1. Introduction

The use of the Vlasov equations is more familiar in the field of plasma physics than in the theory of particle accelerators. There are, however, problems in accelerators for which this theoretical technique can be useful; especially in the field of space-charge phenomena, where the effects of mutual interaction between the particles are the subject under investigation. The first accelerator problem to be treated this way seems to be the "negative mass" instability 1): two recent examples are the papers on resistive-wall instabilities 2)3).

If one takes the Boltzmann equation, neglects entirely the collision integrals, and write down the Maxwell equations for the electric and magnetic fields, one has a set of equations which is known either as the Collisionless Boltzmann Equation or the Vlasov Equations. 4) Since the Boltzmann equation is encountered mainly in the kinetic theory of gases, where it seems to be introduced only for the purpose of having the collision integrals inserted and (at least approximately) evaluated 5), it seems appropriate to enlarge a little on this definition of the Vlasov equations.

2. The Boltzmann Equation

We take as variables three cartesian coordinates \( x, y, z \); and the three associated velocities \( u, v, w \); and the time \( t \). We suppose there is only one type of particle in the system, and we work in terms of its density function in phase-space:

\[
f(x, y, z, u, v, w, t)
\]

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4) This sentence I have quoted almost verbatim from: A. Simon, Collisionless Boltzmann Equation, International Summer course in Plasma Physics 1958, Risø Report No.18, p.63.

5) See for example: Thermodynamique, Y. Rocard, 1952 page 234, or reference 6).
which is the number of particles per unit six-volume at the point \( x, \ldots, x \) at time \( t \).

We do not regard \( f \) as consisting of a six-dimensional delta-function at the location of each particle, but rather as a continuous function of its arguments. This is justified because we are only interested in volumes (or perhaps in averages over intervals of time) which are large enough for very many particles to be involved. \(^1\) Thus we have cut ourselves off from any consideration of

a) fluctuation phenomena - Brownian motion, Schottky noise, etc.
b) the details of individual collisions; for \( f \) does not indicate whether or not there is a particle present at some given position and time.

The Boltzmann equation is the partial differential equation satisfied by \( f \). It is obtained by considering the particles flowing in and out of the six-dimensional "volume" \( dx \, dy \, dz \, du \, dv \, dw \) by way of its six pairs of faces; as, for example, in the reference 5) or 6), and is:

\[
\frac{df}{dt} = - \left( \dot{x} \frac{df}{dx} + \dot{y} \frac{df}{dy} + \dot{z} \frac{df}{dz} \right) - \left( \ddot{u} \frac{df}{du} + \ddot{v} \frac{df}{dv} + \ddot{w} \frac{df}{dw} \right) + \text{Collision integral.}
\]

We use the dots to indicate time differentiation along the particle track, so that \( \dot{x} \) is \( u \), etc. by definition and \( \ddot{u} \), etc. are the components of the particle acceleration. For compactness a 3-vector notation is convenient:

\[
\frac{df}{dt} = - \mathbf{u} \cdot \frac{\partial f}{\partial x} - \dot{\mathbf{u}} \cdot \frac{\partial f}{\partial u} + \text{C.I.}
\]

\(^1\) To deal with this problem of defining a continuous density function \( f \) (for a system that consists of a finite number of discrete particles) in a more rigorous way, one may introduce the concept of probability; essentially by considering a large number (ensemble) of copies of the system under consideration, and calculating with averages over this ensemble.

The term called "Collision integral" covers the effect of short-range forces, by which collisions scatter particles from one part of the velocity distribution to another. We limit ourselves to cases where this phenomenon is either negligible or can be dealt with 2) quite separately and independently of the coherent-collective phenomena that interest us. In what follows we therefore neglect the "Collision integral".

All other forces acting on the particles are supposed to appear in \( (1) \) in the terms containing the acceleration \( \ddot{u} \). If the only significant fields of force present are electromagnetic one may substitute

\[
\ddot{u} = \frac{e}{m} \left( E_x + v E_y - \omega B_y \right)
\]

\( \dot{v} = \text{etc.} \)

To emphasize the fact that \( \ddot{u} \) in \( (1) \) is given in this way by the fields present, we shall write \( \mathcal{X} \) or \( X, Y, Z \), in future for \( \ddot{u} \) etc.

These electric and magnetic fields include both any "externally applied" fields and the fields associated with the particle distribution itself. In this way the long-range particle interactions, Coulomb force and Lorentz force, are included. This self-field is calculated from the charge density:

\[
p = e \iiint f \, d^3u
\]

and the current density:

\[
\mathcal{J} = e \iiint u \, d^3u
\]

together with Maxwell's equations and the appropriate boundary conditions.

Note that \( (2) \) is a non-relativistic formula (See appendix I).

2) The Touschek effect is an example.
The second set of terms on the right-hand side of (1) needs a further remark. The net inflow into our elementary 6-volume through the two faces perpendicular to \( u \) is evidently
\[
\begin{align*}
\left[ \left( \frac{\partial}{\partial u} \right) \left( u \frac{\partial f}{\partial u} \right) \right] & dx dy dz dv dw \\
= & \frac{\partial}{\partial u} \left( u \frac{\partial f}{\partial u} \right) dx dy dz dv dw \\
= & \left( -\frac{2u}{du} f - \frac{u^2 f}{du} \right) dx dy dz dv dw \\
\end{align*}
\]
(5)

Of the two terms in this bracket the second is to be found in (1) while the first seems to be lacking. This is because \( \frac{\partial^2 u}{\partial u} \) is assumed to be zero. Thus the validity of (1) is limited to systems that do not have velocity-dependent forces of the viscous drag or radiation-reaction types. Magnetic forces, being perpendicular to the velocity on which they depend, are admissible, and probably centrifugal and Coriolis forces are admissible too.

Summarizing, the Vlasov equations are
\[
\frac{\partial f}{\partial t} = -u \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial u} 
\]
(6)

where \( \mathcal{J} \) stands for the acceleration given by (2), with the fields to be found from Maxwell's equations. Besides any externally applied fields, the field of the particle distribution itself is included by including the charge density (3) and the current density (4) in Maxwell's equation.
Evidently these self-fields are linear in \( f \), so the last term in (6) makes it a non-linear equation.

3. Methods of Solution

If one regards (6) as an equation to be solved for \( f \), with the fields treated as given, and one regards (3), (4) with Maxwell's equations as a system of equations to be solved for the fields produced by an \( f \) which is treated as given, then the problem is to find a function \( f \) which is consistent (i.e. satisfies equ. (6)) with the field that it produces.

Some solutions of simple cases can be found by assuming an \( f \) of a certain form but specified by one or more undetermined constants. This \( f \) and the fields deduced from it are substituted in (6) and, with luck, one finds that the equation can be satisfied for a certain set or sets of values of these constants.

It is mainly stationary solutions \((\frac{\partial f}{\partial t} = 0)\) that are found by this method.

As soon as an appropriate stationary (or other basic) solution is known it becomes interesting to look at the behaviour of states that differ from it only a little, using a first-order small-amplitude theory. The most important question to be answered in this way is the question whether the basic solution is stable or not.

The problem taken by Vlasov in his original paper\(^7\) was that of the small oscillations of an otherwise neutral plasma. In this case there is no problem
of finding a suitable stationary solution, and his name is associated with the first-order small-amplitude analysis.  

3.1. Small-amplitude theory.

We suppose that $f_0$ is a known solution $\text{iii}$), which we call the unperturbed solution. It follows that

$$\frac{\partial f_0}{\partial t} = -\omega \cdot \frac{\partial f_0}{\partial x} - \omega_0 \cdot \frac{\partial f_0}{\partial u}$$  \hspace{1cm} (7)$$

where $\omega_0$ stands for the acceleration (2), with the electric and magnetic fields consisting of the externally applied fields and those due to the particle density distribution $f_0$.

We now look for a solution in which $f$ is slightly different from $f_0$, say

$$f = f_0 + f_1$$  \hspace{1cm} (8)$$

where $f_1$ is supposed small. We take the case where the externally applied fields are not changed, so the situation that we are studying is one in which the system, capable of existing in the state $f_0$, is left to itself after being started from some perturbed initial conditions.

7) Available in translation called AEC-t 2729; the original paper in Russian is in Zhev. Eksp. i Teoret Fiz. 8 291 1938.

\text{iii}) It is generally accepted that his solution was incorrect 4)8) but this does not diminish the usefulness his method of setting up the first-order problem.

\text{iii}) Commonly, but not necessarily, a stationary one.

Since (3) and (4) and Maxwell's equations are linear, the new accelerations will be

$$\ddot{x} = \ddot{x}_0 + \ddot{x}_1$$  \hspace{1cm} (9)

where the change $\ddot{x}_1$ is to be obtained by putting $f_1$ into (3) and (4) and solving Maxwell's equations.

We substitute (8) and (9) into the Vlasov equation (6), cancel out the zero-order terms that are already in the equation in the unperturbed situation (7) and reject the second-order term

$$\ddot{x}_1 \frac{\partial f_1}{\partial \nu}$$

on the grounds that we can make it as small as we like compared with the others by taking $f_1$ sufficiently small. This gives us an equation whose terms are all first-order:

$$\frac{\partial f_1}{\partial t} = -\nu \frac{\partial f_1}{\partial x} - \dot{x}_0 \frac{\partial f_1}{\partial \nu} - \ddot{x}_1 \frac{\partial f_0}{\partial \nu}$$  \hspace{1cm} (10)

This being a homogeneous linear system, its solutions can be superposed in linear combinations, with coefficients that are arbitrary or are determined by some specified initial conditions. It is therefore usual to attack the problem by expressing the spatial dependence in some sort of Fourier expansion appropriate to the geometry and boundary conditions of the problem, and then consider the time-dependence of the modes by determining eigenfrequencies (in general complex) or by the Laplace transform technique. Reasonably simple examples are studied in 4) and 1), and we give one in Appendix II.
In the reference \textsuperscript{3)} there is given a special method which is sometimes useful for solving (10) when \( f_0 \) is given and we assume \( \chi_1 \) known \textsuperscript{a}). We rewrite (10) in the form

\[
\left( \frac{\partial}{\partial t} + \tilde{z} \frac{\partial}{\partial \tilde{x}} + \tilde{u} \frac{\partial}{\partial \tilde{u}} \right) f_1 = -\chi_1 \frac{\partial f_0}{\partial \tilde{u}} \tag{11}
\]

The left-hand side can be recognised as the total time derivative of \( f_1 \) taken along a path that corresponds to an actual or a possible particle trajectory in the unperturbed system, so we may write (11) as

\[
\frac{d}{dt} \bigg|_{\text{unp.}} f_1 = -\chi_1 \frac{\partial f_0}{\partial \tilde{u}} \tag{12}
\]

Since these trajectories are in practice known, this can be integrated to give:

\[
f_1 = -\int_{\text{unp.}} \chi_1 \frac{\partial f_0}{\partial \tilde{u}} \cdot dt \tag{15}
\]

where the integration is to be performed along these unperturbed trajectories.

As written, the right-hand side of (13) is an indefinite integral, and produces the usual arbitrary constant of integration (which can be a different constant for each different trajectory, so being effectively the arbitrary function which one expects to find in the general solution of a partial differential equation), and in principle this ambiguity has to be resolved by appeal to the initial conditions on \( f_1 \).

This point, the remark at the end of Appendix II, and the difference between the Vlasov and the Landau treatments, are closely connected questions and may perhaps be taken up in a later note.

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\textsuperscript{a}) At this stage of the calculation the form of \( \chi_1 \) is known but it is undetermined in respect to its complex frequency.
Appendix I

Liouville.

Working in a Cartesian coordinate system and non-relativistically, the velocities are the same as the momenta apart from a constant factor.

We may note that the collisionless Boltzmann equation, that is (2) or (1) with C.i. put equal to zero, or (6), can be written:

\[
\frac{df}{dt} = 0
\]  

(16)

where the symbol of total differentiation is used to indicate the rate of variation along the path of a particle. Thus every particle carries with it its local value of the phase-space-density \( f \), unchanged, to wherever it goes. This of course is Liouville's theorem.

Two points of view are possible:

(1) In working out the partial differential equation satisfied by \( f \), for a system containing only conservative forces (page 4) Liouville's theorem has been rediscovered by accident.

(2) We could have saved ourselves some work by knowing Liouville's theorem, (14), and then deducing (1) or (6) from it.

A valuable conclusion that one may draw is that if we want to complicate the problem, say by using relativistic equations of motion, or a polar coordinate system, we can keep the complications to a minimum by working with some canonical system of coordinates and momenta, so that Liouville's theorem is valid and the form

\[
\frac{df}{dt} = -\sum q \frac{\partial f}{\partial q} - \sum p \frac{\partial f}{\partial p}
\]  

(15)

of the collisionless Boltzmann equation inevitable.
There may be cases where we wish to introduce a non-conservative force, for example to take account of radiation damping in an electron accelerator, or to allow in a rough way for the small influence of collisions in a plasma, and then a certain amount of care is required. It is not sufficient to include such a force in $\mathbf{z}$ in (6) : one must go back to (5) and so find that in (6) both

$$-\mathbf{z} \frac{2f}{\partial u} \quad \text{and} \quad -f \left( \frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} + \frac{\partial z}{\partial w} \right)$$

have to be included, in respect of this additional non-conservative force.
Appendix II

An example.

We take as example the longitudinal spacecharge phenomenon for a coating beam, sometimes known as the negative mass instability; this was first examined by Nielsen + Sessler. For our purpose of illustrating the Vlasov-equation method we disregard many important aspects of the physics of the phenomenon, and just look at everything in the simplest possible way: anyone interested in the negative mass instability as such should study the references, especially 9) and 10).

The dynamical variables for a particle are its azimuthal position in the accelerator, $\phi$, and the canonically-conjugate momentum $W$. We define $W$ by

$$W = \frac{U - U_s}{N_s}$$  \hspace{1cm} (17)

where $U_s$ is the energy of a reference particle whose angular velocity is $N_s$. For the purposes of this discussion $W$ can be regarded as the angular momentum, minus that of the reference particle.

The equations of motion are

$$\dot{\phi} = N_s + kW$$  \hspace{1cm} (15)

where $k$ is a constant, positive if we are below transition energy, negative if above, and

$$\dot{W} = eR E(\phi,t)$$  \hspace{1cm} (19)

where $E(\phi,t)$ is the azimuthal electric field.


We now "close the loop" by equating this to the $E$ that we assumed, (24), and so obtain the dispersion relation:

$$1 = -\frac{e^2 n \xi}{R} \int_{-\infty}^{+\infty} \frac{df_0}{dW} \frac{dw}{(\omega - n \Omega_e - nk\omega)} \quad (28)$$

The variable $W$ is a real quantity, and this integral arises just from the summation over all particles that occurs in any field calculation; so the appropriate path of integration is the real axis. One must, however, allow for the possibility of $\omega$ being a complex quantity.

To go further one must make a choice of unperturbed energy-spectrum. We take

$$f_0 = \frac{N}{2\pi w} \quad \text{in} \quad -W < W < +W \quad (29)$$

$$= 0 \quad \text{elsewhere}.$$

This gives a uniform band of particles in the $\Theta, W$ space with $N$ particles total. With this spectrum $df_0/dW$ can be expressed in terms of delta-functions:

$$\frac{df_0}{dW} = \frac{N}{2\pi w} \left\{ \delta(w + W) - \delta(W - W) \right\}$$

Substituting this in (28), the dispersion relation for this spectrum is

$$1 = -\frac{e^2 n \xi}{2\pi \omega} \frac{1}{R} \left\{ \frac{1}{\omega - n \Omega_e + nk\omega} - \frac{1}{\omega - n \Omega_e - nk\omega} \right\} \quad (30)$$

which can also be arranged

$$(\omega - n \Omega_e)^2 = (nk\omega)^2 + \frac{e^2 n^2 \xi N k}{2\pi R} \quad (31)$$
If the right-hand side is positive \( \omega \) has two real roots, but with \( k \) negative (above transition energy) and the space-charge term sufficiently large the roots are complex and one of them corresponds to an antidamped solution. This is the negative-mass instability.

It is perhaps worth pointing out that the choice of an \( f_c \) which is constant in a certain occupied region of the phase-space and zero elsewhere results in an \( f_1 \) which differs from zero only along the boundaries of this occupied region. This makes it interesting to compare the above with the analysis used in the first part (Sections III, IV, V) of reference 10, where a uniformly-occupied region is taken but is studied in terms of the equations of motion at its boundaries.

Another point worth a remark is the fact that, when \( \omega \) is real, the denominator in (27) passess through zero within the domain of integration. In principle, therefore, no meaning can then be attached to this integral.