilon_0^{-2}$$

where C_1, C_2 are positive constants.

We estimate now Q_ϵ in (5.17). One has

$$Q_\epsilon = - \int_0^1 ds \sum_{i < k} m_i m_k \int_0^{2\epsilon_0} dt \frac{\epsilon t^{2/3} \delta_{i,k}(t) + \epsilon^2 w^2(t) |\zeta_i - \zeta_k|^2}{|t^{2/3} [\xi_i(t) - \xi_k(t) + s\epsilon w(t)(\zeta_i - \zeta_k)]|^3} \quad 5.18$$

where the functions $\delta_{i,k}(t)$ are infinitesimal in t . We rescale time by defining τ according to

$$t \equiv \epsilon^{3/2} \tau$$

to obtain

$$Q_\epsilon = -\frac{3}{2} \sqrt{\epsilon} \int_0^1 ds \sum_{i < k} m_i m_k \int_0^{2\epsilon_0 \epsilon^{-3/2}} \epsilon_0 \frac{\tau^{2/3} \tilde{\delta}_{i,k}(\epsilon, \tau) + s \tilde{w}^2 |\zeta_i - \zeta_k|^2}{|\tau^{2/3} [\xi_i(t) - \xi_k(t)] + s \tilde{w}(\epsilon, \tau)(\zeta_i - \zeta_k)|^3} d\tau \quad 5.19$$

Recall that for $0 \leq t \leq 2\epsilon_0$ (and therefore for $0 \leq \tau \leq 2\epsilon_0 \epsilon^{-3/2}$) one has, from (5.9) and (5.17), for positive constants $C > 1$ C_1

$$C^{-1} \leq |\xi_i(t) - \xi_k(t)| < C, \quad |\zeta_i - \zeta_k| > C_i \quad \forall i \neq k$$

$$\sum_{i \neq k} |(\xi_i(t) - \xi_k(t), \zeta_i - \zeta_k)|^2 < \delta^2 \quad 5.20$$

where δ can be made arbitrarily small by choosing ϵ_0 small.

Therefore for $0 \leq \tau \leq \epsilon_0 \epsilon^{-3/2}$

$$\frac{\tau^{4/3}}{2C^2} + \frac{sC_1^2}{2} \leq |\tau^{2/3}(\xi_i(\tau) - \xi_k(\tau)) + s\tilde{w}(\epsilon, \tau)(\zeta_i - \zeta_k)|^2 \leq \frac{2\tau^{4/3}}{C^2} + 2sC_1^2 \quad 5.21$$

This leads to the estimate, for every $i \neq k$

$$\begin{aligned} & \int_0^1 ds \int_0^{2\epsilon_0 \epsilon^{-3/2}} \epsilon_0 \frac{s \tilde{w}^2 C_1^2}{|\tau^{2/3} [\xi_i(t) - \xi_k(t)] + s \tilde{w}(\epsilon, \tau)(\zeta_i - \zeta_k)|^3} d\tau \\ & \geq \int_0^1 ds \int_0^{\epsilon_0 \epsilon^{-3/2}} \epsilon_0 \frac{s C_1^2}{|\frac{\tau^{4/3}}{C^2} + s C_1^2|^{3/2}} d\tau \geq C_2 \epsilon_0 \end{aligned} \quad 5.22$$

uniformly for ϵ small enough, since the integrand is in L^1 and therefore

$$\lim_{\epsilon \rightarrow 0} \int_0^1 ds \int_{\epsilon_0 \epsilon^{-3/2}}^{\infty} \epsilon_0 \frac{s C_1^2}{2^{3/2} |\frac{\tau^{4/3}}{C^2} + s C_1^2|^{3/2}} d\tau = 0$$

On the other hand, using again (5.20), (5.21), for every $i \neq k$

$$\int_0^1 ds \int_0^{2\epsilon_0 \epsilon^{-3/2}} \epsilon_0 \frac{\tau^{2/3}}{2^{-3/2} |\frac{\tau^{4/3}}{C^2} + s C_1^2|^{3/2}} d\tau \leq C_3 \delta(\epsilon_0) \quad 5.23$$

(we have again used the fact that the integrand is in L^1).

Since $\delta(\epsilon_0)$ is infinitesimal in ϵ_0 from (5.22), (5.23) one concludes that, if ϵ_0 is chosen small enough, there is a positive constant C_4 such that

$$Q_\epsilon \leq -C_4 \sqrt{\epsilon} \quad 5.24$$

The first term on the right-hand side in (5.12) is infinitesimal in ϵ of order greater than $\frac{1}{2}$. Therefore, if $\epsilon_0 > 0$ is chosen sufficiently small one can find a positive constant C_5 such that

$$\Phi(x(\cdot)) + \epsilon f(\cdot) - \Phi(x(\cdot)) < -C_5 \sqrt{\epsilon} \|f\| \quad 0 < |\epsilon| < \epsilon_0$$

This concludes the proof of Proposition 5.2 in the case $\tilde{x}(t)$ has only one collision in $[0, T/2)$ which is a total collision ($M = N$) and moreover W is supported outside Δ .

Consider next the case in which there is still only one collision in $[0, T/2)$, W satisfies $(H)_1$ and the collision refers to a cluster of $M < N$ of the bodies. Since we consider only isolated collisions, one can find $t_1 > 0$ such that the distance of this cluster from any other body is bounded away from zero uniformly for $|t| < \epsilon_1$.

Modifying if necessary the trajectory of the M colliding bodies, we may assume that the collision is strongly regular and prove that (5.10) holds.

Choose $\zeta_i \equiv 0, i = M+1, \dots, N$ and f as in (5.14).

Then

$$\Phi(x(\cdot) + \epsilon f(\cdot)) - \Phi(x(\cdot)) \equiv \Psi^1(x(\cdot), \epsilon f(\cdot)) + \Psi^2(x(\cdot), \epsilon f(\cdot))$$

where Ψ_α^1 is given by the right-hand side of (5.11) (with N instead of M), while Ψ^2 has the form

$$\begin{aligned} \Psi^2(x(\cdot), \epsilon f(\cdot)) &= \Xi^2(x(\cdot) + \epsilon f(\cdot)) - \Xi^2(x(\cdot)), \\ \Xi(x(\cdot)) &\equiv \sum_{j>M, k \leq M} \int_{-2\epsilon_0}^{2\epsilon_0} \frac{m_j m_k}{|x_j(t) - x_k(t)|} dt \end{aligned} \quad 5.25$$

Due to the separation between the clusters the function Ξ is of class C^1 in its arguments. With the choice we have made for f one has then

$$|\Psi^2(x(\cdot), \epsilon f(\cdot))| < C \epsilon \epsilon_0^{-\frac{1}{2}} \quad \Psi^1(x(\cdot), f_\epsilon(\cdot)) \leq -C_1 \sqrt{\epsilon}$$

where $C_1 > 0$.

For ϵ_1 sufficiently small and positive one has then also in this case

$$\Phi(x(\cdot) + \epsilon f(\cdot)) - \Phi(x(\cdot)) < -C_5 \sqrt{\epsilon} \|f\|, \quad 0 < |\epsilon| < \epsilon_1$$

The case a general W satisfying $(H)_2$ can be treated similarly; all the new contributions which occur are infinitesimal, when $\epsilon_0 \rightarrow 0$, in comparison with the ones we have discussed in the case in which the support of W does not intersect Δ .

Notice now that the requirement that $\tilde{x}$ has only one collision in $[0, T/2)$ was not required in the proof of *i*) but only in the proof of *ii*). Indeed, the proof of *i*) is obtained "localizing" the function $f(t)$. This concludes the proof of Proposition 5.4. ///

Sharper results have been obtained for $N = 2$. If one considers the Kepler problem and makes the assumption $(H)_1$ on W , under the assumption that x^0 has a collision one can give a good lower bound on the index of x_δ^0 for δ sufficiently small. One obtains $i_\delta(x_\delta^0) \geq 1$ and one can improve this bound to $i_\delta(x_\delta^0) \geq 2$ if the Kepler potential is substituted by a potential of the form $|x|^{-\alpha}$, $1 < \alpha \leq 2$. Together with Proposition 4.12 this leads to the conclusion that x^0 has at most one collision in half-period (recall that $x^0(t)$ satisfies the antiperiodicity condition) if $\alpha \in (0, 1]$ and no collisions if $\alpha \in (1, 2)$.

We shall briefly sketch the proof, [T] which requires a very refined analysis of the behaviour of x_δ^0 at the time of minimal reciprocal distance between the two bodies, if x^0 has a collision. As we will see, relevant to the proof is a rescaling which bears some resemblance with the rescaling introduced in Sections 2 and 3 in the discussion of the behaviour of a solution near a collision. Essential use is made of the fact that the strong force acts as a modification of the angular momentum (see Section 2); In particular at the minimal distance the motion is "almost planar" and this allows the use of scaling arguments to find a lower bound on the number of negative eigenvalues of the Hessian of the functional Φ_T^δ at x_δ^0 .

We shall only discuss the case $\mathcal{B}_T$.

One assumes that the system is invariant under translations so that one can choose a reference system in which the baricenter is at rest. Denote by y the relative coordinate; Newton's equations are then reduced to a system of equations for a point particle in R^3 subject to a force described by a potential

$$V(y) \equiv \frac{1}{|y|} + W(y, t) \quad y \in R^3,$$

The analysis is in fact carried out in [T] for the more general class of potentials of the form

$$V_\alpha(y) \equiv \frac{1}{|y|^\alpha} + W(y, t) \quad 0 < \alpha < 2$$

Notice that the coincidence set is reduced here to a single point, the origin.

On the potential W one makes the same assumptions as in the proof of Proposition 5.4, with the exception of independence on time and of the symmetry requirement $(H)_2$ i.e. one assumes

$$W(y, t) \in C^2((R^3 - \{0\}) \times R, R), \quad W(y, t) = W(y, t + T) \quad i)$$

$$W(y, t) \geq -\frac{1}{|y|} \quad \forall t, \quad \lim_{|y| \rightarrow \infty} W(y, t) = 0 \quad ii)$$

$$\lim_{|y| \rightarrow 0} [|y|^\alpha V(y, t) + |y|^{\alpha+1} |\nabla_y V(y, t)| + |y|^{\alpha+2} \|D_y^2 V(y, t)\| + |y|^\alpha |\partial_t V(y, t)|] = 0 \quad iii)$$

uniformly in $t \in [0, T]$

In [T] the following theorems are proved

Theorem I

Under conditions i) , ii) iii) if $1 < \alpha < 2$ the limit $x^0(t)$ is a classical solution of (0.1)

Theorem II

Under conditions i) , ii) iii) if $0 < \alpha \leq 1$ the limit $x^0(t)$ has at most one collision.

We have already remarked that for $N = 2$ the x_δ^0 have index smaller or equal to one. The proof of Theorems I and II is then obtained by giving a lower bound on the index i_δ of x_δ^0 for δ sufficiently small; this lower bound is obtained through a detailed analysis of the behaviour of x_δ^0 at the minimal distance between the two bodies, under the assumption that x^0 has a collision. The conclusion of this "local" analysis is that $i_\delta \geq p(\alpha)\nu$ where ν is the number of collisions in $x(t)$ and $p(\alpha)$ is the integer part of $\frac{2}{2-\alpha}$; this leads immediately to Theorems I and II.

A more detailed analysis [CZ.1] shows that, assuming that $y(t)$ has a collision at $t = 0$ the vectors

$$\xi^\pm \equiv \lim_{t \rightarrow \pm 0} t^{-\frac{2}{2+\alpha}} y(t)$$

are in fact colinear (this is consistent with a similar proof given in Section 2 for the two-body Kepler problem, if one recalls that addition of a strong force is, for $N = 2$, equivalent to a redefinition of angular momentum).

The proof of Theorem II can therefore be given also along the lines of the proof of Proposition 5.4; here step i) is not needed since for $N = 2$ the number of collisions is finite [A,CZ.1] and each is strongly regular.

We give here a very brief sketch of the proofs of Theorems I and II given in [T] to exhibit the role played by the local analysis, and in particular of a rescaling which, as one can verify, is the counterpart for "near-collision" solutions of the Mc Gehee change of time scale for collision solutions. In fact it should be expected that a unified formulation can be obtained if one considers the regularized two-body flow.

In [T] the strong force modification is obtained through a potential $\delta \frac{1}{|y|^4}$. The choice of the inverse fourth power is a matter of convenience, to obtain an estimate of the form

$$\inf_{t \in [0, T]} y_\delta^0(t) \geq C\epsilon^{1/2}$$

where $C \equiv \Phi_T^\delta(y_\delta^0)$. If a strong force potential inversely proportional to the square of the distance were introduced (as in the previous sections) the estimate on the minimal distance would be proportional to the inverse of $|\log \delta|$.

As in the general approach, minimax techniques are introduced to show the existence of the critical points x_δ^0 with index not larger than one, and estimates deriving from the form of the action functional are used to prove convergence of these critical points, in $H^1([0, t], R^3)$ and pointwise, to a weak solution x^0 of (0.1).

The next step is a rescaling argument.

Define

$$x_\delta^0(t) \equiv \delta^{-1} y_\delta^0(\delta^{\frac{\alpha+2}{2}} t + t_\delta)$$

where t_δ is the time at which $|y_\delta^0(t)|$ attains its minimum value (time of closest approach between the two bodies).

The equations of motions, and the energy estimates for $y_\delta^0(t)$, are then converted into equations and inequalities for $x_\delta^0(t)$. Moreover one has

$$|x_\delta(0)| = 1 \quad x_\delta(0) \perp \dot{x}_\delta(0)$$

the latter being a simple consequence of the fact that the distance is a differentiable function of t .

Estimates (as function of δ) of the coefficients of the terms $|x_\delta^0|^{-(\alpha+2)}$ and $|x_\delta^0|^{-6}$ which appear in the equation for x_δ^0 lead to the conclusion that $x_\delta^0(t)$ converge, possibly after extracting a subsequence, to a function $z(t)$ which satisfies

$$\ddot{z}(t) + \frac{\alpha z}{|z|^{\alpha+2}} + 4c_0 \frac{z}{|z|^6} = 0, \quad z(0) = \hat{k}_1, \quad \dot{z}(0) = \sqrt{2(1+c_0)}\hat{k}_2$$

where $\hat{k}_1$ and $\hat{k}_2$ are orthogonal unit vectors.

If one writes now the Hessian of Φ_T^δ at y_δ^0 in terms of the functions $x_\delta^0(t)$ and takes into account all estimates and the convergence of $x_\delta^0(t)$ to $z(t)$ one concludes that, after a rescaling by a factor $\delta^{\frac{2-\alpha}{2}}$, the Hessians converge strongly to an operator J which, when evaluated on functions $\phi(s)\hat{k}_3$ (where $\hat{k}_3$ is perpendicular both to $\hat{k}_1$ and to $\hat{k}_2$) gives

$$J(\phi) \equiv \int_{-l}^l \left[|\dot{\phi}(s)|^2 - \frac{\alpha |\phi|^2(s)}{|z(s)|^{\alpha+2}} - \frac{4c_0 \phi(s)^2}{|z(s)|^6} \right] ds \quad 5.26$$

On the constant c_0 , l it is only known that they are nonnegative. Therefore in estimating the dimension of the negative eigenspace of the quadratic form (5.26) one must set $c_0 = 0$. A detailed estimate, given in $[T]_1$, leads to the conclusion that this eigenspace has dimension at least one if $\alpha \in (0, 1)$ and at least two if $\alpha \in [1, 2)$. This proves theorems I and II.

A corresponding detailed analysis would be very interesting but has not been made so far for the case $N \geq 3$. One can expect major differences for $N \geq 3$ as compared to $N = 2$ as a consequence of the vastly more irregular behaviour of x^0 near collisions. Notice however that the strong force is no longer equivalent to a modification of the total angular momentum nor of the angular momentum of subsets of the N bodies relative to their baricentre.

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