THE DEPENDENCE OF THE VISCOSITY IN ACCRETION DISCS ON

THE SHEAR/VORTICITY RATIO

Marek Abramowicz$^{1,2}$    Axel Brandenburg$^2*$    Jean-Pierre Lasota$^{3,4}$


2. Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.

3. Belkin Visiting Professor, Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel.

4. UPR 176 du CNRS; DARC, Observatoire de Paris, Section de Meudon, 92195 Meudon, France.

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*After Feb. 1996: Dept. of Mathematics and Statistics, Univ. of Newcastle upon Tyne, NE1 7RU, UK.

NORDITA - Nordisk Institutfor Teoretisk Fysik

Blegdamsvej 17  DK-2100 København Ø  Danmark
The dependence of the viscosity in accretion discs on
the shear/vorticity ratio

Marek Abramowicz1,2, Axel Brandenburg2* and Jean-Pierre Lasota3,4
1 Department of Astronomy and Astrophysics, Chalmers University of Technology, 41296 Gothenburg, Sweden
2 Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
3 Helin Visiting Professor, Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76105, Israel
4 UPR 178 du CNRS; DARC, Observatoire de Pontoise, Section de Meudon, 92194 Meudon, France

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ABSTRACT
We estimate the Shakura-Sunyaev viscosity parameter alpha for different values of the shear/vorticity ratio, \( \sigma/\omega \), using local simulations of dynamo-generated turbulence. We find that the time average of \( \alpha \) is approximately proportional to \( \sigma/\omega \) (at least for \( \sigma/\omega < 10 \)). We point out that this result may have important implications for the properties of thick accretion discs, because there \( \omega \) is small and \( \alpha \) would then tend to be large. Our result may also be important for accretion discs around black holes, because \( \sigma/\omega \) becomes large in the inner ten Schwarzschild radii due to relativistic effects.

Key words: accretion discs: viscosity – black holes

1 INTRODUCTION
There is now strong evidence that the turbulent viscosity in accretion discs is caused by a magnetic instability (Balbus & Hawley 1991). Three-dimensional local turbulence models are now available that have enabled us for the first time to “measure” the normalized disc viscosity \( \alpha \), which is the turbulent viscosity divided by the sound speed and the scale height. Previously, \( \alpha \) was often taken as a free parameter in global models. However, we are now in a position to determine the dependence of \( \alpha \) on local properties of the disc.

It is plausible that the strengths of large-scale shear and vorticity play important roles in determining the magnitude of \( \alpha \). In accretion discs the angular velocity \( \Omega \) varies with radius like \( \Omega \sim r^{-\gamma} \), where \( \gamma = 3/2 \) for Keplerian rotation. The growth rate of the Balbus-Hawley instability is proportional to \( q \) (Balbus & Hawley 1992). Furthermore, for \( q \to 2 \) the disc approaches a state of constant angular momentum and becomes unstable by Rayleigh’s criterion. Therefore we expect \( \alpha \) to increase with \( q \). Indeed, there are several circumstances where \( q > 3/2 \), especially in the inner parts of accretion discs, as we shall see in §3.

However, \( q \) is a coordinate dependent quantity and therefore not useful in more general circumstances, for example near black holes, where the coordinate \( r \) has no dynamical significance. Therefore, we shall express \( \alpha \) in terms of shear and vorticity of the background rotation.

Since the magnetic shear instability is local, it is possible to estimate the viscosity using a local model either with an externally applied magnetic field (Hawley et al. 1995, Matsumoto & Tajima 1995) or without (Brandenburg et al. 1995). Here we consider only the latter case, where a magnetic field is generated self-consistently by dynamo action. This removes the ambiguity due to the otherwise ill-defined magnetic field strength. This is the relevant case for discs where the central object has no, or only a weak, magnetic field, and where the field is generated entirely within the disc on the scales resolved in the simulation.

In a local model all radial gradients are ignored except for the angular velocity. This is an important approximation that becomes invalid near the central object. Effects due to the converging accretion flow, for example, are ignored. When we talk about effects due to deviations from a Keplerian flow near the central object, we have to keep in mind that we only refer to effects that show up locally. We assume therefore that they are the dominant ones.

2 SHEAR AND VORTICITY IN A SHEARING BOX
In the shearing box approximation the total velocity is decomposed into three components: rigid rotation, linear shear, and a turbulent component. Near a reference point \((R,\phi_0)\) (in cylindrical polar coordinates) the background velocity (i.e. without the turbulent component) is

\[
U = \Omega_0 \times r + u_{\text{shear}},
\]

where \( \Omega_0 = 2\Omega_0 \) with \( \Omega_0 = \Omega(R) \) is the angular velocity at the reference radius, and \( r = (R + x,y,0) \) is the cylindrical radius vector in local Cartesian coordinates, \( z = r - R \).
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\[ y = (\phi - \phi_s)R, \] and \( z \). This flow results from the balance between the radial component of gravity, the centrifugal force, the Coriolis force, and possibly an extra body force \( f = \rho \Omega^2 \). In the local approximation, \( z/R \ll 1 \), we have

\[ \frac{GM}{R^3} \left( 1 - \frac{2}{R} \right) + \Omega^2 R \left( 1 + \frac{z}{R} \right) + 2\dot{\theta} \Omega \sigma^{(0)} + pf^2 z = 0. \] (2)

This yields the value of the basic rotation \( \Omega_0 = GM/R^3 \) and the shear flow, \( \nu_{\text{shear}} = \nu_0 \sigma^{(0)}(z) \), with

\[ \sigma^{(0)}(z) = -q \Omega_0 z, \] (3)

where \( q = (3 + p)/2 \). For \( p = 0 \) we have ordinary Keplerian rotation, \( q = 3/2 \). This velocity field may be described by shear and vorticity tensors,

\[ \sigma_{ij} = \frac{1}{2}(\partial U_i/\partial x_j + \partial U_j/\partial x_i) - \frac{1}{2} \partial \phi/\partial x, \] (4)

\[ \omega_{ij} = \frac{1}{2}(\partial U_i/\partial x_j - \partial U_j/\partial x_i). \] (5)

In the following we characterize the flow by \( \sigma^2 \) and \( \omega^2 \), which are coordinate independent invariants of the flow. In our case their moduli are

\[ \sigma = |\partial U_i/\partial x_j + \partial U_j/\partial x_i|/\sqrt{2} = q \Omega_0/\sqrt{2}, \] (6)

\[ \omega = |\partial U_i/\partial x_j - \partial U_j/\partial x_i|/\sqrt{2} = (2 - q) \Omega_0/\sqrt{2}. \] (7)

Using this background velocity we computed models similar to those in Brandenburg et al. (1996), where certain curvature terms have been restored that allow for a finite mass accretion rate. In the present paper we use models for different values of \( q \) and compute the turbulent viscosity \( \nu_{turb} \), which describes the magnitude of the total horizontal stress relative to the large scale shear, i.e.,

\[ \nu_{turb} = -\nu_0 \sigma^{(0)}(z) \frac{\Omega}{\Omega_0}. \] (8)

We then express \( \nu_{turb} \) in units of \( \sigma_0 \) and \( H_0 \), i.e. \( \nu_{turb} = \alpha \sigma H_0 \) and use \( \sigma^{(0)}(z) = -q \Omega_0 \).

The value of \( \alpha \) is not constant, but depends on the magnetic field strength which, in turn, varies in a cyclic manner (Brandenburg et al. 1995). Therefore we give here the mean value obtained by averaging over approximately one cycle. This introduces uncertainty, because the cycle length varies somewhat in time and from case to case. In a few cases we have run for up to three magnetic cycles. We should also point out that \( \alpha \) increases somewhat with increasing numerical resolution (Brandenburg et al. 1996). In order to perform a survey in parameter space we restrict ourselves to modest resolution of \( 31 \times 63 \times 32 \) meshpoints.

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Table 1. Summary of the runs. In addition to the value of \( \alpha \) and its variance the normalized Maxwell stress, the inverse plasma beta, and the ratio of magnetic to turbulent kinetic energy are also given. For some values of \( \sigma/\omega \) there are two different results corresponding to runs with different initial conditions.

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The solutions for different values of \( q \) have been obtained by restarting the simulation from a previous snapshot that was obtained for a larger or smaller value of \( q \). It takes some time for the solution to settle, but after less than one orbit \( \alpha \) has adjusted roughly to the new value. For \( q \leq 1 \) it became increasingly difficult to find a unique value of \( \alpha \) because of rather long transients. In this case \( \alpha \) seems to depend also on the initial conditions. For example, in a case with \( q = 0.5 \) the magnetic field was modified such that \( B_y = 0 \) initially. In this case \( (B_y) \) began to evolve away from zero in an oscillatory manner, but \( \alpha \) was roughly similar to the values found before. However, at much later times (more than 100 orbits) \( \alpha \) became much smaller. We are not sure whether this field was very slowly decaying, or whether there is possibly a dynamo even in the limit of vanishing shear, driven possibly by the effects of buoyancy (Różycka et al. 1995). However, for small negative values of \( q \) \( (q = -0.1) \) we found that \( \alpha \) decayed very quickly to zero.

The results are summarized in Table 1. Note that \( \alpha \) increases monotonously with \( q \). The values of \( \alpha \) are well represented by a linear dependence on the ratio \( \sigma/\omega \),

\[ \alpha = \alpha_0 \frac{\sigma}{\omega}, \] (9)

see Fig. 1. The case of Keplerian rotation, \( q = 3/2 \), corresponds to \( \sigma/\omega = 3 \). From the simulations we find \( \alpha_0 \approx 0.0015 \). This simple linear dependence, obtained here from a fit to the results of fully three-dimensional simulations, is quite striking and one wonders whether this linear dependence could be understood in terms of more basic physics. The fact that \( \alpha \) increases with \( \sigma/\omega \) suggests that shear enhances the turbulence, or makes it least more
effective in transporting angular momentum. Using general arguments based on comparing relevant length scales, Godon (1995) also found that \( \alpha \) should depend on the shear.

Equation (9) would imply an infinitely large value of \( \alpha \) for discs of constant angular momentum \( (\omega = 0) \). It is not clear what the relevant mechanism for the saturation of the shear instability for \( q \rightarrow 2 \) is, but we expect that the linear dependence implied by Eq. (9) would break down as \( \sigma/\omega \rightarrow \infty \).

The actual values of \( \alpha \) are probably larger than those given here, because of a relatively low resolution used here. Nevertheless, the linear dependence \( \alpha \) on \( \sigma/\omega \) is remarkable and probably robust. In the following we discuss possible implications of this new result.

3 APPLICATIONS

The dimensionless viscosity coefficient \( \alpha \) is defined by the equation, \( \nu_t = \alpha c_s H \), where \( \nu_t \) is the turbulent kinematic viscosity coefficient, \( c_s \) is the local sound speed, and \( H \sim c_s/\Omega \) is the local pressure scale height. In this formula (together with Eq. (9)) all the quantities on the right hand side are defined locally by coordinate independent quantities. We suggest therefore that this formula may be applied in a much wider range of hydrodynamical calculations than it was possible with our local models.

In the following we discuss two different cases where \( \sigma/\omega \) can be larger than 3, the asymptotic value for Keplerian rotation. The first concerns thick accretion discs, where additional pressure support can make the rotation profile steeper. Another important case are the inner parts of thin accretion discs, where even for Keplerian rotation \( \sigma/\omega \gg 3 \).

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3.1 Thick accretion discs

Thick accretion discs have been proposed in early 1980s (see Abramowicz, Calvani & Nobili 1980) as possible models for quasars and other active galactic nuclei. One of the particularly characteristic features of the thick discs is the presence of long and narrow funnels along the axis of rotation. It is believed that these funnels may play a crucial role in the collimation of relativistic jets emerging from quasars.

Thick accretion discs have been mostly studied in the case where the angular momentum distribution is uniform and the vorticity zero. Such discs are only possible for very small values of \( \alpha \), because otherwise viscosity would lead to a rapid redistribution of angular momentum, resulting in a state where angular momentum increases outwards. This scenario is now in conflict with Eq. (9), because for \( \omega \rightarrow 0 \) our formula predicts large viscosity, in contrast to the original assumption. We therefore expect that a subsequent redistribution of angular momentum leads to values of \( \alpha \) small enough until the angular momentum distribution is no longer affected. The value to which \( \alpha \) will relax may determine the shape of the flow.

Using 2-D time-dependent hydrodynamical simulations Igumenshchev et al. (1996) have demonstrated, for two different values of \( \alpha (10^{-2} \quad \text{and} \quad 10^{-3}) \), that the funnels do not disappear when the angular momentum distribution changes during the viscous evolution. It is not known what happens if the value of \( \alpha \) is much larger, and strongly varies in the flow. It would be interesting to see whether this leads to some steady "canonical" angular momentum distribution, and whether the empty funnels along the axis of rotation survive the rapid viscous evolution close to the axis. Recent calculations by Chen et al. (1996) and Narayan et al. (1996), based on stationary models employing Newtonian formulse in the vertical direction and pseudo-Newtonian in the radial one, seem to indicate that when \( \alpha \) is constant and \( > 0.1 \) everywhere, the funnels do not survive (Narayan et al. 1996). We plan to study this question using 2-D, fully relativistic, time-dependent hydrodynamics using, however, our improved formula (9) for the disc viscosity.

3.2 Thin and slim accretion discs around black holes

For the Schwarzschild (non-rotating) black hole with the mass \( M \) and gravitational radius \( r_g = 2GM/c^2 \), the Keplerian angular velocity of rotation in terms of the Schwarzschild radial coordinate \( r \) is \( \Omega = GMr^{-3/2} \), and the Keplerian angular momentum is \( \ell = \Omega R^2 \).
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with \( R^2 = r^2/(1 - r_\text{G}/r) \) being the radius of gyration. From these formulae one deduces that the Keplerian angular momentum is not a monotonic function of radius, but it has a minimum at \( r = 3r_\text{G} \). For \( r < 3r_\text{G} \) no stable Keplerian orbits are possible. In thin and slim (Abramowicz et al. 1988) accretion discs around Schwarzschild black holes, matter loses its angular momentum due to the action of a viscous torque and spirals down towards the central black hole. In thin accretion discs, for \( r > 3r_\text{G} \) the inward radial component of velocity is much smaller than the azimuthal component, which is almost Keplerian. For \( r < 3r_\text{G} \), both components are comparable because the matter is radially free-falling. Thus, the general distribution of the angular momentum in the standard disc corresponds to almost Keplerian for \( r > 3r_\text{G} \), and almost constant for \( r < 3r_\text{G} \). Somewhere at \( 2r_\text{G} < r < 3r_\text{G} \) the character of the flow changes, and for this reason this place is often called the inner edge of the disc.

Relativistic expressions for the shear and vorticity for a general rotational motion in a static spacetime read (see Abramowicz 1982),

\[
\sigma^2 = \frac{1}{4}(1 - \Omega^2)^{-2}\nabla^2\Omega/
u\Omega, \\
\omega^2 = \frac{1}{4}(1 - \Omega^2)^{-2}\nabla^2\nu/
u, 
\]

(10)
(11)

From this we derive that in the case of the Keplerian rotation around the Schwarzschild black hole the shear/vorticity ratio equals

\[
\frac{\sigma}{\omega} = 3 \frac{r - r_\text{G}}{r - 2r_\text{G}}. 
\]

(12)

This is shown in Fig. 2 as a function of the radial coordinate \( r \) by the heavy line. For comparison, the shear/vorticity ratio for Keplerian rotation of the pseudo-Newtonian potential is shown by the dashed line. The pseudo-Newtonian potential was proposed by Paczyński & Wiita (1980) as the Newtonian model for a non-rotating black hole, and found to be an excellent approximation in most hydrodynamical applications (see e.g. Artemova, Björnsson & Novikov, 1996).

In theoretical studies of accretion discs it is customary to assume that \( \alpha \) is constant through the disc, or that it varies as a certain power of \( H/r \), where \( H \) is the vertical pressure scale height (or the vertical half-thickness of the disc). Our viscosity formula implies, however, that the value of \( \alpha = \alpha_{\text{crit}} \approx 0.005 \). At \( r = 10r_\text{G} \) for strictly Keplerian flow one has \( \sigma/\omega = 4.1 \) and \( \alpha = 0.007 \), and for \( r = 5r_\text{G} \), one has \( \sigma/\omega = 7.0 \), and \( \alpha = 0.012 \). Note that the disc radiates most of the energy from the region \( 10r_\text{G} > r > 3r_\text{G} \) and thus a knowledge of the precise value of \( \alpha \) there is of direct observational interest.

For slim accretion discs and optically thin advection dominated flows (see Abramowicz 1996; Narayan 1996 for recent reviews) the azimuthal motion of the matter is non-keplerian even for \( r > 10r_\text{G} \). For low values of \( \alpha (\ll 0.01) \) the specific angular momentum in the inner disc is practically constant and matter is accreted via 'relativistic Roche–loke overflow'. For higher values of \( \alpha \) the subkeplerian specific angular momentum distribution is steeper and matter undergoes viscous accretion. These results are obtained assuming a constant \( \alpha \). Like in the case of thick discs, discussed above, and for the same reasons, this behaviour is in contradiction with our results concerning the dependence of viscosity on the shear/vorticity ratio. One can expect that if this dependence is included into the model the resulting flow will be different from the one obtained assuming a constant \( \alpha \).

In addition, from theoretical studies it follows that the various qualitative properties of the flow depend on whether the value of \( \alpha \) is larger or smaller than the critical value. For several of these properties \( \alpha_{\text{crit}} \approx 0.1 \). For example, the sonic point for flows with \( \alpha < \alpha_{\text{crit}} \) has saddle topology and for those with \( \alpha > \alpha_{\text{crit}} \) nodal topology, which relates to the presence of the unstable standing acoustic mode (for references see Kato et al., 1996). Similarly, the possible existence of a range of radii with a hot, optically thin, thermally overstable accretion flow cooled by advection depends on whether \( \alpha < \alpha_{\text{crit}} \) (see Abramowicz et al. 1995, Chen
The dependence of the viscosity in accretion discs on the shear/vorticity ratio et al., 1995, 1996). Our result suggests a rather high value of $\alpha$, which most probably is greater than the critical one in both cases discussed above.

4 CONCLUSIONS

We have demonstrated, using numerical simulations, that the $\alpha$-viscosity due to dynamo-generated turbulence is proportional to the shear/vorticity ratio. On one hand, this provides a starting point for two fundamental theoretical questions: (1) why it is so and (2) how do nonlinear effects influence the formula in the limit of vanishing vorticity? On the other hand, our formula may be used in a variety of large-scale numerical hydrodynamical simulations, because it uses only invariant quantities that could easily be defined and calculated in any particular situation.

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