Abstract: We link, by means of a semiclassical approach, the fractional statistics of particles obeying the Haldane exclusion principle to the Tsallis statistics and derive a generalized quantum entropy and its associated statistics.

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The generalized non-extensive statistics and the fractional exclusion statistics have excited great interest because of the deep insights on the classical and quantum behavior of many different physical systems and because of the wide range of their applications. The notion of generalized entropy, based on multifractal concepts, has been introduced by Tsallis [1],

$$S_q = \sum_{i=1}^{W} S_q^i \quad S_q^i = \frac{k_B}{q-1} n_i (1 - n_i^{q-1}) \ , \quad (1)$$

where $S_q^i$ is the entropy per state, $k_B$ is the Boltzmann's constant, $W$ is the total number of possible microscopic configurations and $\{n_i\}$ are the associated probabilities. The limit $q \to 1$ of $S_q$ yields the well-known Shannon entropy $S_1 = k_B \sum_i n_i \log n_i$. For general values of $q$, $S_q$ satisfies the usual properties of non-negativity, equiprobability, irreversibility and concavity. One of the most important differences between $S_q$ and the Shannon entropy is that $S_q$, for $q \neq 1$, is non-extensive. Such a property plays a relevant rôle in many physical systems where long-range interactions are present. Furthermore, it has been observed [2] that non-extensive behavior is strictly connected to quantum groups ($q$-deformation algebra, $q$-oscillators). The generalized Tsallis statistics (TS) has recently received great attention; many authors have analyzed the thermodynamic properties of TS [3–5] and the Tsallis equilibrium distribution has been used to study gravitational systems [6], solar neutrino problem [7] anomalous diffusion [8,9], optimization algorithms, statistical inference and probability theory [10]. Stariolo [11] has shown that the TS can be derived as the asymptotic equilibrium distribution of a Langevin and a Fokker-Planck equation containing a properly defined generalized potential. Furthermore, considering a gas of non-interacting quantum particles in equilibrium condition, the Fermi-Dirac (FD) and the Bose-Einstein (BE) distributions have been generalized à la Tsallis [12].

In his formulation of exclusion statistics, Haldane defined a generalized Pauli exclusion principle (HE) introducing the dimension $d_\alpha$ of the Hilbert space for single particle states as a finite and extensive quantity that depends on the number $N$ of particles contained in the system [13]. The exclusion principle implies that the number of available single particle states decreases as the occupation number increases

$$\Delta d_\alpha = -g \Delta N \ , \quad (2)$$

where $g$ is the parameter that characterizes the complete or partial action of the exclusion principle and makes possible the interpolation between the BE ($g = 0$) and FD ($g = 1$) statistics. Following Haldane's formulation, several papers [14–18] have been devoted to study the equilibrium distribution function associated to this approach and the thermodynamic properties of many different physical systems. In particular, many authors have suggested the intrinsic connection between the fractional statistics and the interpretation of the fractional quantum Hall effect and anyonic physics within the Calogero-Sutherland model and the Luttinger model [19–21].

Very recently, Rajagopal [22] has considered the Tsallis's entropy by introducing the quantum degeneracy Haldane factor that defines the generalized exclusion principle and has deduced the resulting quantum distribution function when $q$ is very close to one.

In this paper we show that the extension of the TS to the quantum FD, BE and HE statistics can be obtained in a remarkably simple formalism by means of a kinetic approach, recently proposed by us [23,24]. This semiclassical formalism allows us to deduce the quantum distribution in a very natural mode without approximations.

It is well known that, in the decorrelation approximation, valid in the thermodynamic limit, the level mean occupation $n_i(t) = n(t, E_i)$ obeys the following master equation

$$\frac{dn_i(t)}{dt} = \pi(t, E_{i+1} \to E_i) + \pi(t, E_{i-1} \to E_i)$$

$$- \pi(t, E_i \to E_{i+1}) - \pi(t, E_i \to E_{i-1}) \ , \quad (3)$$

where $\pi(t, E_i \to E_{i+1})$ is the transition probability from the state $E_i$ to the state $E_{i+1}$. The Eq.(3) describes, in the nearest neighbor interaction approximation, the
change of the population of the level $E_i$ due to the transitions to and from the levels $E_{i\pm 1}$. The particle kinetics is completely defined by means of the transition probability which we postulate [23] given by

$$\pi(t,E_i \to E_{i+1}) = \tau(t,E_i,\Delta E) \varphi(n_i) \psi(n_{i+1}), \quad (4)$$

where $\Delta E = E_{i+1} - E_i$, $\tau(t,E_i,\Delta E)$ is the transition rate from the state $E_i$ to the state $E_{i+1}$, $\varphi(n_i)$ is a function depending on the occupational distribution at the initial state $E_i$ and $\psi(n_{i+1})$ depends on the arrival state $E_{i+1}$. The functions $\varphi(n_i)$ and $\psi(n_i)$ must satisfy the following conditions: $\varphi(0) = 0$ because the transition probability is equal to zero if the initial state is empty, $\psi(0) = 1$ because the transition probability is not modified if the arrival state is empty.

The definition of the functions $\varphi(n_i)$ and $\psi(n_i)$ is a crucial point of this formalism since these two functions can inhibit or enhance the transition probability from a site to another one. In this sense Eq.(4) defines a generalized exclusion-inclusion principle which governs the particle kinetics. Let us say that the function $\varphi(n_i)$ is proportional to the probability of finding in the state $E_i$ the occupation number $n_i$, and $\psi(n_i)$ is proportional to the probability of introducing an extra particle into a state with occupational number $n_i$. Therefore, these functions can be interpreted as the semiclassical analogues of the quantum creation and annihilation operators matrix elements in second quantization; this analogy can be shaped in the following expressions

$$\varphi(n_i) \propto |< n_i - 1 | \hat{a}_{n_i} | n_i > |^2, \quad (5)$$

$$\psi(n_i) \propto |< n_i + 1 | \hat{a}_{n_i}^\dagger | n_i > |^2 . \quad (6)$$

Defining the particle current $j_i(t) = j(t, E_i)$ as

$$j_i(t) = [\pi(t, E_i \to E_{i+1}) - \pi(t, E_{i+1} \to E_i)] \Delta E , \quad (7)$$

the master equation (3) can be written as a continuity equation for the distribution function $n_i(t)$ [23]

$$\frac{dn_i(t)}{dt} + \frac{j_i(t) - j_{i-1}(t)}{\Delta E} = 0 . \quad (8)$$

In stationary conditions, in the limit $t \to \infty$, the particle current vanishes and posing $\varphi_i = \varphi(n_i)$, $\psi_i = \psi(n_i)$, $\tau_i = \tau(t \to \infty, E_i | \Delta E]$) we obtain the following balance equation

$$\tau_i \varphi_i \psi_{i+1} = \tau_{i+1} \varphi_{i+1} \psi_i . \quad (9)$$

In this case, if we consider Brownian particles, the transition rate can be written as [23]

$$n_{i} = ce^{\beta E_i} , \quad (10)$$

where $\beta = 1/(k_B T)$, $E_i = 1/2 m \nu_i^2$ is the single particle kinetic energy and $c$ is a constant.

Then, if we take into account Eq.(10), we obtain from Eq.(9)

$$\frac{\psi_i}{\varphi_i} = e^{\epsilon_i} , \quad (11)$$

where $\epsilon_i = \beta (E_i - \mu)$ is the dimensionless single particle energy defined up to an additive constant $\mu$, the chemical potential, that can be determined fixing the particle number of the system.

The above equation has a very simple and general form that defines intrinsically the statistical distribution and links two different quantities. In the r.h.s. the single particle energy appears and defines the particle interaction in the mean field approximation: in fact, the single particle energy $\epsilon_i$ can be determined from the drift and the diffusion coefficients of the system [23]. In l.h.s., the $\varphi_i$ and $\psi_i$ functions contain the classical or quantum behavior of the system.

The Eq(11) defines a family of statistics for several choices of the function $\varphi_i$ and $\psi_i$.

In the classical case, the transition probability does not depend on the occupational distribution of the arrival site, hence $\varphi_i = n_i$, $\psi_i = 1$ and we obtain the classical Maxwell-Boltzmann (MB) distribution

$$n_i = e^{-\epsilon_i} . \quad (12)$$

For $\varphi_i = n_i$ and $\psi_i = 1 + \kappa n_i$, we obtain the quantum fractional distribution [23]

$$n_i = \frac{1}{\exp \epsilon_i - \kappa} , \quad (13)$$

where $\kappa \in [-1,1]$ is a parameter connected with the exchange statistical parameter $\alpha$ appearing in the quantum phase $e^{i\pi \alpha}$ [25]. For $\kappa = -1,0$ and 1 one obtains the FD, MB and BE distribution, respectively.

In this case the function $\psi_i$ depends on the particle distribution of the arrival site. If $\kappa > 0$ (boson-like particle) the transition probability is enhanced, if $\kappa < 0$ (fermion-like particle) the transition probability is inhibited. Hence, the parameter $\kappa$ may be interpreted as the degree of indistinguishability or the degree of classicality of the particles under consideration.

As we have previously outlined, the HE statistics contains a generalized exclusion Pauli principle that influences the transition probability from a site to another. The function $\psi_i$ reflects these quantum properties. According to Karabali and Nair paper [26], where the algebra of creation and annihilation operator of particles obeying the Haldane exclusion statistics has been considered, it appears consistent to postulate, in consequence of the semiclassical interpretation of the function $\psi_i$ given by Eqs.(5) and (6), the following relation

$$\psi_i = [1 - gn_i]^g[1 + (1 - g)n_i]^{1-g} . \quad (14)$$
Inserting the above expression in Eq.(11) with \( \varphi_i = n_i \), we obtain the HE distribution for the fractional exclusion statistics [13-15]

\[
[1 - g n_i]^g [1 + (1 - g) n_i]^{1-g} = n_i e^{\xi_i}.
\]

(15)

The Haldane’s parameter \( g \) defines the generalized exclusion Pauli principle [13] and for the particular cases \( g = 0,1 \) one recovers the BE and FD distributions, respectively.

We have seen that the MB, FD, BE and HE statistics can be obtained from a semiclassical kinetic equation in the case of Brownian particles. Now we want to link the above statistical distributions to the Tsallis distribution [1]

\[
n_i = \{q[1 - (1 - q)e_i]\}^{1/(1-q)}.
\]

(16)

We wish to observe that the Druyvenstein distribution [27] can be seen as the limit \( q \approx 1 \) (weak collective interaction) of the Tsallis distribution. In fact, expanding the r.h.s. of Eq.(16) at the first order of \((q - 1)\), we obtain:

\[
n_i \approx e^{-\xi_i - n_i^q},
\]

where we have defined \( s = (1-q)/2 \). This description may play a relevant rôle in many physical systems; in Ref. [7] we have applied this distribution to the solar core in order to solve the solar neutrino problem.

The TS describes the statistics of classical particles and can be obtained from Eq.(11) using \( \varphi_i = n_i \), \( \psi_i = 1 \) (classical transition probability) and posing, in place of \( \epsilon \), the generalized energy [1]

\[
\hat{\epsilon}_i = \frac{1}{q-1} \log \{q[1 - (1 - q)e_i]\},
\]

(17)

then Eq.(16) can be written as

\[
n_i = e^{-\hat{\epsilon}_i}.
\]

(18)

In the limit \( q \to 1 \) we recover the MB distribution.

The functional modification of the single particle energy \( \hat{\epsilon}_i \) may be interpreted as an effect of many-body effective interactions or of a collective interaction.

If we want to extend the classical TS to the quantum case, it is necessary to introduce the generalized energy \( \hat{\epsilon}_i \) of Eq.(17), in the Eq.(11)

\[
\frac{\psi_i}{\varphi_i} = e^{\hat{\epsilon}_i}.
\]

(19)

The functions \( \varphi_i \) and \( \psi_i \) suitable to the definition of the transition probability are strongly related to the quantum nature of the particles under consideration.

The TS of Eq.(16) has been obtained extremizing the generalized entropy \( S_q \) defined in Eq.(1), enforcing the constraints of fixed energy \( E = \sum_i E_i n_i^q \) and of particle number \( N = \sum_i n_i^q \)

\[
\delta S_q - \beta \delta E + \beta \mu \delta N = 0.
\]

(20)

The above variational problem produces the following equation for \( n_i \)

\[
\frac{\partial}{\partial n_i} (S_q^i - \beta E_i n_i^q + \beta \mu n_i^q) = 0.
\]

(21)

Comparing Eqs.(19) and (21), it is possible to obtain a generalized Tsallis’s entropy containing an exclusion-inclusion principle, defined by means of the functions \( \psi_i \) and \( \varphi_i \) as

\[
S_q^i = \frac{k_n}{q-1} \left[ \int n_i^q - 1 \left( \frac{\psi_i}{\varphi_i} \right)^q - n_i^q \right].
\]

(22)

In the classical case \( \varphi_i = n_i, \psi_i = 1 \) and Eq.(22) reduces to the Tsallis’s entropy of Eq.(1).

Let us describe the quantum generalization of the classical TS; we fix \( \varphi_i = n_i \), the exclusion-inclusion principle must be contained into the expression of the function \( \psi_i \). If we choose \( \psi_i = 1 + \kappa n_i \), we obtain the following statistics

\[
n_i = \frac{1}{\{q[1 - (1 - q)e_i]\}^{1/(q-1)} - \kappa},
\]

(23)

which contains the generalized BE \((\kappa = 1)\) and the FD \((\kappa = -1)\) quantum version of the TS derived in Ref. [12].

In this case the entropy can be obtained inserting the expression \( \psi_i = 1 + \kappa n_i \) into Eq.(22)

\[
S_{q,\kappa}^i = \frac{k_n}{q-1} \left[ \frac{(1 + \kappa n_i)^q}{\kappa q} - n_i^q \right] - \frac{1}{\kappa q(q-1)}.
\]

(24)

In the limit \( q \to 1 \) of the above equation, one obtains the well known entropy for particles obeying BE \((\kappa = 1)\) and FD \((\kappa = -1)\) statistics.

Analogously, we observe that, inserting the expression of \( \psi_i \) of Eq.(14), which describes the exclusion principle introduced by Haldane, in the entropy of Eq.(22), it is possible to write the generalized Haldane-Tsallis exclusion (HTE) entropy \( S_{q,g} = \sum_i S_{q,g}^i \) (which depends on the two parameters \( q \) and \( g \)) in terms of a hypergeometric function \( F \) [28] as

\[
S_{q,g}^i = \frac{k_n}{1-q} \left[ \frac{(1 - g n_i)^q}{\alpha g^{1-\beta}} F(\alpha,\beta;\alpha + 1; (1-g)(1-g n_i))
\right.
\]

\[
\left. + n_i^q - \frac{1}{\alpha g^{1-\beta}} F(\alpha,\beta;\alpha + 1; 1-g) \right],
\]

(25)

where \( \alpha = 1 + g(q-1), \beta = (g-1)(q-1) \).

We observe that, in the limit \( q \to 1 \) of the Eq.(25), one obtains the von Neumann entropy per state of the HE statistics [17].

3
The quantum TS for Brownian particles can be obtained by the HTE distribution statistics
\[\left[1 - gn_i\right]^q \left[1 + (1 - g)n_i\right]^{1-q} = n_i \{q[1 - (1 - q)c_i]\}^{1/(q-1)},\]

(26)

which, in the limit \(q \rightarrow 1\), produces the HE distribution given by Eq.(15).

The statistics defined by means of Eq.(26) describes a system of quantum particles with an intermediate behavior between BE and FD statistics and, at the same time, characterized by a non-extensive entropy.

In conclusion, we have shown that the classical and the quantum TS for Brownian particles can be obtained as stationary states of a linear or a non-linear (in the quantum case) master equation if an exclusion-inclusion principle is introduced in the expression of the transition probability. The classical or the quantum nature of the particles is taken into consideration by the functions \(\varphi_i\) and \(\psi_i\), this one enhances or inhibits the transition from a state to another. The use of this semiclassical approach allows us to acquire a quantum generalization of the TS and to derive a generalized entropy together its associated statistics, linking the quantum behavior of particles obeying the fractional exclusion principle introduced by Haldane to the non-extensive behavior of the TS. Situations of this kind can be present in different fields of physics from quantum cosmology to condensed matter. As observed in Ref. [2,30] the non-extensive behavior of the system could be explain (part of) the discrepancy at the origin of the dark matter. Furthermore, the quantum entropy of Eq.(22) and its associated distribution statistics, may play a relevant rôle in those systems with long range interaction where the quantum properties are not negligible as: neutron stars, black holes, astrophysical dense plasma, quark gluon plasma. We quote in addition, as possible candidates, stellar polytropes, plasma heated by inverse bremsstrahlung, non thermal components of ions in fusion reactors and laboratory plasma, gases of particles in condensed matter correlated by many-body forces or long-range forces. Of course, actually, studies on the thermodynamic properties of systems of particles obeying the HTE statistics have not yet been accomplished, nevertheless we think that this will be a field of advanced research in the near future.