On Representations of $U_q\mathfrak{osp}(1|2)$ when q is a Root of Unity

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Abstract

The infinite dimensional highest weight representations of $U_q\mathfrak{osp}(1|2)$ for the deformation parameter q being a root of unity are investigated. As in the cases of q -deformed non-graded Lie algebras, we find that every irreducible representation is isomorphic to the tensor product of a highest weight representation of $\mathfrak{sl}_2(\mathbb{R})$ and a finite dimensional one of $U_q\mathfrak{osp}(1|2)$. The structure is investigated in detail.

1. After Jimbo [1] and Drinfeld [2], quantum groups or quantum deformations are widely investigated by a lot of theoretical physicists and mathematicians. The first and the simplest example was $U_q\mathfrak{su}(2)$, the q -deformation of the universal enveloping algebra of $\mathfrak{su}(2)$, and its method of deformation has been applied to other Lie algebras and graded Lie algebras (superalgebras) as well. In some of these works, the deformation parameters are taken to be real and some mathematical structures have been investigated. Further, some authors have constructed toy models possessing quantum group structure for real q . In spite of these published works in this direction, however, it seems to us that they are of purely mathematical interest and fail to give new features to the physical world.

In contrast, it has been shown that quantum groups with q a root of unity are related to some interesting physical models, especially two dimensional physics.[3] In these models, finite dimensional representations of $U_q\mathfrak{sl}_2(\mathbb{C})$ or $U_q\widehat{\mathfrak{sl}_2}$ play an important but implicit role. Recently one of the authors (T.S.)[4] has constructed infinite dimensional highest weight representations of $U_q\mathfrak{sl}_2(\mathbb{C})$, i.e., $U_q\mathfrak{su}(1,1)$ when

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q is a root of unity. A remarkable feature is that, on representation spaces, they have *necessarily* the structure

$$U_q \mathfrak{su}(1, 1) = U \mathfrak{sl}_2(\mathbb{R}) \otimes U_q \mathfrak{su}(2), \quad (1)$$

and, thanks to the decomposition, it is shown that $U_q \mathfrak{su}(1, 1)$ at the second root of unity is isomorphic to the superalgebra $\mathfrak{osp}(1|2)$. He further applied the result to quantum Liouville theory[5] and showed that the decomposition worked well in a geometrical sense.

With keeping these facts in mind, we will study highest weight representations of $U_q \mathfrak{osp}(1|2)$ in the case when q is a root of unity. This is another simple and worth-studying example of q -deformations of superalgebra. Historically, $U_q \mathfrak{osp}(1|2)$ was first obtained by Kulish [6] in relation with the graded Yang-Baxter equation[7]. Then, Saleur [8] used $U_q \mathfrak{osp}(1|2)$ algebra to build the vertex-model-solution of the graded Yang-Baxter equation and the associated interacting round a face model. He also found a hidden $U_q \mathfrak{osp}(1|2)$ symmetry in some models whose continuum limit belongs to the $N = 1$ superconformal series. In this letter, infinite dimensional representations of $U_q \mathfrak{osp}(1|2)$ are investigated with $q^p = 1$. It is interesting for us to find a similar decomposition formula (1) in the $U_q \mathfrak{osp}(1|2)$ case.

The organization of this paper is as follows; In section 2, a brief review of $U_q \mathfrak{osp}(1|2)$ algebra is exhibited and its highest weight representations with generic q are summarized as well. Irreducible highest weight representations at a root of unity are constructed in section 3. In section 4, we investigate the irreducible representations and find the tensor product structures for the $U_q \mathfrak{osp}(1|2)$ algebra.

Finally, it is convenient to list here some notations we will make use of. For a number $x \in \mathbb{R}$, the q -number $[x]$ is defined by $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$ as usual. In the graded case, another type of q -number $\llbracket x \rrbracket$ defined by

$$\llbracket x \rrbracket = \frac{q^x - (-)^{2x} q^{-x}}{q^{1/2} + q^{-1/2}}. \quad (2)$$

is needed. This should be called q -fermionic number hereafter. Notice that q -fermionic number $\llbracket x \rrbracket$ can be defined only for $x \in \mathbb{N}/2$. In the classical limit $q \rightarrow 1$, $\llbracket x \rrbracket$ goes to 1 when $2x$ is odd or 0 when $2x$ is even. Moreover, two types of q -deformed binomial coefficients are introduced,

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]![r]!}, \quad \left[\begin{matrix} n + \frac{1}{2} \\ r + \frac{1}{2} \end{matrix} \right] = \frac{\llbracket n + \frac{1}{2} \rrbracket!}{\llbracket n - r - \frac{1}{2} \rrbracket! \llbracket r + \frac{1}{2} \rrbracket!}, \quad (3)$$

where $\llbracket n + \frac{1}{2} \rrbracket! = \llbracket n + \frac{1}{2} \rrbracket \llbracket n - \frac{1}{2} \rrbracket \cdots \llbracket \frac{1}{2} \rrbracket$.

2. This section presents a brief summary of the q -deformed universal enveloping algebra $U_q \mathfrak{osp}(1|2)$ of the superalgebra $\mathfrak{osp}(1|2)$ together with the discussion of its highest weight representations with generic q .

Let us first define the q -deformed universal enveloping algebra $U_q\mathfrak{osp}(1|2)$. It is generated by $\mathcal{G}_+$, $\mathcal{G}_-$, H satisfying the following relations,

$$\{\mathcal{G}_+, \mathcal{G}_-\} = -[2H], \quad [H, \mathcal{G}_\pm] = \pm \frac{1}{2}\mathcal{G}_\pm, \quad (4)$$

where $\{\cdot, \cdot\}$ is the ordinary anti-commutator. The Hermitian conjugation properties of these generators are given as

$$\mathcal{G}_+^\dagger = -\mathcal{G}_-, \quad \mathcal{G}_-^\dagger = -\mathcal{G}_+, \quad H^\dagger = H. \quad (5)$$

Let us turn our attention to constructing representations of $U_q\mathfrak{osp}(1|2)$. We will restrict our concern only to the highest weight representations, since they are important for physical applications. There are two types of highest weight representations, one is of finite dimension and the other is of infinite dimension. Which type of representation appears depends on the representation of the subalgebra $U_q\mathfrak{sl}_2(\mathbb{C})$ of $U_q\mathfrak{osp}(1|2)$. That is, if a finite (resp. infinite) dimensional representation of the $U_q\mathfrak{sl}_2(\mathbb{C})$ part is chosen, then, as the result, one gets a finite (resp. infinite) dimensional representation of $U_q\mathfrak{osp}(1|2)$. In the following, we will restrict our discussions to the infinite dimensional representations.

The infinite dimensional highest weight module V_h with highest weight h is built on the highest weight vector $|h; 0\rangle$ satisfying $\mathcal{G}_-|h; 0\rangle = 0$, $H|h; 0\rangle = h|h; 0\rangle$, and is given as the direct sum of *even-module* V_h^{even} and *odd-module* V_h^{odd} as follows,

$$V_h = V_h^{even} \oplus V_h^{odd}, \quad (6)$$

$$V_h^{even} := \{|h; 2n\rangle \mid n \in \mathbb{Z}_{\geq 0}\}, \quad V_h^{odd} := \{|h; 2n+1\rangle \mid n \in \mathbb{Z}_{\geq 0}\}. \quad (7)$$

To represent $U_q\mathfrak{osp}(1|2)$ in more detail, we first define the even-vector $|h; 2n\rangle \in V_h^{even}$ and the odd-vector $|h; 2n+1\rangle \in V_h^{odd}$ as,

$$|h; 2n\rangle = \frac{(-)^n}{[n]!\llbracket n - \frac{1}{2} \rrbracket!} \mathcal{G}_+^{2n} |h; 0\rangle, \quad |h; 2n+1\rangle = \frac{(-)^{n+1}}{[n]!\llbracket n + \frac{1}{2} \rrbracket!} \mathcal{G}_+^{2n+1} |h; 0\rangle. \quad (8)$$

Once these vectors are defined in terms of $\mathcal{G}_+$, one can calculate the actions of the generators, $\mathcal{G}_+$, $\mathcal{G}_-$ and H on the spaces V_h^{even} , V_h^{odd} and they are

$$\mathcal{G}_+|h; 2n\rangle = -\llbracket n + \frac{1}{2} \rrbracket |h; 2n+1\rangle, \quad \mathcal{G}_+|h; 2n+1\rangle = [n+1]|h; 2n+2\rangle, \quad (9)$$

$$\mathcal{G}_-|h; 2n\rangle = -\llbracket 2h + n - \frac{1}{2} \rrbracket |h; 2n-1\rangle, \quad \mathcal{G}_-|h; 2n+1\rangle = [2h+n]|h; 2n\rangle, \quad (10)$$

$$H|h; 2n\rangle = (n+h)|h; 2n\rangle, \quad H|h; 2n+1\rangle = (h+n+\frac{1}{2})|h; 2n+1\rangle. \quad (11)$$

It should be emphasized that the above actions (9)–(11) are consistent with the relation (4) *only when the highest weight h is half-integral*. It is also noticed that

these actions have the correct classical limit for such values of h , otherwise the action of $\mathcal{G}_-$ on V_h^{even} vanishes in the classical limit. Upon eqs.(9,10) and Hermitian conjugation (5), the norms of these vectors are calculated as

$$\| |h; 2n\rangle \|^2 = \begin{bmatrix} 2h+n-1 \\ n \end{bmatrix} \left[\begin{matrix} 2h+n-\frac{1}{2} \\ n-\frac{1}{2} \end{matrix} \right], \quad (12)$$

$$\| |h; 2n+1\rangle \|^2 = \frac{[2h+n]}{[n+\frac{1}{2}]} \| |h; 2n\rangle \|^2, \quad (13)$$

where the normalization $\| |h; 0\rangle \|^2 = 1$ is made use of.

When q is real, there is no problem and every module V_h is unitary and irreducible by itself. Indeed, V_h is isomorphic to the module V_h^{cl} of the classical algebra $\mathfrak{osp}(1|2)$. However, as we will see in the next section, we have some problems in constructing irreducible modules when q is a root of unity, and owing to these problems the resulting irreducible highest weight modules are quite different from V_h .

3. Let us proceed to infinite dimensional representations of $U_q\mathfrak{osp}(1|2)$ when q is a root of unity. From now on we choose q as the primitive p -th root of unity, i.e.,

$$q = e^{2\pi i \frac{1}{p}}. \quad (p \geq 3) \quad (14)$$

As in the case of $U_q\mathfrak{sl}_2(\mathbb{C})$, in the constructions of irreducible highest weight modules, we come across some problems originated from the facts that $[x] = 0$ and $\llbracket y \rrbracket = 0$ for some numbers x and y . Therefore, before going to the construction, it is important to estimate what values give $[x] = 0$ and $\llbracket y \rrbracket = 0$. It is easy to find the answer for $[x] \sim \sin \frac{2\pi}{p}x$. On the other hand, we should consider the q -fermionic number $\llbracket y \rrbracket$ case by case in p . The answer is the following,

$$\begin{aligned} [x] &= 0, \quad \text{for } x = \frac{p}{2} \mathbb{N}, \\ \llbracket y \rrbracket &= 0, \quad \text{for } y = \begin{cases} p \mathbb{N} & \text{when } p = \text{odd} \\ \frac{p}{2} \mathbb{N} & \text{when } p = 4m \ (m \in \mathbb{N}) \\ \frac{p}{4} \mathbb{N} & \text{when } p = 4m+2 \ (m \in \mathbb{N}). \end{cases} \end{aligned} \quad (15)$$

We then construct irreducible highest weight modules by the following steps;

Step 1 DETERMINATION OF HIGHEST WEIGHT h

In order to obtain well-defined highest weight modules, the values of the highest weight h should be parameterized by two integers μ and ν as

$$(i) \quad \text{when } p = \text{odd} \quad h_{\mu\nu} = \frac{1}{2} (p\nu - \mu + 1) \quad 1 \leq \mu \leq p, \quad (16)$$

$$(ii) \quad \text{when } p = 4m \quad h_{\mu\nu} = \frac{1}{2} \left(\frac{p}{2}\nu - \mu + 1 \right) \quad 1 \leq \mu \leq 2m, \quad (17)$$

$$(iii) \quad \text{when } p = 4m + 2 \quad h_{\mu\nu} = \begin{cases} \frac{1}{2} \left(\frac{p}{2}\nu - \mu + 1 \right), & 1 \leq \mu \leq m + 1 \\ \frac{1}{2} \left(\frac{p}{4}(2\nu - 1) - \mu + \frac{1}{2} \right), & 1 \leq \mu \leq m \end{cases} \quad (18)$$

In all cases, the parameter ν runs $\mathbb{N}$.

PROOF The first problem we encounter in the construction of highest weight module is that vectors having infinite norms, being called singular vectors, appear. Let us start with the case (i). The first singular vector is the vector $|h; 2p\rangle$. The divergence comes from the factor $[p]$ in the denominator of the norm (12). Thus, a highest weight module is not necessarily well-defined in the sense that vectors $|h; r\rangle$, $r \geq 2p$ are singular. In order to avoid these singular vectors, we have to require that the weight h should satisfy the condition $[2h + \mu - 1] = 0$ with some integer μ , $1 \leq \mu \leq p$, so that it cancel the factor $[p]$ in the denominator. Upon eq.(15), this condition is just eq.(16). The case (ii) is the same as the case (i) except that the first singular vector is $|h; p\rangle$ instead of $|h; 2p\rangle$. Therefore the integer μ is bounded as $1 \leq \mu \leq p/2 = 2m$. Note here that, in the cases (i) and (ii), no factor $[\cdot]$ in the norms vanishes as we can easily see from (15) together with the fact $2h$ is now an integer. For the case (iii), however, the situation is different. Some factor $[\cdot]$ can be zero. The first singular vector appears at the level $2m + 1$ because of the factor $[m + \frac{1}{2}]$ in the denominator of $\| |h; 2m + 1\rangle \|^2$. There are two choices to cancel the factor, namely, the one is $[2h + \mu - 1] = 0$ and the other is $[2h + \mu - \frac{1}{2}] = 0$ providing, respectively, the first and the second equations in (18).

Step 2 HIGHEST WEIGHT MODULES

With the values given in (16)–(18), we have no more singular vectors but, as the expense of this, some zero-norm vectors appear. The highest weight modules are depicted by the following diagrams;

(i) The case when $p = \text{odd}$

$$\begin{array}{ccccccccccc} \bullet & \rightleftharpoons & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \dots & \rightleftharpoons & \circ & \leftarrow \\ 0 & & 1 & & & & 2\mu - 2 & & 2\mu - 1 & & & & 2p - 1 & \\ & & & & & & \leftarrow & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \dots \\ & & & & & & & 2p & & & & & 2(p + \mu) - 1 & & \end{array} \quad (19)$$

(ii) The case when $p = 4m$

$$\begin{array}{ccccccccccc} \bullet & \rightleftharpoons & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \dots & \rightleftharpoons & \circ & \leftarrow \\ 0 & & 1 & & & & 2\mu - 2 & & 2\mu - 1 & & & & p - 1 & \\ & & & & & & \leftarrow & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \dots \\ & & & & & & & p & & & & & p + 2\mu - 1 & & \end{array} \quad (20)$$

(iii-a) The case when $p = 4m + 2$ and $h_{\mu\nu} = \frac{1}{2}((2m + 1)\nu - \mu + 1)$

$$\begin{array}{ccccccccccccccc}
\bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \circ & \rightleftharpoons & \dots & \rightleftharpoons & \circ & \leftarrow & \bullet & \rightleftharpoons & \dots \\
0 & & & & 2\mu - 2 & & 2\mu - 1 & & 2\mu & & & & 2m & & 2m + 1 & & \\
& & & & & & & & & & & & & & & & \\
& \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \circ & \rightleftharpoons & \dots & \rightleftharpoons & \circ & \leftarrow & \bullet & \rightleftharpoons & \dots & & \\
& & & & 2(m + \mu) & & & & & & & & p - 1 & & p & &
\end{array} \tag{21}$$

(iii-b) The case when $p = 4m + 2$ and $h_{\mu\nu} = \frac{1}{2} \left(\frac{2m+1}{2}(2\nu - 1) - \mu + \frac{1}{2} \right)$

$$\begin{array}{ccccccccccccccc}
\bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \dots & \rightleftharpoons & \circ & \leftarrow & \bullet & \rightleftharpoons & \dots \\
0 & & & & 2\mu - 2 & & 2\mu - 1 & & 2\mu & & & & 2m & & 2m + 1 & & \\
& & & & & & & & & & & & & & & & \\
& \rightleftharpoons & \bullet & \rightleftharpoons & \bullet & \rightarrow & \circ & \rightleftharpoons & \dots & \rightleftharpoons & \circ & \leftarrow & \bullet & \rightleftharpoons & \dots & & \\
& & & & 2(m + \mu) & & & & & & & & p - 1 & & p & &
\end{array} \tag{22}$$

Here $\bullet$ and $\circ$ stand for a vector with non-zero norm and a vector with zero-norm, respectively. The arrow $\rightarrow$ corresponds to the action of $\mathcal{G}_+$ and $\leftarrow$ does to $\mathcal{G}_-$. Note that the highest weight modules for the case (iii-a) and (iii-b) are different from each other: The vectors at the levels $2(kp + \mu) - 1$, $k = 0, 1, \dots$ in the case (iii-a) are of zero-norm but, on the contrary, they are of finite norms in the case (iii-b). The essential feature of the modules is that, as one can see from the above diagrams, they are no longer irreducible. For example, the vector at the level $2\mu - 1$ in the diagram (19) is not only the zero-norm vector but also the highest weight vector, because the action of $\mathcal{G}_-$ on the vector disappears. Let us call such a vector *null vector*. On a null vector, a submodule is constructed. In order to obtain an irreducible module, we have to obtain all submodules and subtract all of them correctly. This is the next step.

Step 3 IRREDUCIBLE HIGHEST WEIGHT MODULES

Let $V_{\mu,\nu}$ be the highest weight module with highest weight $h_{\mu\nu}$, and $\mathbf{V}_{\mu,\nu}$ be the irreducible module obtained from $V_{\mu,\nu}$. The irreducible module for each case is depicted in the following diagrams,

$$\text{(i) } \mathbf{V}_{\mu,\nu} = \bigoplus_{k=0}^{\infty} \mathfrak{V}_k, \quad \text{where } \mathfrak{V}_k : \begin{array}{ccc} \bullet & \rightleftharpoons & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet \\ 2kp & & 2kp + 1 & & & & 2(kp + \mu) - 2 \end{array} \tag{23}$$

$$\text{(ii) } \mathbf{V}_{\mu,\nu} = \bigoplus_{k=0}^{\infty} \mathfrak{V}_k, \quad \text{where } \mathfrak{V}_k : \begin{array}{ccc} \bullet & \rightleftharpoons & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet \\ kp & & kp + 1 & & & & kp + 2\mu - 2 \end{array} \tag{24}$$

$$\text{(iii-a) } \mathbf{V}_{\mu,\nu} = \bigoplus_{k=0}^{\infty} \mathfrak{V}_k, \quad \text{where } \mathfrak{V}_k : \begin{array}{ccc} \bullet & \rightleftharpoons & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet \\ \frac{p}{2}k & & \frac{p}{2}k + 1 & & & & \frac{p}{2}k + 2\mu - 2 \end{array} \tag{25}$$

$$\text{(iii-b) } \mathbf{V}_{\mu,\nu} = \bigoplus_{k=0}^{\infty} \mathfrak{V}_k, \quad \text{where } \mathfrak{V}_k : \begin{array}{ccc} \bullet & \rightleftharpoons & \bullet & \rightleftharpoons & \dots & \rightleftharpoons & \bullet \\ \frac{p}{2}k & & \frac{p}{2}k + 1 & & & & \frac{p}{2}k + 2\mu - 1 \end{array} \tag{26}$$

PROOF To obtain the irreducible highest weight modules $\mathbf{V}_{\mu,\nu}$, we have to estimate all submodules appearing in the highest weight module $V_{\mu,\nu}$. Let us study the case (i) as an example. As we have already discussed, a submodule appears on the first null vector $|h_{\mu\nu}; 2\mu - 1\rangle$. Because the weight of the vector is $\frac{1}{2}(p\nu + \mu)$, the submodule can be written as $V_{1-\mu,\nu}$, that is, $V_{\mu,\nu} \supset V_{1-\mu,\nu}$. It is not, however, the end of the story. The submodule $V_{1-\mu,\nu}$ again includes a submodule $V_{\mu,\nu+2}$, since the vector $|h_{1-\mu,\nu}; 2p - 2\mu + 1\rangle \in V_{1-\mu,\nu}$ is the first null vector having the weight $\frac{1}{2}(p(\nu+2) - \mu + 1)$, i.e., $V_{\mu,\nu} \supset V_{1-\mu,\nu} \supset V_{\mu,\nu+2}$. Repeating this procedure, we obtain the following infinite series of submodules,

$$V_{\mu,\nu} \supset V_{1-\mu,\nu} \supset V_{\mu,\nu+2} \supset \cdots \supset V_{\mu,\nu+2k} \supset V_{1-\mu,\nu+2k} \supset V_{\mu,\nu+2(k+1)} \supset \cdots \quad (27)$$

Therefore, by subtracting all of them, we finally obtain the irreducible module (23). Similarly, one obtains irreducible highest weight modules for the other cases. It should be emphasize that all zero-norm vectors have disappeared by the subtraction. In other words, irreducible highest weight modules consist of finite-norm vectors only.

4. Now we have obtained irreducible highest weight modules of $U_q\mathfrak{osp}(1|2)$ at a root of unity. A remarkable point is that every irreducible module is parameterized by two integers μ and ν and it has the block-wise structure as shown in (23–26). The final task is to investigate the irreducible modules in more detail, especially it is quite important to make the structures clear. Recall that, in the case of infinite dimensional representations $U_q\mathfrak{sl}_2(\mathbb{C})$ at a root of unity, such a block-wise structure parameterized by two-integers brought us to the decomposition (1). For our graded Lie algebra $U_q\mathfrak{osp}(1|2)$ case, the following structure theorem holds.

THEOREM *On the irreducible highest weight module $\mathbf{V}_{\mu,\nu}$, the algebra $U_q\mathfrak{osp}(1|2)$ at a root of unity has the following tensor-product structure:*

$$U_q\mathfrak{osp}(1|2) \cong U\mathfrak{sl}_2(\mathbb{R}) \otimes U_q\mathfrak{osp}(1|2) \quad (28)$$

Equivalently, each irreducible module is decomposed into two irreducible modules as

$$\mathbf{V}_{\mu,\nu} \cong V_{\zeta}^{cl} \otimes \mathfrak{V}_J \quad (29)$$

where V_{ζ}^{cl} and $\mathfrak{V}_J$ are, respectively, the highest weight module of the classical Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ with the highest weight $\mathfrak{z}$ and the finite dimensional module of $U_q\mathfrak{osp}(1|2)$ with spin J , i.e.,

$$\begin{aligned} V_{\mathfrak{z}}^{cl} &:= \{ |\mathfrak{z}; k\rangle \mid k = 0, 1, \cdots \} \\ \mathfrak{V}_J &:= \{ |J : -J\rangle, |J : -J + \frac{1}{2}\rangle, \cdots, |J : J\rangle \} \end{aligned}$$

The values $\mathfrak{z}$ and J are given by the parameters μ, ν as follows; for the cases (i), (ii), $\mathfrak{z} = \frac{\nu}{2}$, $J = \frac{1}{2}(\mu - 1)$, for the case (iii-a), $\mathfrak{z} = \nu$, $J = \frac{1}{2}(\mu - 1)$, and for the case (iii-b), $\mathfrak{z} = \nu - \frac{1}{2}$, $J = \frac{1}{2}(\mu - \frac{1}{2})$.

PROOF Proof of this theorem consists of the following observations.

Observation 1 As we can see in (23–26), the irreducible highest weight modules $\mathbf{V}_{\mu,\nu}$ have the block-wise structures, namely, they are composed of an infinite number of blocks $\mathfrak{V}_k$, $k = 0, 1, \dots$ which consist of a finite number of vectors indicated by $\bullet$. Because of this fact, one can expect to find tensor product structure in each irreducible module $\mathbf{V}_{\mu,\nu}$ as in the case of $U_q \mathfrak{sl}_2(\mathbb{C})$.

Observation 2 Notice that the arrows connecting the final vector in $\mathfrak{V}_k$ and the first vector in the next block $\mathfrak{V}_{k+1}$ are missing. We should, therefore, ask what generators cause the movements from $\mathfrak{V}_k$ to $\mathfrak{V}_{k+1}$ and back. The answer is that they are an analog of the Lusztig operators [9] in $U_q \mathfrak{sl}(2, \mathbb{C})$. In our case, they are introduced according to the value of the deformation parameter p ,

$$(i) \quad E_+ := \frac{\mathcal{G}_+^{2p}}{[p - \frac{1}{2}]![p]!}, \quad E_- := \frac{\mathcal{G}_-^{2p}}{[p - \frac{1}{2}]![p]!}, \quad E_0 := \frac{1}{2} \begin{bmatrix} 2H + p - 1 \\ p \end{bmatrix}, \quad (30)$$

$$(ii) \quad E_+ := \frac{\mathcal{G}_+^p}{[\frac{p}{2} - \frac{1}{2}]![\frac{p}{2}]!}, \quad E_- := \frac{\mathcal{G}_-^p}{[\frac{p}{2} - \frac{1}{2}]![\frac{p}{2}]!}, \quad E_0 := \frac{1}{2} \begin{bmatrix} 2H + \frac{p}{2} - 1 \\ \frac{p}{2} \end{bmatrix}, \quad (31)$$

$$(iii-a) \quad E_+ := \frac{\mathcal{G}_+^{\frac{p}{2}}}{[\frac{p}{4}]![\frac{p}{4} - \frac{1}{2}]!}, \quad E_- := \frac{\mathcal{G}_-^{\frac{p}{2}}}{[\frac{p}{4}]![\frac{p}{4} - \frac{1}{2}]!}, \quad E_0 := \begin{bmatrix} 2H + \frac{p}{2} - 1 \\ \frac{p}{2} \end{bmatrix}, \quad (32)$$

$$(iii-b) \quad E_+ := \frac{\mathcal{G}_+^{\frac{p}{2}}}{[\frac{p}{4}]![\frac{p}{4} - \frac{1}{2}]!}, \quad E_- := \frac{1}{2} \frac{\mathcal{G}_-^{\frac{p}{2}}}{[\frac{p}{4}]![\frac{p}{4} - \frac{1}{2}]!}, \quad E_0 := \frac{1}{2} \begin{bmatrix} 2H + \frac{p}{4} \\ \frac{p}{4} \end{bmatrix}. \quad (33)$$

Needless to say that the above generators make sense only on irreducible modules. Then, in each case, we need six generators to define $U_q \mathfrak{osp}(1|2)$ at a root of unity. The next observation is the heart in the proof of this theorem.

Observation 3 Taking into account the structures of $\mathbf{V}_{\mu,\nu}$ as shown in eqs. (23–26) together with the fact that each finite dimensional space $\mathfrak{V}$ is mapped to each other by the above new generators $E_{\pm}, E_0$, one expects that every highest weight module $\mathbf{V}_{\mu,\nu}$ has a tensor product structure. To find the structure, it is essential to know the relations among $E_{\pm}$ and E_0 . Upon $\mathbf{V}_{\mu,\nu}$ one can easily show that, up to some constant and sign factors, they provide the classical Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ in every case, i.e., $[E_+, E_-] = 2E_0$, $[E_0, E_{\pm}] = \pm E_{\pm}$.

Thus we find that $\mathbf{V}$ is the tensor product of V^{cl} , a representation of the $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{V}$, a finite dimensional representation of $U_q \mathfrak{osp}(1|2)$. More precisely, there exist isomorphisms $\rho : \mathbf{V} \rightarrow V^{cl} \otimes \mathfrak{V}$, such that

$$\rho(|h_{\mu\nu}; k\bar{p} + s\rangle) = |\mathfrak{z}; k\rangle \otimes |J; \bar{M}\rangle, \quad k = 0, 1, 2, \dots, \quad (34)$$

where $\bar{p} = (i) 2p$, (ii) p and (iii) $p/2$, and $\bar{M} = -J + \frac{s}{2}$, $s = 0, 1, \dots, 2\mu - 1$ for the case (iii-b) and $s = 0, 1, \dots, 2\mu - 2$ for the others. We further have other isomorphisms $\hat{\rho}$ induced by ρ as $\rho(\mathcal{O}|\cdot\rangle) = \hat{\rho}(\mathcal{O})\rho(|\cdot\rangle)$ where $\mathcal{O} \in U_q \mathfrak{osp}(1|2)$. After some calculations one easily sees that $\hat{\rho}$ maps $U_q \mathfrak{osp}(1|2)$ into $U \mathfrak{sl}_2(\mathbb{R}) \otimes U_q \mathfrak{osp}(1|2)$

such as

$$\begin{aligned}\hat{\rho}(E_{\pm}) &= \pm L_{\mp 1} \otimes \mathbf{1}, & \hat{\rho}(E_0) &= L_0 \otimes \mathbf{1}, \\ \hat{\rho}(\mathcal{G}_{\pm}) &= \mathbf{1} \otimes \pm \mathcal{G}_{\pm \frac{1}{2}}, & \hat{\rho}(q^H) &= \mathbf{1} \otimes \mathcal{K}.\end{aligned}\tag{35}$$

Let us first examine the $\mathfrak{sl}_2(\mathbb{R})$ sector. The actions of $L_{\pm 1,0}$ on the space $V_{\mathfrak{z}}^{cl}$ are straightforwardly calculated as,

$$\begin{aligned}L_1|\mathfrak{z}; k\rangle &= (2\mathfrak{z} + k - 1)|\mathfrak{z}; k - 1\rangle, & L_{-1}|\mathfrak{z}; k\rangle &= (k + 1)|\mathfrak{z}; k + 1\rangle, \\ L_0|\mathfrak{z}; k\rangle &= (\mathfrak{z} + k)|\mathfrak{z}; k\rangle.\end{aligned}\tag{36}$$

It is easy to check that these generators $L_{\pm 1}, L_0$ satisfy the relations of algebra $\mathfrak{sl}_2(\mathbb{R})$,

$$[L_n, L_m] = (n - m)L_{n+m}, \quad n = 0, \pm 1.\tag{37}$$

We next study the $U_q\mathfrak{osp}(1|2)$ sector, the finite dimensional representation. From the actions (9–11) together with the isomorphism (34), it is easy to obtain, for all the cases,

$$\begin{aligned}\mathcal{G}_{\pm \frac{1}{2}}|J; M\rangle &= \mp [J \pm M + \frac{1}{2}]|J; M \pm \frac{1}{2}\rangle, & \mathcal{K}|J; M\rangle &= q^M|J; M\rangle, \\ \mathcal{G}_{\pm \frac{1}{2}}|J; M \pm \frac{1}{2}\rangle &= [J \pm M + 1]|J; M \pm 1\rangle, & \mathcal{K}|J; M + \frac{1}{2}\rangle &= q^{M+\frac{1}{2}}|J; M + \frac{1}{2}\rangle.\end{aligned}$$

Here the number $\overline{M}$ which takes the values in half-integer is decomposed into $\overline{M} = (M, M + \frac{1}{2})$ with M taking integer values. It is also easy to check that the above actions satisfy the $U_q\mathfrak{osp}(1|2)$ relations (4). We have now reached the theorem. $\square$

5. In summary, we studied infinite dimensional highest weight representations of $U_q\mathfrak{osp}(1|2)$ when q is the primitive p -th root of unity. As in the non-graded Lie algebra $U_q\mathfrak{sl}_2(\mathbb{C})$, we have shown that each infinite dimensional representation is isomorphic to the tensor product of two representations, an infinite dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ and a finite dimensional one of $U_q\mathfrak{osp}(1|2)$. It is interesting to stress that, in our $U_q\mathfrak{osp}(1|2)$ case, the classical sector appearing on irreducible representations is not the graded algebra $\mathfrak{osp}(1|2)$ but the non-graded $\mathfrak{sl}_2(\mathbb{R})$. Thus not only the effects of q -deformation but also graded structures entirely come from the finite dimensional sector. The case (iii), i.e., when $q = \exp \frac{\pi i}{2m+1}$ is notable. In this case, there are two possible highest weights (18) which yield two distinct irreducible highest weight modules. Both of them have the above mentioned tensor product structures but the finite dimensional sector of each is different from the other. Although the dimensions of $\mathfrak{V}_J$ for both cases can be written as $\dim \mathfrak{V}_J = 4J + 1$, the number of vectors in $\mathfrak{V}_J$ in the case (iii-a) is odd, whereas, it is even in the case (iii-b). Notice here that the spin J in the case (iii-b) is not half-integral but $(2\mathbb{N} + 1)/4$.

In this letter, only algebraic structures of $U_q\mathfrak{osp}(1|2)$ are concerned. It is, however, important to consider our result from the physical viewpoints. This is the next job and will appear elsewhere.

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