UNIFICATION OF INTERACTIONS WITHIN THE FRAMEWORK OF POLAR
COUPLING CONSTANTS

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ABSTRACT

The partial differential equations with respect to the polar rather than the (usual)
cartesian coupling constants for the $S$ matrix are found to be useful for carrying out the
unification of interactions. Particularly useful is the coupling constant integrability
condition that allows the $S$ matrix to be integrated in the $n$-dimensional coupling
constant space with respect to just one—the radial-coupling constant. Here it is found that
at least three types of unifications exist: (1) the true unification when, in the $n$
interactions (with $n$ coupling constants), the $n$ spherical neutral vector bosons have
absorbed $n - 1$ coupling angles, leaving the radial coupling constant as the only bona fide
coupling parameter; (2) the partial true unification when, in the $m$ interactions (and
coupling constants), the $n$ spherical neutral vector bosons ($n < m$) have again absorbed
$n - 1$ coupling angles, however, leaving now in addition to the radial coupling constant,
also $m - n$ coupling angles as bona fide coupling parameters; and (3) the unified
description of interactions (without involving the neutral vector bosons) in terms of the
radial coupling constant, where, although passive, the coupling angles are still formally
coupling parameters. Unlike in previous cases, here the coupling angles need not
diagonalize the Lagrangian mass density terms. The spontaneously broken $SU(2) \times U(1)$
model ($n = 2$), which unifies electromagnetic and weak interactions, is an example of true
unification. On the other hand, because the $SU(3)$ quantum chromodynamics lacks the
neutral vector boson, the $SU(3) \times SU(2) \times U(1)$ gauge model ($m = 3, n = 2$) represents just the partial true unification of electromagnetic, weak, and strong interactions. Finally, interactions with gauge coupling constants (dimensionless in the natural system of units) can always be given a unified description.
1 Introduction

The original standard model of electroweak interactions containing just leptons was based on the spontaneously broken gauge group $SU(2) \times U(1)$ [1,2]. With the introduction of weak mixing angle [3], in addition to weak and electromagnetic interactions, the model contained also the neutral current interactions. The spontaneous symmetry breakdown generates the masses of gauge bosons and fermions [4-7].

However, because the standard model involves two separate gauge coupling constants associated with the $SU(2)$ and $U(1)$ gauge groups, respectively, there have been some questions whether it represents a true unification or just unified description of electromagnetic and weak interactions. One usually contrasts this with some grand unified theory (GUT), as for example, with the $SU(5)$ gauge theory [8], which is truly unified at some high energy unification scale. Namely, before the various spontaneous symmetry breakdowns occur, bringing this $SU(5)$ GUT down to accessible energies, it had just one coupling constant, the $SU(5)$ gauge coupling constant.

In this article, however, we wish to determine to which extent the many particle interactions at accessible energy scales can be considered truly unified, regardless of whether they can be related to some GUT at an extreme high energy scale. The formalism of the partial differential equations with respect to coupling constants (PDECC) [9], which goes beyond just gauge principles, will provide such criteria; here, one can distinguish between the true unification and the unified description, say, of $n$ interactions at accessible energy scales.

These criteria are easiest to obtain if the polar, $r, \theta_1, \ldots, \theta_{n-1}$, rather than Cartesian, $g_1, \ldots, g_n$, coupling constants are used within the PDECC formalism [9]. One finds, while solving for the $S$ matrix, that the coupling constant integrability freedom is a prerequisite to have either the unified description or true unification of $n$ interactions. In fact, with this freedom one can always achieve the unified description of $n$ interactions: The
PDECC for the $S$ matrix need to be integrated with respect to just $r$, the radial coupling constant. The coupling angles, although passive, are still coupling parameters, so we still have $n$ coupling constants. This is the reason we say that the $n$ interactions have only the unified description.

The true unification of $n$ interactions into a single interaction occurs when the $n - 1$ coupling angles become mixing angles, mixing the $n$ "Cartesian" vector boson-vector components into the $n$ "polar" physical vector boson-vector components; as such, the $n - 1$ coupling angles become part of the free system (and cease being the coupling parameters), leaving the radial coupling constant $r$ as a only bona fide coupling parameter. Therefore, from the formal point of view, $r$ describes a truly unified single interaction. Of course, here it is assumed that to each Cartesian (polar) coupling constant there corresponds a Cartesian (polar) vector boson-vector component of the neutral vector boson-vector in the $n$ dimensional coupling constant space.

If in the $m$ dimensional coupling constant space we have $m$ interactions, however, with only $n$ Cartesian neutral vector bosons, where $m > n$, the true unification cannot be achieved. This is so because with the $n$ Cartesian neutral vector bosons one cannot define a neutral vector boson-vector in the $m$ dimensional coupling constant space. However, with the same procedure as before, with the $n$ Cartesian neutral vector bosons treated as Cartesian components of a neutral vector boson-vector in the $n$ dimensional coupling constant space, we again, by defining the coupling angle dependent physical polar neutral vector boson-components, eliminate $n - 1$ coupling angles as coupling parameters. Hence, the number of bona fide coupling parameters is now $1 + m - n$. This we call partial true unification of $m$ interactions. Of course, one may give a unified description to the $1 + m - n$ interactions, while, treating the radial coupling constant from the $n$ dimensional coupling constant space as a Cartesian coupling constant.

The $SU(2) \times U(1)$ and $SU(3) \times SU(2) \times U(1)$ electroweak models belong to the cases with $m = n = 2$ and $m = 3, n = 2$, respectively. Both theories have two physical neutral
vector bosons, the photon and Z-boson. The $SU(2) \times U(1)$ theory represents the true unification of electromagnetic and weak interactions. However, because the $SU(3)$ quantum chromodynamics lacks the neutral vector boson, the $SU(3) \times SU(2) \times U(1)$ theory represents only the partial true unification of strong, weak, and electromagnetic interactions.

In the case of true unification, the masses of vector bosons are either zero or depend on coupling angles. Without this dependence, the Lagrangian mass terms of vector bosons could not be properly diagonalized; the mass generating spontaneous symmetry breaking is consistent with this view.

The coupling constant integrability freedom, strictly speaking, is valid in the absence of masses in the theory. The effect of the coupling angle-dependent masses is that this integrability freedom is restricted when integrating the PDECC for the $S$ matrix: only "straight" paths connecting $r = 0$ to $r = r$ are allowed, because they respect the physical values of vector boson masses. However, we may ignore the mass terms when integrating the PDECC for the $S$ matrix and take them into account at the "end" of calculations.

In Section 2 we discuss the PDECC for the $S$ matrix in terms of Cartesian and polar coupling constant components. Here we also describe the meaning of the unified description of $n$ interactions. The true unifications are discussed in Section 3. In Section 4 we discuss the conditions that the standard models of electroweak interactions represent true unified theories. Conclusions and discussion are given in Section 5.

2 The PDECC formalism with the Cartesian and polar coupling constants: the unified description of interactions

The $n$ coupling constants (dimensionless in the natural system of units), $g_1, \ldots, g_n$, that characterize $n$ interactions, are treated as Cartesian coordinates within the PDECC formalism [9]. The total Lagrangian density is written as
\[ L(x) \equiv L(\phi(x), \partial_\mu \phi(x); \bar{g}) , \quad (2.1) \]

where \( \phi(x) \) denotes a set of Heisenberg (interacting fields). Now, for a non-Abelian gauge theory, the Lagrangian density may contain space-time derivative coupling terms. The \( S \) matrix will involve noncovariant \( T \) products and the noncovariant Hamiltonian density. The \( T \) contractions contain both covariant and noncovariant contractions, the later coming from field derivatives. However, these noncovariant terms are canceled by the noncovariant terms coming from the noncovariant interaction Hamiltonian, so that at the end the \( S \) matrix is a covariant object expressible in terms of covariant \( T^* \) products [9]. The \( T^* \) product has only covariant contractions, which are simply the \( T \) contractions in which the pure noncovariant terms are left out [9]. This statement, however, is equivalent to requiring that the four-dimensional delta functions [9-11] and their derivatives (first paper in [9]) vanish at zero space-time points. On the formal level, these requirements can be justified by employing the dimensional regularization [9-11] and demanding the consistency of the PDECC formalism with the canonical formalism (first paper in [9]). As a consequence, the \( S \) matrix may be written in these equivalent forms (first paper in [9]):

\[
\frac{1}{i} \frac{\partial}{\partial g_i} S = S \int d^4 x L_{i}^*(x) , \quad (2.2a)
\]

\[
\frac{1}{i} \frac{\partial}{\partial g_i} S = \int d^4 x T S L_{i}^*(x) = \int d^4 x T^* S L_{i, in}^* (x) . \quad (2.2b,c)
\]

Here the notation is such that,

\[
L_{i}^*(x) \equiv \frac{\partial \cdot L(x)}{\partial g_i}, \quad \frac{\partial \cdot \phi(x)}{\partial g_i} = \ldots = \frac{\partial \cdot \partial_\mu \ldots \partial_\nu \phi(x)}{\partial g_i} = 0, \{ \frac{\partial \cdot}{\partial g_i}, \partial_\mu \} = 0 ; \quad (2.3)
\]

\[
F(x) \equiv F(\phi(x), \partial_\mu \phi(x); \bar{g}) , \quad (2.4a)
\]

\[
(F(x))_{in} \equiv F(\phi_{in}(x), (\partial_\mu \phi(x))_{in}; \bar{g}) , \quad (2.4b)
\]
\[ F_m(x) \equiv F(\phi_m(x), \partial_{\mu} \phi_m(x); \bar{g}) \] \quad (2.4c)

The kinetic, \( \mathbf{L}(K; \phi) \), and mass, \( \mathbf{L}(m; \phi) \), Lagrangian density terms (where m is the physical mass), associated with a specific field \( \phi \), satisfy

\[
\frac{\partial}{\partial g_i} \mathbf{L}(K; \phi) = 0, \quad \frac{\partial}{\partial g_i} \mathbf{L}(m; \phi) = 0. \tag{2.5a,b}
\]

Relations (2.2) to (2.4) assume that as \( t_x \to -\infty, \phi(x) \to \phi_m(x), \)
\( \partial_{\mu} \phi(x) \to (\partial_{\mu} \phi(x))_m, \) etc. All the field vacuum expectation values are zero; information about possible spontaneous symmetry breakings is present in the Lagrangian density already.

When solving the PDECC for the \( S \) matrix, say, in form (2.2c), we need the following types of PDECC's [9]:

\[
\frac{1}{i} \frac{\partial}{\partial g_i} T^\ast (F(x)G(y) \ldots) = \int d^4 z (T^\ast (F(x)G(y) \ldots) \mathbf{L}_i^\ast (z))
\]

\[
-\mathbf{L}_i^\ast (z) T^\ast (F(x)G(y) \ldots) + \frac{1}{i} \frac{\partial}{\partial g_i} T^\ast (F(x)G(y) \ldots). \tag{2.6}
\]

Here one assumes that at \( t \to -\infty, \phi_m(x), \partial_{\mu} \phi_m(x), \) etc. are independent of \( \bar{g} \):

\[
\frac{\partial \phi_m(x)}{\partial g_i} = 0, \quad \frac{\partial \partial_{\mu} \phi_m(x)}{\partial g_i} = 0. \tag{2.7}
\]

### 2.1 Polar coupling constants

The polar coupling constants are defined through these relations:

\[
g_1 = \theta_0 \cos \theta_1,
\]

\[
g_2 = \theta_0 \sin \theta_1 \cos \theta_2,
\]

\[
\vdots
\]

\[
g_{n-1} = \theta_0 \sin \theta_1 \ldots \sin \theta_{n-2} \cos \theta_{n-1},
\]

\[
g_n = \theta_0 \sin \theta_1 \ldots \sin \theta_{n-2} \sin \theta_{n-1},
\]
\[ \theta_0^2 = \sum_{i=1}^{n} g_i^2, \theta_0 \equiv r. \] (2.8)

The non-zero (diagonal) components of the metric tensor in the polar coupling constant space are

\[ \eta_{\theta, e} = 1, \eta_{\theta, \theta} = \theta_0^2, \]
\[ \eta_{e, e} = \sum_{i=1}^{n} \left( \frac{\partial g_i}{\partial \theta} \right)^2 = \theta_0^2 \sin^2 \theta_1 \sin^2 \theta_{n-1}, \alpha = 2, \ldots, n-1. \] (2.9)

The components of the vector path element are, respectively,

\[ d\vec{r} = (d g_1, d g_2, \ldots, d g_n) = (d \theta_0, (\eta_{\theta, \theta})^{1/2} d \theta_1, \ldots, (\eta_{e, e})^{1/2} d \theta_{n-1}) \],

where \( d \theta_0 \equiv d r \). Next, we define the transformation coefficients as

\[ C_{i \alpha} = \left( \eta_{\theta, \theta} \right)^{-1/2} \frac{\partial g_i}{\partial \theta_\alpha} = \left( \eta_{e, e} \right)^{1/2} \frac{\partial \theta_\alpha}{\partial g_i}, \] (2.10a, b)

where the equality (2.11a)=(2.11 b) follows from,

\[ \sum_{i=1}^{n} d g_i^2 = \sum_{\alpha=0}^{n-1} \eta_{\theta, \theta} d \theta_\alpha^2. \]

From equivalent expressions (2.11) one can directly verify the orthonormality relations:

\[ \sum_{i=1}^{n} C_{i \alpha} C_{j \beta} = \delta_{\alpha \beta}, \sum_{\alpha=0}^{n-1} C_{i \alpha} C_{j \alpha} = \delta_{ij}. \] (2.12a, b)

Consequently, the Cartesian and polar unit vectors, \( \hat{g}_i \) and \( \hat{\theta}_\alpha \), are related as

\[ C_{\theta, \theta} = \hat{g}_1, \hat{\theta}_\alpha, \] (2.13)

\[ \hat{g}_1 = \sum_{\alpha=0}^{n-1} \left( \frac{\partial \theta_\alpha}{\partial g_1} \right) \left( \eta_{\theta, \theta} \right)^{1/2} \hat{\theta}_\alpha = \sum_{\alpha=0}^{n-1} C_{i \theta_0} \hat{\theta}_\alpha, \] (2.14a)

\[ \hat{\theta}_\alpha = \sum_{i=1}^{n} \left( \frac{\partial g_i}{\partial \theta_\alpha} \right) \left( \eta_{e, e} \right)^{-1/2} \hat{g}_i = \sum_{i=1}^{n} C_{i \theta_0} \hat{g}_i. \] (2.14b)

In order to evaluate \( C_{i \theta_0} \) one uses relations (2.8) and (2.11a) to obtain:

\[ C_{i \theta_0} = \frac{g_i}{\theta_0} = \sin \theta_1 \cdots \sin \theta_{i-1} \cos \theta_i, i = 1, \ldots, n-1, \]

\[ C_{n \theta_0} = \frac{g_n}{\theta_0} = \sin \theta_1 \cdots \sin \theta_{n-1}, \] (2.15a)

\[ C_{i \theta_\alpha} = \theta_0 \left( \eta_{e, e} \right)^{-1/2} \frac{\partial C_{i \theta_0}}{\partial \theta_\alpha} \quad \alpha = 1, \ldots, n-1; i = 1, \ldots, n. \] (2.15b)

With the help of relations (2.2) to (2.4), one can write \( \mathbf{L}_{i,m} \) as
\[ \mathbf{L}_{\gamma,\eta}(x) = \frac{\partial}{\partial g_i} \mathbf{L}_{\gamma}(\phi^{\eta}(x), \phi^{\eta}(x); \tilde{g}) \equiv \Lambda_{\gamma}(x; \tilde{g}) \quad (2.16) \]

where \( \Lambda_{\gamma} \) are the Cartesian components of vector \( \tilde{\Lambda}(x; \tilde{g}) \) in the coupling constant space: \( \Lambda_{\gamma} = \tilde{g} \cdot \tilde{\Lambda} \). Clearly, Cartesian, \( \Lambda_{\gamma} \), and polar, \( \Lambda_{\theta_{\alpha}} = \tilde{\theta}_{\alpha} \cdot \tilde{\Lambda} \), are related as

\[ \Lambda_{\theta_{\alpha}}(x; \tilde{g}) = \sum_{\gamma=1}^{n} C_{\gamma,\alpha} \Lambda_{\gamma}(x; \tilde{g}) \quad (2.17a) \]

\[ = (\eta_{\theta_{\alpha}} \theta_{\alpha})^{1/2} \frac{\partial}{\partial \theta_{\alpha}} \mathbf{L}_{\gamma}(\phi^{\eta}(x), \phi^{\eta}(x); \tilde{g}), \quad \alpha = 0, \ldots, n-1 \quad (2.17b) \]

\[ \Lambda_{\gamma}(x; \tilde{g}) = \sum_{\alpha=0}^{n} C_{\gamma,\alpha} \Lambda_{\theta_{\alpha}}(x; \tilde{g}) \quad (2.17c) \]

where \( \Lambda_{\theta_{\alpha}} = \Lambda_{\gamma} \) is the radial component, while \( \Lambda_{\theta_{\alpha}}, \alpha = 1, \ldots, n-1 \) are the angular components of the interaction Lagrangian vector \( \tilde{\Lambda} \), respectively.

The PDECC for the \( S \) matrix can be written alternately as

\[ \frac{1}{i} \frac{\partial}{\partial \theta_{\alpha}} S = \int d^4x \ T^* S \Lambda_{\gamma}(x; \tilde{g}), \quad \alpha = 0, \ldots, n-1 \quad (2.18a) \]

\[ \frac{1}{i} \frac{\partial}{\partial \theta_{\alpha}} S = \int d^4x \ T^* S \ (\eta_{\theta_{\alpha}} \theta_{\alpha})^{1/2} \Lambda_{\theta_{\alpha}}(x; \tilde{g}), \quad \alpha = 0, \ldots, n-1 \quad (2.18b) \]

The assumption is that when one solves these equations, the kinetic and mass Lagrangian density terms do not contribute. This assumption, which is already present in Equations (2.5) and (2.7), can also be written as

\[ \Lambda_{\gamma}(K; \phi^{\eta}) = \frac{\partial}{\partial g_i} \mathbf{L}_{\gamma}(K; \phi) = 0, \quad \Lambda_{\gamma}(m; \phi^{\eta}) = \frac{\partial}{\partial g_i} \mathbf{L}_{\gamma}(m; \phi) = 0 \quad (2.19a,b) \]

\[ \Lambda_{\theta_{\alpha}}(K; \phi^{\eta}) = \tilde{\Lambda}(K; \phi^{\eta}) \cdot \tilde{\theta}_{\alpha} = 0, \quad \Lambda_{\theta_{\alpha}}(m; \phi^{\eta}) = \tilde{\Lambda}(m; \phi^{\eta}) \cdot \tilde{\theta}_{\alpha} = 0 \quad (2.20a,b) \]

### 2.2 Coupling constant integrability freedom

In the absence of the spontaneous symmetry breakdown, the physical masses are either zero or put in by hand. In either case, as indicated by relations (2.19) and (2.20), they do not depend on coupling constants. As a consequence, it is expected that \( \mathbf{L}_{\gamma}(x; \tilde{g}) \) is a nonsingular single valued function in \( \tilde{g} \) so that the relation...
\[ \frac{1}{c} d \bar{g} \cdot \tilde{\Lambda}(x; \bar{g}) = \frac{1}{c} d \mathbf{L}_{in}(x; \bar{g}) = 0 \]  

(2.21a)

should hold, where \( C \) is a closed path that goes through \( \bar{g} = 0 \). Relation (2.21a) is an example of a stronger version of the integrability freedom. It allows us to solve relations (2.18) as

\[ S(\bar{g}) = T^* \exp \{ i \int_{\partial P} d^{\alpha} \vec{g}' \cdot \tilde{\Lambda}(x; \vec{g}') \} , \]

(2.21b)

where \( P \) is an arbitrary path connecting \( \vec{g}' = 0 \) with \( \vec{g}' = \bar{g} \). Because of (2.21a) the solution (2.21b) is unique.

Next we derive an infinitesimal version of a stronger version of the integrability freedom. In (2.21a) choose the closed path \( \Delta C \) to run as follows (consult Figure 1):

\( (\vec{0}) \to (\theta_0 \equiv r, \theta_1, \ldots, \theta_{\alpha}, \ldots, \theta_{n-1}) \to (\theta_0 \equiv r, \theta_1, \ldots, \theta_{\alpha} + \Delta \theta_\alpha, \ldots, \theta_{n-1}) \to (\vec{0}) \). This gives

\[ \int_{\Delta C} d \bar{g} \cdot \tilde{\Lambda}(x; \bar{g}) = \Delta \theta_\alpha \left( (\eta_{\theta_0}, \eta_{\alpha})^{1/2} \Lambda_{\theta_\alpha}(x; \bar{g}) + \frac{\partial}{\partial \theta_\alpha} \int_0^1 dr \Lambda_r(x; \bar{g}) \right) + O((\Delta \theta_\alpha)^2) \]  

(2.22a)

\[ = \frac{1}{2} d \mathbf{L}_{in}(x; \bar{g}) \bigg|_{r=0, \theta_{\alpha} \to \theta_0} - \mathbf{L}_{in}(x, \bar{g}) \bigg|_{r=0, \theta_\alpha \to \theta_0} - \mathbf{L}_{in}(x, \bar{g}) \bigg|_{r=0, \theta_\alpha \to \theta_0} . \]  

(2.22b)

Upon evaluating relation (2.22b), one obtains

\[ \int_{\Delta C} d \mathbf{L}_{in}(x; \bar{g}) = \Delta \theta_\alpha \frac{\partial \mathbf{L}_{in}(x; r = 0, \theta_1, \ldots, \theta_{\alpha-1})}{\partial \theta_\alpha} + O((\Delta \theta_\alpha)^2) . \]  

(2.22c)

Using expressions (2.17), one easily verifies that (2.22a) = (2.22c). Now the requirement (2.21a) when applied to the (2.22b,c) for a finite and arbitrary \( \Delta \theta_\alpha \), yields

\[ \frac{\partial \mathbf{L}_{in}(x; r = 0, \theta_1, \ldots, \theta_{\alpha-1})}{\partial \theta_\alpha} = 0, \alpha = 1, \ldots, n-1 , \]

(2.23)

which implies that \( \mathbf{L}_{in} \big|_{r=0} \) is independent of \( \theta^\prime s \). As we see, another aspect of the stronger version of the integrability freedom is to stress the importance of \( \theta_0 \equiv r \).

One may perform some consistency checks. For example, taking \( \Delta \theta_\alpha \) to be infinitesimal, differentiating (2.22a) with respect to \( r \) and remembering that (2.22c) = 0, we obtain

\[ \frac{\partial}{\partial r} (\eta_{\theta_\alpha})^{1/2} \Lambda_{\theta_\alpha}(x; \bar{g}) = \frac{\partial}{\partial \theta_\alpha} \Lambda_r(x; \bar{g}) , \alpha \geq 1 . \]

(2.24a)
which is a weaker version of the integrability freedom expressed in the polar coordinate system, and derivable from the one in the Cartesian coordinate system:

\[
\frac{\partial}{\partial g_i} \Lambda_j(x; \tilde{g}) = \frac{\partial}{\partial g_j} \Lambda_i(x; \tilde{g}) .
\]  

(2.24b)

One can easily see that relations (2.24) also follow directly from definitions of \( \Lambda \)'s, relations (2.16) and (2.17).

A very important stronger version of integrability freedom is obtained upon integrating (2.24a) with the help of (2.23):

\[
(\eta_{\theta_\alpha \theta_\alpha})^{1/2} \Lambda_{\theta_\alpha}(x; \tilde{g}) = \int_0^r dr \frac{\partial}{\partial \theta_\alpha} \Lambda_r(x; \tilde{g}) , \alpha \geq 1 ,
\]

(2.25)

which simply states that any \( \Lambda_{\theta_\alpha} , \alpha \geq 1 \), can be expressed in terms of \( \Lambda_r \); that is, all that counts in the unification scheme is \( \Lambda_r \equiv \Lambda_{\theta_\alpha} \).

### 2.3 The unified description of interactions

The choice of the path \( P \) as the "straight" line from \( \tilde{g} = 0 \) to \( \tilde{g} = \tilde{g} \) in the direction of \( \hat{\theta}_0 \equiv \hat{r} \), when solving the PDECC (2.18b) for the \( S \) matrix, yields the unified description of, say, \( n \) interactions in terms of the radial coupling constant \( \theta_0 \equiv r \):

\[
S(\tilde{g}) = T^r \exp \{ i \int d^4x \int_0^r dr \Lambda_r(x; r, \theta_1, ..., \theta_{n-1}) \} .
\]

(2.26)

The expansion is in terms of \( r \) where, consistent with (2.23), \( \theta_\alpha , \alpha \geq 1 \) are unchanged as \( r \to 0 \). What about the PDECC with respect to \( \theta_\alpha , \alpha \geq 1 \)? First of all, using relations (2.18), they can be written as

\[
\frac{1}{i} \frac{\partial}{\partial \theta_\alpha} S = \int d^4x \int_0^r dr \frac{\partial}{\partial \theta_\alpha} \Lambda_r(x; \tilde{g}) , \alpha \geq 1 .
\]

(2.27a)

Although these relations tell us how the \( S \) matrix changes with \( \theta_\alpha , \alpha \geq 1 \), they are nevertheless superfluous; the reason is that for any path connecting the origin to \( \tilde{g} \), one always obtains the relation (2.26) for the \( S \) matrix. Namely, for \( \alpha \geq 1 \), we have
\[
\frac{\partial}{\partial \theta_n} \Lambda_n (x; r, \theta_1, \ldots, \theta_{n-1})
\]

which proves the point.

Relation (2.26) for the $S$ matrix, although integrated only over just the radial coupling constant $r$, represents only the unified description (rather than the true unification) because, as suggested by relations (2.10), (2.19), and (2.20), we still have for any physical field $\phi_n (x)$, that

\[
\frac{\partial}{\partial \theta_n} \phi_n (x) = 0, \alpha = 1, \ldots, n-1.
\]

$\theta_n, \alpha \geq 1$, although passive, are still coupling parameters even though they do not appear any more as the PDECC coupling variables.

3 Coupling angles as a part of the free system: The true unification of interactions

Here we discuss the true unification of $n$ interactions, say, at some "accessible" energy scale. This will happen when the $n-1$ coupling angles cease being coupling parameters; the $n$ interactions are governed by just one coupling constant, the radial coupling constant $r = \theta_n$. This will happen if to each Cartesian coupling constant $g_i$ we associate a neutral vector boson $\xi_i, Z^\mu_{n_i}$, with $\xi_i$ allowing for the sign difference, $\xi_i^2 = 1$; notice that the standard $SU(2) \times U(1)$ gauge model [1-3] has these properties. These $n$ neutral vector boson fields define $n$ Cartesian components of the neutral vector boson field vector $\tilde{Z}^\mu_n$ in the $n$-dimensional coupling constant space. The polar components of $\tilde{Z}^\mu_n$, which are given through the mixings of the Cartesian components with coupling angles as mixing angles, are identified with the physical neutral vector bosons. Thus the coupling angles become the part of the free system and cease being the coupling parameters. The $n$
interactions become truly unified into a single interaction associated with the single
coupling constant the radial coupling constant \( r = \theta_0 \).

Utilizing relations (2.12) for unit vectors, the Cartesian and polar components of \( \tilde{Z}^\mu_{in} \) are
related as
\[
\xi_i Z^\mu_{in} = \tilde{Z}^\mu_{in} \cdot \hat{\xi}_i = \sum_{a=0}^{n-1} C_{i,\theta_a} Z^\mu_{\theta_a, in} , \tag{3.1}
\]
\[
Z^\mu_{r, in} = \tilde{Z}^\mu_{in} \cdot \hat{\theta}_0 = \sum_{a=1}^{n} C_{i,\theta_a} \xi_i Z^\mu_{\theta_a, in} , \tag{3.2a}
\]
\[
\alpha \geq 1 : \quad Z^\mu_{\theta_a, in} = \tilde{Z}^\mu_{in} \cdot \hat{\theta}_\alpha = r (\eta_{\theta_a, \theta_0})^{-1/2} \frac{\partial}{\partial \theta_\alpha} Z^\mu_{r, in} . \tag{3.2b}
\]
Relation (3.2b) takes into account the properties of C-coefficients, relations (2.15). We can view (3.2b) as a weaker version of the integrability freedom (compare with (2.24a)) for the polar components \( Z^\mu_{\theta_a, in} \), \( \alpha \geq 1 \); they are simply derived from \( Z^\mu_{r, in} \). This procedure is, of course, consistent with the canonical commutation relations. In fact, relations (3.1) - (3.2) are nothing but concisely written mixings among the neutral vector bosons that one uses when identifying the mass spectrum in the unification models. Here, however, the mixing angles are identified with the coupling angles which, from the mathematical point of view, is correct thing to do.

The formal reason why the identification of \( n-1 \) coupling angles as mixing angles represents a true unification of \( n \) interactions comes from the fact that, with relations (3.2), we have
\[
\frac{\partial}{\partial r} Z^\mu_{\theta_a, in} = 0, \alpha \geq 0 ; \quad \frac{\partial}{\partial \theta_\beta} Z^\mu_{\theta_a, in} \neq 0, \alpha \geq 0, \beta \geq 1 . \tag{3.3}, (3.4)
\]
Relations (3.4) are the consequence of \( C_{i,\theta_0} \) being independent of \( \theta_0 \equiv r \), and the fact that \( \eta_{\theta_a, \theta_0} \), \( \alpha \geq 1 \), are proportional to \( r^2 \); these relations clearly show that the coupling angles must be part of the free system. On the other hand, relation (3.3) clearly shows that \( r \) is a
bona fide coupling constant (compare with (2.7)); the \( n \) interactions are now described as a single interaction in terms of a single coupling constant \( r \).

\[13\]
The masses of physical vector bosons, through the diagonalization procedure, will generally depend on coupling (mixing) angles, since the physical (polar) neutral vector boson fields also depend on them. This dependence is consistent with masses being generated either through the spontaneously broken symmetry (mediated by one or more Higgs particles) or through a massive gauge theory [12]; the massive gauge theories are in a sense spontaneously broken gauge theories with the physical Higgs field set to zero [13]. With masses dependent on coupling angles, the integrability freedom becomes restricted. In what follows, the masses will be assumed to be zero; the integrability freedom needs to be verified only for the interaction and kinetic parts of the Lagrangian density. The effect of masses will be discussed last. Once the true unification is achieved in terms of the single coupling constant \( r = \theta_0 \), one may solve either the PDECC or the ordinary equation of motion perturbatively just in terms of \( r \).

3.1 Effect of true unification on some interaction Lagrangian density terms

For the simplicity, only the neutral part of gauge interactions with fermions and the kinetic gauge field terms are considered.

The neutral part of gauge interactions involving fermions (at some finite energy unification scale) is linear in coupling constant and has the form [14]:

\[
\mathbf{L}(NC) = \sum_{i=1}^{5} g_i \lambda_i (NC) \quad ,
\]

\[
\lambda_i (NC) = Z_{j,im}^\mu J_{\mu,i}^m \cdot J_{\mu,j}^m = \overline{\psi}_m \gamma_{\mu} t_i \psi_m \quad .
\]

(3.5a,b,c)

Similar to the standard \( SU(2) \times U(1) \) electroweak model, here \( t_i \) are matrices that represent Hermitian electrically neutral generators of the unification group with \( \psi \) representing appropriate fermion multiplets; \( \gamma^\mu \) are the usual Dirac matrices. Generally vectors \( \tilde{\Lambda}(NC) \) and \( \tilde{Z}_m^\mu \) are not proportional to each other. Nevertheless, one can express \( \Lambda_{\theta_0} (NC) \) as functions of \( Z_{\theta_0,m}^\mu \) in two steps:
\[ \Lambda_{\theta_\alpha}(NC) = \hat{\theta}_\alpha \cdot \check{\Lambda}(NC) = \sum_{i=1}^{n} C_{i\theta_\alpha} \Lambda_i(NC) \]  
(3.6a)

\[ = \sum_{i=1}^{n} \sum_{\beta=0}^{m} C_{i\theta_\alpha} C_{i\theta_\beta} \xi_i J_{\mu,\beta} \mu \cdot Z_{\beta_{ij} \mu} \]  
(3.6b)

It is easily seen that \( \Lambda_{\theta_\alpha}, \alpha \geq 1 \), satisfy the integrability freedom. In general the forms of neutral currents, which couple to the physical neutral vector bosons, are rather complex.

Namely, writing

\[ \Lambda_{\theta_\alpha}(NC) = \sum_{\beta=0}^{n-1} J_{\mu,\beta} \mu \cdot Z_{\beta_{ij} \mu} \]  
(3.7a)

from (3.6b), we obtain for currents the following expressions:

\[ J_{\mu,\beta} \mu = \sum_{i=1}^{n} C_{i\theta_\alpha} C_{i\theta_\beta} \xi_i J_{\mu, \beta} \mu \]  
(3.7b)

Clearly, the current has the symmetry:

\[ J_{\mu, \beta} \mu = J_{\mu, \beta} \alpha \beta \]  
(3.8)

A current with \( \alpha \neq \beta \) is called universal; it couples with the same strength to \( Z_{\theta_\alpha} \) as it does to \( Z_{\theta_\beta} \). As it will be seen later, the electromagnetic current and the current associated with the third component of the weak isospin are such currents. One should note that, because of the stronger version of the integrability freedom, the physical coupling constants are determined from \( \Lambda_i \) in the S matrix (2.26).

The individual kinetic terms of the Lagrangian density depend on angles \( \theta_\alpha, \alpha \geq 1 \), when expressed in terms polar neutral vector bosons. The polar components of \( \tilde{\Lambda} \) satisfy, in general,

\[ \Lambda_i(K; Z_{\theta_\alpha \mu}) = 0, \Lambda_{\theta_\alpha}(K; Z_{\theta_\beta \mu}) \neq 0, \alpha = 1, \ldots, n-1; \beta = 0, \ldots, n-1 \]  
(3.9)

However, with these and relations (3.1) to (3.2), or directly from (2.17b), one sees that the following relations hold:

\[ \Lambda_{\theta_\alpha}(K; \text{Total}) = (\eta_{\theta_\alpha, \theta_\alpha})^{-1/2} \frac{\partial \mathbf{L}_{\alpha}(K; \text{Total})}{\partial \theta_\alpha} = 0 \]  
(3.10)

and \( \mathbf{L}_{\alpha}(K; \text{Total}) \) does not spoil the integrability freedom.
3.2 True partial unification of interactions

If the number of gauge (Cartesian) coupling constants, \( m \), is larger than the number of neutral (Cartesian) vector bosons, \( n; \ m > n \), then one can have a partial true unification; one carries out the true unification on the \( n \) gauge interactions only. Again the \( n \) Cartesian neutral vector bosons are transformed into the \( n \) physical spherical neutral vector bosons which, in turn, "absorb" the \( n - 1 \) coupling angles; the \( n \) coupling constants \( g_1, \ldots, g_n \) are reduced to single coupling constant \( \theta_0 \equiv r = (g_1^2 + \cdots + g_n^2)^{1/2} \). Hence, the partial true unification of \( m \) interactions into \( 1 + m - n \) interactions is accomplished; they are described by \( \theta_0 \) and \( g_{n+1}, \ldots, g_m \) coupling constants. Of course, these can be given a unified description where, however, also \( \theta_0 \) would have to be treated now as a Cartesian coupling constant.

The true partial unification also allows universal fermion current density that couples to each of the \( n \) physical spherical neutral vector bosons; such a current density, the third component of the weak isospin, occurs in the standard \( SU(3) \times SU(2) \times U(1) \) quantum chromodynamics-electroweak model.

3.3 Effect of the masses

As long as the \( n \) coupling constants are all bona fide coupling constants; that is, no attempt is made at any kind of unification, the integrability freedom is easy to satisfy. This is so because when the physical masses are independent of coupling constants one has: \( \Lambda_i (m; \phi) = 0, \Lambda_\alpha (m; \phi) = 0; i = 1, \ldots, n; \alpha = 0, \ldots, n - 1 \), and no physical mass Lagrangian density enters into the \( S \) matrix. However, in the case of true unification, the polar (physical) components of the neutral vector boson-vector depend on coupling angles; hence, at least some masses must depend on coupling angles if all the mass terms are to be properly diagonalized.
Specifically, when the masses are generated through the spontaneous symmetry breaking, the parameters of this breaking must be constrained in such a way that the resulting masses, while depending on some coupling angles, do not depend on \( r \); otherwise \( r \) could not be a coupling constant of the truly unified theory. The same thing applies for masses that are associated with massive gauge theories. Therefore, for a specific field \( \phi \), the Lagrangian mass term density \( \mathbf{L}_{\alpha} (m; \phi) \) will satisfy
\[
\Lambda_{\alpha} (m; \phi) \equiv \frac{\partial \mathbf{L}_{\alpha} (m; \phi)}{\partial r} = 0 ,
\]
(3.13)
since \( \partial m / \partial r = 0 \) and \( \partial \phi_{\alpha} / \partial r = 0 \). However, for \( \partial m / \partial \theta_{\alpha} = 0, \alpha \geq 1 \), there are two possibilities:
\[
\Lambda_{\theta_{\alpha}} (m; \phi) = (\eta_{\theta_{\alpha}})^{-1/2} \frac{\partial \mathbf{L}_{\alpha} (m; \phi)}{\partial \theta_{\alpha}} \neq 0, = 0 ,
\]
(3.14a,b)
if \( \partial \phi_{\alpha} / \partial \theta_{\alpha} \neq 0, = 0 \), respectively.

Finally, one also has
\[
\Lambda_{\phi_{\alpha}} (m; \phi) = (\eta_{\phi_{\alpha}})^{-1/2} \frac{\partial \mathbf{L}_{\alpha} (m; \phi)}{\partial \phi_{\alpha}} \neq 0 ,
\]
if \( \partial m / \partial \theta_{\alpha} \neq 0 \), and \( \partial \phi_{\alpha} / \partial \theta_{\alpha} \neq 0, = 0; \alpha \geq 1 \).
(3.15)

The physical mass terms cannot appear in the \( S \) matrix. The integrability freedoms from relations (2.21a), (2.22), and (2.23) have to be restricted; the allowed paths \( C, P, \Delta C \), etc., have to be deformed in such a way that \( \Lambda_{\theta_{\alpha}} (m) \neq 0, \alpha \geq 1 \), do not contribute. The easiest way to achieve this is to fix those \( \theta \)'s, at their physical values, for which \( \Lambda_{\theta_{\alpha}} (m) \neq 0, \alpha \geq 1 \), so that formally \( d \theta_{\alpha} = 0 \). This is the essence of the restricted integrability freedom for coupling angle-dependent Lagrangian mass density terms: it amounts simply to ignoring the mass terms when integrating the PDECC for the \( S \) matrix.

4 Applications to electroweak interactions
The electroweak interactions will be used to discuss and demonstrate: (1) the
integrability freedom, (2) true and partial true unifications, and (3) effects of masses
(Higgs mechanism) and neutral vector boson-mixings on the integrability freedom and
true unification of interactions. Point (1), since it has to be satisfied for unified
description, true, and partial true unification, will be discussed from the general point of
view. With points (2) and (3), however, one has to be more specific: neutral and charged
vector bosons have to be clearly identified. Thus, it is necessary to give a brief description
of electroweak interactions.

4.1 Brief summary of the standard electroweak model

In terms of Cartesian coupling constants, the $SU(2) \times U(1)$ electroweak gauge group
Lagrangian density can be written as [1-3]

$$\mathbf{L} = \mathbf{L}_S + \mathbf{L}_H,$$

(4.1)

where $\mathbf{L}_H$ contains the Higgs boson coupling, whose main role is to generate vector
boson and fermion masses. The "symmetric" part $\mathbf{L}_S$ involves only gauge bosons and
fermions:

$$\mathbf{L}_S = -\frac{1}{4} \sum_{A=1}^{3} W_{\mu}^{A} W_{\nu}^{A} - \frac{1}{4} B_{\mu} B_{\nu} + \sum_{\psi} \bar{\psi} i \gamma^\mu D_\mu \psi .$$

(4.2)

The fermion fields are described through their left-handed and right-handed components:

$$\psi = \psi_L + \psi_R , \quad \psi_{L,R} = P_{L,R} \psi , \quad \psi_{L,R} = \psi \Gamma_{L,R} ,$$

$$P_{L,R} = \frac{1}{2} (1 + \gamma_5) , \quad P_{L,R}^2 = P_{L,R} , \quad P_{L,R} P_{R,L} = 0 ,$$

$$\gamma_5^2 = 1, \quad \gamma_5^* = \gamma_5 .$$

(4.3)

The Dirac matrices are the same as in reference [15], except that $\gamma_5$ differs by a sign; $\psi_L$
are lepton and quark doublets with weak $SU(2)_L$ isospin assignments, respectively,
\[ \psi_L : l_L = \begin{pmatrix} V_r \\ e_L \\ \mu_L \\ \tau_L \end{pmatrix}, \quad \psi_R : q_L = \begin{pmatrix} u_L \\ d^\prime_L \\ c_L \\ s^\prime_L \end{pmatrix}, \quad \bar{q}_L = \frac{t}{b_L} \]  

Here \( d^\prime, s^\prime, \) and \( b^\prime \) are linear combinations of mass eigenfields \( d, s, \) and \( b, \) obtained from the latter by a unitary transformation, while \( \psi_R \) are weak \( SU(2)_L \) singlets:

\[ \psi_R : l_R = e_R, \mu_R, \tau_R ; \quad q_R = u_R, d^\prime_R, \bar{c}_R, s^\prime_R, \bar{b}_R, \]

Abelian \( U(1) \) group is associated with weak hypercharge \( y. \) The electric charge operator is defined as

\[ Q = T_{L3} + \frac{Y}{2}, \quad Y = Y_L + Y_R, \]  

where \( Y_{L,R} \) and \( T_{L,A} (A = 1,2,3) \) are the hypercharge and \( SU(2) \) generators, respectively.

The charge eigenvalues are: \( q(v, \nu) = 0, q(e, \mu) = -1, q(u,c) = 2/3, \) and \( q(d^\prime, s^\prime) = -1/3. \)

Consistent with relations (4.5) the hypercharge assignments then are:

\[ y_L(v, \nu, \tau, e, \mu, \tau) = -1, \quad y_R(e, \mu, \tau) = -2, \quad y_L(u,c,t,d^\prime, s^\prime, b^\prime) = 1/3, \]

\[ y_R(u,c,t) = 4/3, \) and \( y_R(d^\prime, s^\prime, b^\prime) = -2/3. \)

Gauge fields \( W^A_{\mu} \) and \( B_{\mu} \) are associated with \( SU(2) \) and \( U(1) \) gauge groups, respectively.

The corresponding anti symmetric gauge tensors, appearing in (4.2), are

\[ W^A_{\mu \nu} = F^A_{\mu \nu} + g_1 \varepsilon^{ABC} W^B_{\mu} W^C_{\nu}, \quad F^A_{\mu \nu} = \partial_{\mu} W^A_{\nu} - \partial_{\nu} W^A_{\mu}; \]

\[ B_{\mu} = \partial_{\mu} \psi - \psi \partial_{\nu} B_{\mu}, \]

where \( \varepsilon_{ABC} \) are the \( SU(2) \) group structure constants coinciding with the totally anti-symmetric Levi-Civita tensor.

The covariant derivatives for the fermion fields are given as

\[ D_{\mu} \psi = \left[ \partial_{\mu} - ig_1 \sum_{A=1}^3 t^A_{\mu} W^A_{\mu} - ig_2 \frac{y}{2} B_{\mu} \right] \psi, \]

\[ t^A = t^A_L + t^A_R, \quad t^A_L = \frac{P^0 \tau^A}{2}, \quad t^A_R = 0, y = y_L + y_R. \]

where $\tau^A$ are Pauli matrices. Consequently, for the symmetric Lagrangian density one writes,

$$L_s = L_{w,b} + L_f,$$

(4.8a)

where

$$L_{w,b} = -\frac{1}{4} W_{\mu}^{A} W_{\nu}^{A} - \frac{1}{4} (B_{\mu})^2,$$

(4.8b)

$$L_f = \sum_{\nu} \bar{\psi} i\gamma^\mu \left[ \partial_{\mu} - ig_3 \sum_{A=1}^{2} \frac{P_L \tau^A}{2} W_{\mu}^A - ig_2 \frac{y}{2} B_{\mu} \right] \psi.$$

(4.8c)

The Higgs part has the form,

$$L_H = -|D_{\mu} h|^2 - V(|h|^2) - \left\{ \sum_{i} G_{i} \tilde{l}_{i} h l_{i} + \tilde{q}_{L}^{0} \left[ G_{s} \tilde{u}_{R} + G_{s} h d_{R} \right] \\
+ \tilde{q}_{R}^{0} \left[ G_{c} \tilde{c}_{R} + G_{t} h s_{R} \right] + \tilde{q}_{L}^{0} \left[ G_{t} \tilde{t}_{R} + G_{s} h b_{R} \right] + H.C. \right\}, \quad l = e, \mu, \tau :$$

$$V = -\mu^2 |h|^2 + \lambda (|h|^2)^2, \mu^2 > 0, \lambda > 0 ;$$

$$D_{\mu} h = \left[ \partial_{\mu} - ig_3 \sum_{A=1}^{2} \frac{\tau^A}{2} W_{\mu}^A - ig_2 \frac{y}{2} B_{\mu} \right] h,$$

(4.9a)

where

$$h = \begin{pmatrix} h_u \\ h_d \end{pmatrix}, \quad \tilde{h} = i \tau_2 h^*, \quad \tilde{q}_L^0 = \begin{pmatrix} u \\ d \end{pmatrix}_L,$$

(4.9b)

The spontaneous symmetry breakdown takes place where the Higgs potential has the minimum; that is, at $h_u = 0$ and $|h_d| = v / \sqrt{2}, v = \sqrt{\mu^2 / \lambda}$. Thus, in the unitary gauge [16] one writes for the Higgs doublet

$$h(x) = \frac{v + H(x)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

(4.10)

where $H(x)$ is known as the Higgs field, which can be associated with $H_{in}(x)$.

Collecting all the terms from $L_H, L_{w,b},$ and $L_f$, one can rewrite the total $L$ as the sum of the kinetic, mass, and interaction terms:

$$L = L(K) + L(m) + L(int).$$

(4.11)

$$L(K) = L(K; W_\gamma) + L(K; W_\delta) + L(K; B) + L(K; \psi) + L(K; H),$$

(4.12)
\[ L(K; W) = -\frac{1}{2} F^\mu_\nu F^{*\mu\nu}, W^\pm_\mu = \frac{1}{\sqrt{2}} \left( W^\mu_+ \mp i W^\mu_- \right) \],

\[ L(K; W'_\gamma) = -\frac{1}{4} (F^\mu_\nu)^2, L(K, B) = -\frac{1}{4} (B^\mu_\nu)^2 \],

\[ L(K; \psi) = \sum_\psi \overline{\psi} \gamma^\mu \partial_\mu \psi, \psi = \nu, e, \ldots, u, d, \ldots \]

\[ L(K, H) = -\frac{1}{2} \left( \partial_\mu H \right)^2 \]

\[ L(m) = L(m; W) + L(m; W'_\gamma, B) + L(m; \psi) + L(m; H) \]

\[ L(m; W) = \frac{(g, \nu)^2}{2} W^\mu_\nu W^{*\mu} \]

\[ L(m; W'_\gamma, B) = -\frac{\nu^2}{2} \left( g, W^\gamma_\mu - g_2 B_\mu \right)^2 \]

\[ L(m; \psi) = -\frac{\nu}{\sqrt{2}} G_\nu \overline{\psi} \psi, \psi = l, q; \]

\[ l = e, \mu, \tau; q = u, d, c, s, t, b \]

\[ L(m; H) = -\frac{1}{2} \left( 2 \nu \lambda \right) H^2 \]

\[ L(int) = L(int; \psi, W) + L(int; \psi, W'_\gamma, B) \]

\[ + L(int; W), L(int; W'_\gamma, W'_\gamma) + L(int; W'_\gamma, H) \]

\[ + L(int; W'_\gamma, B, H) + L(int; \psi, H) + L(int; H) \]

\[ L(int; \psi, W) = \frac{g_1}{\sqrt{2}} \left( J^\mu_\nu W^\mu_\nu + J^-\mu W^-_\mu \right) \]

\[ J^\mu_\nu = \sum_\psi \overline{\psi} \gamma^\mu \frac{\tau^3 P_\nu}{2} \psi, J^-_\mu = J^\mu_\mu \pm i J^\mu_\nu \]

\[ L(int; \psi, W'_\gamma, B) = g_1 J^\mu_\nu W^3_\mu + g_2 J^\mu_\nu B \]

\[ J^\mu_\mu = J^3_\mu + J^\pm_\mu, J^\pm_\mu = \sum_\psi \overline{\psi} \gamma^\mu \frac{\tau^\pm}{2} \psi \]

\[ L(int; W'_\gamma) = \frac{g_1^2}{2} W^\mu_\nu W^\nu_\mu (W^-_\mu W^\mu W^- + W^\mu W^- W^-) \]

\[ L(int; W'_\gamma, W'_\gamma) = g_1 \left( i F^-_\mu W^3_\mu W^-_\nu + i F^-_\mu W^3_\mu W^-_\nu \right) \]

\[ - \left( W^-_\mu W^\mu W^- - W^\mu W^- W^- \right) \partial_\mu W^3_\nu \]

\[ + g_1^2 \left( W^\nu_\mu W^3_\nu W^{3\nu} - W^\nu_\mu W^3_\nu W^{3\nu} \right) \]

\[ L(int; W'_\gamma, H) = -\frac{g_1^2}{4} \left( 2\nu H + H^2 \right) W^\mu_\mu W^\mu_\mu \]

\[ L(int; W'_\gamma, B, H) = -\frac{1}{8} \left( 2\nu H + H^2 \right) \left( g, W^3_\mu - g_2 B_\mu \right)^2 \]
\[ \mathbf{L}^{\text{int}}(\psi, H) = -\frac{H}{\sqrt{2}} \sum_{\psi} G_\psi \bar{\psi} \psi, \psi = l, q ; \]

\[ l = e, \mu, \tau ; \ q = u, d, c, s, t, b . \]  

(4.17h)  

\[ \mathbf{L}^{\text{int}}(H) = -\left( \lambda v H^3 + \frac{\lambda}{4} H^4 \right) . \]  

(4.17i)  

The description of the standard electroweak model, given above, is in the unitary gauge. It results from the spontaneously broken \( SU(2) \times U(1) \) gauge group. The quantum chromodynamics based on the \( SU(3) \) group is for now left out; its effect on the unification will be discussed later. The unitary gauge has the advantage of exhibiting the particle spectrum clearly, but the propagators have bad high-energy behavior. On the other hand, the R-gauges \([16, 17]\) have better behaved propagators at high energy, but with a complicated particle spectrum. For discussing the unification criteria, the unitary gauge is preferable since in it allows the fields at \( t \to -\infty \) to be associated with real particles.

In what follows, the fields are all free in fields; for the sake of simplicity, the subscript in will be omitted.

### 4.2 Integrability freedom in the standard electroweak model

For the standard electroweak model based on the \( SU(2) \times U(1) \) spontaneously broken gauge group the Cartesian coupling constants are simply \( g_1 \) and \( g_2 \). As indicated by relations (2.11)-(2.14), they are related to the polar coupling constants \( r = \theta_0 \) and \( \theta_1 = \theta \) as

\[ g_1 = r \cos \theta, \ g_2 = r \sin \theta , \]  

(4.18a)  

\[ \theta_0 = r = \sqrt{g_1^2 + g_2^2} ; \ \eta_{\theta_0, \theta_0} = 1, \ \eta_{\theta_0, \theta_1} = r ; \]  

(4.18b,c)  

\[ C_{1\theta_0} = \cos \theta, \ C_{2\theta_0} = \sin \theta, \ C_{1\theta_1} = -\sin \theta, \ C_{2\theta_1} = \cos \theta . \]  

(4.18d)  

As mentioned in Section 2.2, the integrability freedom is discussed with the assumption that the physical masses are either zero or independent of coupling constants. Effectively
this means that one takes by definition $\mathbf{L}_\mu = 0$, so that only terms from the Lagrangian density $\mathbf{L}_\gamma$, (4.2), enter into discussion; these terms are: (4.8a)-(4.8c), (4.13a)-(4.13d), and (4.17a)-(4.17e).

In view of (2.7), from the kinetic terms (4.13a)-(4.13d), one has that
\[
\Lambda_i(K; W_\mu) = \Lambda_i(K; B) = \Lambda_i(K; \psi) = 0, \; i = 1, 2, \tag{4.19a}
\]
\[
\Lambda_{\theta_\alpha}(K; W_\mu) = \Lambda_{\theta_\alpha}(K; B) = \Lambda_{\theta_\alpha}(K; \psi) = 0, \; \alpha = 0, 1, \tag{4.19b}
\]
as it should be, since these terms cannot contribute to the $S$ matrix.

As to the interaction terms, one starts with $\mathbf{L}(\text{int}; W_\mu)$:
\[
\Lambda_i(\text{int}; W_\mu) = g_i W_\mu^+ W_\nu^+ (W^{-\mu} W^{+\nu} - W^{-\mu} W^{-\nu}),
\]
\[
\Lambda_2(\text{int}; W_\mu) = 0; \tag{4.20a}
\]
\[
\Lambda_{\theta_\alpha}(\text{int}; W_\mu) \equiv \Lambda_{\theta_0}(\text{int}; W_\mu) = C_{\theta_0} \Lambda_1(\text{int}; W_\mu)
\]
\[
= r(\cos^2 \theta) \ W_\mu^+ W_\nu^+ (W^{-\mu} W^{+\nu} - W^{+\mu} W^{-\nu}),
\]
\[
\Lambda_{\theta}(\text{int}; W_\mu) = C_{\theta} \Lambda_1(\text{int}; W_\mu)
\]
\[
= -r(\sin \theta \cos \theta) \ W_\mu^+ W_\nu^+ (W^{-\mu} W^{+\nu} - W^{+\mu} W^{-\nu}). \tag{4.20b}
\]

One easily verifies the weaker and stronger versions of the integrability freedom, relations (2.24) and (2.25).

Next, one discusses $\mathbf{L}(\text{int}; \psi, W_\mu, B)$ from relation (4.17b). One has
\[
\Lambda_1(\text{int}; \psi, W_\mu, B) = J^{3\mu} W_\mu^3, \Lambda_3(\text{int}; \psi, W_\mu, B) = J^\mu B_\mu; \tag{4.21a}
\]
\[
\Lambda_1(\text{int}; \psi, W_\mu, B) = \cos \theta J^{3\mu} W_\mu^3 + \sin \theta J^\mu B_\mu,
\]
\[
\Lambda_\theta(\text{int}; \psi, W_\mu, B) = -\sin \theta J^{3\mu} W_\mu^3 + \cos \theta J^\mu B_\mu
\]
\[
= \frac{\partial}{\partial \theta} \Lambda_1(\text{int}; \psi, W_\mu, B). \tag{4.21b}
\]

The weaker version of the integrability freedom, relations (2.24), are easily verified, since obviously $\Lambda_\theta = \partial \Lambda_1 / \partial \theta$. With this, the stronger version of the integrability freedom in the form (2.25) also follows. The integrability freedoms of other terms can be verified in a similar manner.
As one can see, as long as masses can be considered to be independent of coupling constants (or zero) and if \( W_{\mu}^3 \) and \( B_{\mu} \) were physical particles, then the \( SU(2) \times U(1) \) electroweak model could be given a unified description with \( \theta_0 = \pi \) as a main coupling constant and \( \theta \) as a passive angle; the \( S \) matrix would have form (2.26). The same thing holds for the quantum chromodynamics-electroweak model based on the gauge group \( S(3) \times SU(2) \times U(1) \), except that now one would have two passive angles.

### 4.3 True and partial true unifications of the electroweak model

One still assumes that \( \mathbf{L}_{\mu} = 0 \). The physical neutral vector bosons correspond to the neutral vector boson-vector components \( Z_{\nu}^\mu \equiv \xi_1 \cos \theta Z_1^\mu - \sin \theta Z_\theta^\mu \) (radial) and \( Z_\theta^\mu \) (spherical). With relations (3.1), (3.2), (4.18d), and (4.21a), one has

\[
W_3^\mu \equiv Z_1^\mu = \xi_1 \left( \cos \theta Z_1^\mu - \sin \theta Z_\theta^\mu \right),
\]

\[
B^\mu \equiv Z_2^\mu = \xi_2 \left( \sin \theta Z_1^\mu + \cos \theta Z_\theta^\mu \right),
\]

\[
Z_1^\mu = \cos \theta \xi_1 Z_1^\mu + \sin \theta \xi_2 Z_2^\mu, \quad Z_\theta^\mu = \frac{\partial Z_\mu}{\partial \theta}.
\]

(4.22a)

(4.22b)

Explicitly, one verifies,

\[
\frac{\partial}{\partial \theta} Z_1^\mu = -Z_{1,\mu}, \quad Z_\theta^\mu = -\frac{\partial Z_\theta^\mu}{\partial \theta}.
\]

(4.22c,d)

The coupling angle \( \theta \) enters into the definitions of physical \( Z_1^\mu \) and \( Z_\theta^\mu \); this means that \( \theta \) cannot be considered as a coupling parameter any more (compare with relations (3.4) and the discussion thereafter). The integrability freedom from relations (2.21), however, states that in the \( S \) matrix any integration path is equivalent to the straight path from \( r = 0 \) to \( r = r \); the \( S \) matrix is in the form (2.26), except that now \( W_{\mu}^3 \) and \( B_{\mu} \) have to eliminated in favor of \( Z_1^\mu \) and \( Z_\theta^\mu \) according to (4.22a).
Currents that couple to $Z_\rho^\mu$ and $Z_\theta^\mu$ have to be physical (compare with relations (3.5) to (3.10)). In fact, applying relations (3.5) to (3.7) to $\mathbf{L}(\mathrm{int}; \psi, W, B)$ from relations (4.21), one has ($\Lambda_{\theta, \rho} \equiv \Lambda_r$)

\begin{align*}
\Lambda_r (\text{int}; \psi, W, B) &= Z_\theta^\mu \cos \theta \sin \theta \left[ \xi_2 J_\mu^{em} - (\xi_1 + \xi_2) J_\mu^{3L} \right] \\
+ & Z_\rho^\mu \left[ (\xi_1 \cos^2 \theta - \xi_2 \sin^2 \theta) J_\mu^{3L} + \xi_2 \sin^2 \theta J_\mu^{em} \right], \quad \text{(4.23a)} \\
\Lambda_\theta (\text{int}; \psi, W, B) &= \frac{\partial}{\partial \theta} \Lambda_r (\text{int}; \psi, W, B) \\
= & Z_\rho^\mu \cos \theta \sin \theta \left[ \xi_2 J_\mu^{em} - (\xi_1 + \xi_2) J_\mu^{3L} \right] \\
+ & Z_\theta^\mu \left[ (\xi_1 \sin^2 \theta - \xi_2 \cos^2 \theta) J_\mu^{3L} + \xi_2 \cos^2 \theta J_\mu^{em} \right]. \quad \text{(4.23b)}
\end{align*}

Relation (4.23b) reflects the stronger version of the integrability freedom (2.25), where one takes into account that the coupling space is two-dimensional and that $\mathbf{L}(\text{int})$ is linear in both coupling constants.

Consistent with relation (3.8), relations (4.23) reveal two universal currents $J_\mu^{em}$ and $J_\mu^{3L}$; for example $J_\mu^{em}$ couples with the same strength to $Z_\rho^\mu$ and $Z_\theta^\mu$ in relations (4.23a) and (4.23b), respectively. One reaches similar conclusions with $J_\mu^{3L}$ as long as $\xi_1$ and $\xi_2$ are arbitrary.

However, the physical coupling constants are determined only from $\Lambda_r$ upon integration over $r$ in the $S$ matrix, relation (2.26). Defining the electromagnetic coupling constant as $-e$, $e > 0$, then, with $Z_\theta^\mu = A^\mu$ as the electromagnetic potential, one chooses $\xi_2 = 1$ so that the first term in (4.23a) becomes $-e J_\mu^{em} A_\mu$ in the $S$ matrix; with $\theta$ now fixed at the physical value $\theta_w$ (the Weinberg angle) one has explicitly $e = r \cos \theta_w \sin \theta_w$ with the implication: $r \neq 0, e \neq 0 \rightarrow 0 < \theta_w < \pi/2$. Furthermore, it is known that $A^\mu$ does not couple just to $J_\mu^{3L}$. Hence, in order to exclude this term from (4.23a), one has to choose $\xi_1 = -\xi_2 = -1$. With these then, from relation (4.23a), one obtains that the neutral vector boson $Z_\rho^\mu \equiv Z_\mu$ couples to the fermion neutral current

\begin{equation}
J_\mu^{NC} = J_\mu^{3L} - \sin^2 \theta_w J_\mu^{em}, \quad \text{(4.24)}
\end{equation}
with the strength $g_{NC} = r$. Furthermore, from relation (4.17a), the interaction of weak charge-changing current with $W^\pm_\mu$ is characterized by $r \cos \theta_w / \sqrt{2}$.

The inclusion of $SU(3)$ gauge invariant quantum chromodynamics with three colors, which is necessary because the quarks have fractional charges, spoils the true unification. This is so because among the $SU(3)$ gauge bosons there are no neutral ones to mix with the neutral vector bosons from the $SU(2) \times U(1)$ gauge group. Hence, the quantum chromodynamics-electroweak model based on gauge group $SU(3) \times SU(2) \times U(1)$ can have only true partial unification. Or, treating the $SU(3)$ quantum chromodynamics and the truly unified $SU(2) \times U(1)$ electroweak interactions as two separate interactions, each with its separate "Cartesian" coupling constant, one can give them a unified description.

### 4.4 Effects of masses and neutral vector boson mixings on unification and integrability freedom

Individual kinetic Lagrangian density terms are independent of coupling constants. Therefore, the mixings of neutral vector bosons, despite the fact that some individual terms may be coupling angles dependent, the total kinetic Lagrangian density is independent of $r$ and $\theta_w, \alpha \geq 1$; as such, it does not interfere with the integrability freedom. One can demonstrate this on two relevant terms from relation (4.2):

\[
\mathbf{L}(K; W_\pm) + \mathbf{L}(K; B) = -\frac{1}{4} \left[ (W^3_{\mu\nu})^2 + (B_{\mu\nu})^2 \right] \quad (4.25a)
\]

\[
= \mathbf{L}(K; Z) + \mathbf{L}(K; B) = -\frac{1}{4} \left[ (Z_{\mu\nu})^2 + (A_{\mu\nu})^2 \right] . \quad (4.25b)
\]

One has explicitly that $\partial \mathbf{L}(K; W_\pm) / \partial \theta = \partial \mathbf{L}(K; B) / \partial \theta = 0$. The same is not true for $\mathbf{L}(K; Z)$ and $\mathbf{L}(K; Z)$. However, $\partial \mathbf{L}(K; Z) / \partial \theta = -\partial \mathbf{L}(K; A) / \partial \theta$, which proves the point. In fact all other non-Higgs Lagrangian density terms, (4.17a), (4.17d), and (4.17e) satisfy stronger and weaker versions of integrability freedom; that is, for each of them, all that is needed is the corresponding $A_\pm$. 

When $L_{\mu} \neq 0$, the situation is different. First, one discusses the mass terms (the discussion is the same if vector boson masses are associated with a massive gauge theory [12]). Consistent with relations (3.3) for any mass $m$ one must have, $\partial m/\partial r = 0$. From relations (4.15) it then follows that $v$ and $\lambda$ have to depend on $r$:

$$v(r) = \frac{\nu_0}{r}, \quad \lambda(r) = \frac{\nu_0}{r^2}, \quad G_{\nu}^0(r) = G_{\nu}^0 r,$$

(4.26a,b,c)

Then, the corresponding tree-level masses are

$$m_w(\theta)^2 = \frac{(v_0 \cos \theta)^2}{4}, \quad m_{\nu}^2 = \frac{v_0^2}{4},$$

(4.27a,b)

$$m_{\nu} = \frac{G_{\nu}^0 v_0}{\sqrt{2}}, \quad m_{\mu}^2 = 2 v_0^2 \lambda_0,$$

(4.27c,d)

where, consistent with relations (4.22), one identifies $-r Z_\mu(\theta) = g_1 W_\mu^3 - g_2 B_\mu$ and $\nu_0, \lambda_0,$ and $G_{\nu}^0$ are independent from both $r$ and $\theta$. Hence, the mass terms become

$$L(m; W_\perp) = -m_w(\theta)^2 W_\mu^- W^{\mu}, \quad L(m; A) = 0;$$

(4.28a,b)

$$L(m; W_\perp, B) = L(m; Z(\theta)) = -\frac{m_{\nu}^2}{2} (Z^\mu(\theta))^2;$$

(4.28c)

$$L(m; \psi) = -m_{\nu} \bar{\psi} \psi, \quad L(m; H) = -\frac{m_{\mu}^2}{2} H^2.$$

(4.28d,c)

One notices that $v(r) \to \infty$ as $r \to 0$; formally, the spontaneous symmetry breakdown shifts the singularity in the field $h$ at $r = 0$.

One sees that $L(m; W_\perp)$ and $L(m; Z)$ behave like examples (3.15) and (3.14a), respectively. For these terms only, one restricts the integrability freedom by fixing $\theta = \theta_\nu, d\theta = 0$, so that the allowed integration path from $r = 0$ to $r = r$ excludes these terms from appearing in the $S$ matrix (see the discussion after relation (3.15)). No restrictions on $\theta$ are necessary for fermion and Higgs mass terms when integrating PDECC for the $S$ matrix, since for these terms $\Lambda_\theta = 0$.

The interaction Lagrangian density involving the three-gauge couplings, relations (4.17d) and (4.17e), and the ones involving the couplings of Higgs field to gauge bosons, relations (4.17f)-(4.17i), satisfy weaker and stronger versions of the integrability freedom, particularly relations (2.25) and (2.27b).
Therefore, the standard (spontaneously broken $SU(2) \times U(1)$) electroweak model represents a truly unified theory with just one coupling constant $r$. Hence, regardless of the integration path in the relation (2.21b), the result for the $S$ matrix always is the same:

$$S(r) = T^* \exp \left\{ i \int d^4 x [ \int_0 r \Lambda, (\text{int}; x) ] \right\}$$

$$= T^* \exp \left\{ i \int d^4 x \mathbf{L}(\text{int}; x; r) \right\} .$$

(4.29a,b)

Here $\mathbf{L}(\text{int}; x; r)$ is numerically the same as $\mathbf{L}(\text{int})$ from relation (4.16), except that the only bona fide coupling constant is $r$ and all the neutral vector boson fields are the polar components $Z_\mu(\theta_w)$ and $A_\mu(\theta_w)$, the neutral vector boson and the photon. The mixing angle $\theta = \theta_w$ is no longer a coupling parameter but is rather a free system parameter.

Another thing to point out is that in most applications $T^* = T$; the only time $T^* \neq T$ is when the interaction terms involve the space-time derivatives of fields [9].

Now in the equations of motion one has to use new Lagrangian density $\mathbf{L}(x; r)$, which numerically is the same as the old one except that it contains only one bona fide coupling constant $r$. The perturbative solutions for fields and the $S$ matrix would utilize only $r$ as an expansion parameters.

5 Discussion and conclusion

Clearly, among other things, what one has shown is that the simplest electroweak model, the one based on the spontaneously broken $SU(2) \times U(1)$ gauge group, is indeed a truly unified theory of electromagnetic and weak interactions. This was possible simply because, in general, one defined the boson vector in the $n$ dimensional coupling constant space with the neutral vector bosons as the components. Of course, it is impossible to use charged and neutral vector bosons together as components of a boson vector in the $n$ dimensional coupling constant space, since this would lead to problems with charge conservation.
An important question is how to reformulate quantum chromodynamics so that when it is added to the electroweak model, that is based on the spontaneously broken $SU(2) \times U(1)$ gauge group (which may or may not be already truly unified), it yields truly unified strong and electroweak interactions.

References


Figure caption

Figure 1: The path $C$ is shown that was used to derive special cases of the stronger version of the integrability freedom, relations (2.22) and (2.25). The broken line can be used as $g_1$ and $g_2$ axes when demonstrating the integrability freedom for electroweak interactions, with $\theta_{\alpha} \equiv \theta$. 
Figure 1