414

HERMITIAN VECTOR BUNDLES ON ARITHMETIC VARIETIES

Christophe SOULÉ

2009613

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Février 1996

IHES/M/96/15

Lectures at the AMS Summer Research Institute, “Algebraic Geometry”,
Santa Cruz, 1995.

HERMITIAN VECTOR BUNDLES ON ARITHMETIC VARIETIES

by Christophe SOULÉ

We present in these notes a geometric approach to algebraic varieties over $\mathbf{Z}$ known as Arakelov theory because of the fundamental papers by Arakelov in dimension two [3] [4].

Given a regular, projective and flat scheme X over $\mathbf{Z}$, we shall study the *hermitian vector bundles* over X . By this we mean pairs (E, h) consisting of an algebraic vector bundle E on X and a C^∞ hermitian metric h on the corresponding holomorphic vector bundle $E_{\mathbf{C}}$ on the analytic complex manifold $X(\mathbf{C})$ (h is also required to be invariant under complex conjugation).

We shall try to persuade the reader that hermitian vector bundles enjoy properties which are completely parallel to those of algebraic vector bundles on smooth projective complex manifolds. Among other things we shall give arithmetic versions of Chern classes, Schubert calculus, Grothendieck groups, Riemann-Roch theorems, ampleness, standard conjectures, semi-stability and vanishing theorems. For developing these notions and these results we will need to combine algebraic constructions on X with complex analytic methods on $X(\mathbf{C})$. While the analytic work is often the hardest part in the proofs, the statements are chosen to resemble as much as possible their algebro-geometric counterparts.

Of course there are limits to the analogy between function fields and number fields, geometry and arithmetic. A basic difference is that the smallest Krull dimension of a (flat) variety over $\mathbf{Z}$ is one and not zero as for complex manifolds. A related one is the absence of absolute differentials. In the first section, which is about geometry of numbers, we will discuss how to start in dimension one, by pointing out the analogy between Minkowski's classical theorems on euclidean lattices and standard facts about vector bundles on curves. This analogy will lead us to interesting questions in higher dimension. Apart from section one, most results will be valid in arbitrary dimension.

Our main goal is to understand the structure of the free $\mathbf{Z}$ -module $H^0(X, E)$ of global sections of E on X , when its real span $H^0(X, E) \otimes_{\mathbf{Z}} \mathbf{R}$ is equipped with the norm induced by h . Typically, we look for *small* sections in $H^0(X, E)$. As it turns out, this is an important technical tool for the study of diophantine equations, i.e.,

the determination of the set $X(\mathbf{Z})$. Indeed, most proofs in transcendental number theory and diophantine approximation require the use of “auxiliary polynomials” with integral coefficients of bounded size and degree. Vojta’s approach to the Mordell’s conjecture (first proved by Faltings [37]) demonstrates that in diophantine geometry small sections on arithmetic varieties play the same role as auxiliary polynomials [104]. His method led to the proof of diophantine conjectures of Lang on abelian varieties [39]. It is still not known today - and this is a major difference with the function field case - how to get an effective proof of the Mordell’s conjecture and how to go beyond curves and abelian varieties.

Here, instead of focusing on results in number theory which make use of Arakelov theory (see §6.4 for a brief summary), we take a geometric perspective and develop systematically the analogy mentioned above. As we shall see, this approach has led to new developments in complex geometry (Green currents, secondary characteristic classes, Quillen’s metric). There are many open problems, some of which we will mention on the way. And the reader is likely to discover a new and interesting question by just asking for the arithmetic analogue of her (his) favorite statement in classical algebraic geometry.

I thank V.Maillot, A.Silverberg, Y.Tschinkel, P.Vojta and S.Zhang for helpful comments on a first draft of this paper.

Table of contents

1. Geometry of numbers
 - 1.1. Minkowski’s theorems
 - 1.2. The analogy between function fields and number fields
 - 1.3. Higher dimensions
2. Arithmetic intersection theory
 - 2.1. Green currents
 - 2.2. Arithmetic Chow groups
 - 2.3. The intersection pairing
 - 2.4. Variants and complements
3. Chern classes
 - 3.1. Bott-Chern secondary characteristic classes
 - 3.2. Arithmetic characteristic classes
 - 3.3. Chern numbers
4. Riemann-Roch
 - 4.1. The determinant of cohomology
 - 4.2. A new characteristic class
 - 4.3. An arithmetic Riemann-Roch theorem
 - 4.4. Variants and complements
5. Ampleness
 - 5.1. Effectivity
 - 5.2. Ampleness
 - 5.3. Standard conjectures
6. Stability

- 6.1. Semi-stable hermitian vector bundles
- 6.2. Vanishing
- 6.3. Stability of projective varieties
- 6.4. Applications to number theory, a short survey.

1. Geometry of numbers.

We start our discussion with schemes of dimension one. Hermitian vector bundles over $\text{Spec}(\mathbb{Z})$ are the same as euclidean lattices. We will discuss the classical theorems of geometry of numbers about successive minima in such lattices. We then argue that these results are similar to the Riemann-Roch and Serre's duality theorems for vector bundles on curves. This analogy, when extrapolated to higher dimensions, leads to a qualitative description of some of the results to be discussed in later sections.

1.1. Minkowski's theorems.

1.1.1. Geometry of numbers is the study of euclidean lattices, i.e. discrete subgroups $\Lambda \subset \mathbb{R}^N$ of maximal rank in euclidean spaces (so Λ is a free abelian group of rank N). There are many results about these [59], but the most important one is the following theorem by Minkowski :

THEOREM 1.1. *Let $B \subset \mathbb{R}^N$ be a compact, convex, centrally symmetric set. Let μ be the Lebesgue measure on $\mathbb{R}^N$ and μ the quotient measure on $\mathbb{R}^N/\Lambda$. Assume that the following inequality holds :*

$$\mu(\mathbb{R}^N/\Lambda) \leq 2^{-N} \mu(B) .$$

Then there exists $s \in \Lambda \cap B$, $s \neq 0$.

1.1.2. Minkowski's theorem on successive minima is stronger than Theorem 1.1. We shall state it for hermitian modules on algebraic integers. So let K be a number field, $\mathcal{O}_K$ its ring of algebraic integers, $S = \text{Spec}(\mathcal{O}_K)$, and $\overline{E} = (E, h)$ an hermitian vector bundle over S , in the sense of the Introduction. In other words, E is a torsion free $\mathcal{O}_K$ -module of finite rank N and, for any complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, the corresponding N -dimensional complex vector space $E_\sigma = E \otimes_{\mathcal{O}_K} \mathbb{C}$ is equipped with an hermitian scalar product h_σ . We let

$$\|v\|_\sigma = \sqrt{h_\sigma(v, v)} \quad , \quad v \in E_\sigma ,$$

be the corresponding norm. The invariance of h under the complex conjugation F_∞ acting upon $S(\mathbb{C}) = \{\sigma : K \hookrightarrow \mathbb{C}\}$ means that if $\overline{\sigma}$ is the complex conjugate of σ , and if v lies in E_σ , then $\|v\|_\sigma = \|\overline{v}\|_{\overline{\sigma}}$.

A fundamental invariant of such a pair $\overline{E} = (E, h)$ is the *arithmetic degree*

$$\widehat{\deg}(\overline{E}) \in \mathbb{R} .$$

It is defined as follows. First, when (L, h) is an hermitian line bundle on S , take $s \in L$ to be any nontrivial section, and denote by $[L : \mathcal{O}_K s]$ the index in L of the submodule spanned by s . Then

$$(1.1) \quad \widehat{\deg}(L, h) = \log[L : \mathcal{O}_K s] - \sum_{\sigma \in S(\mathbb{C})} \log \|s\|_{\sigma}.$$

One checks that this definition does not depend on the choice of s . In general, let $\Lambda^N E$ be the top exterior power of E . The metric h gives a metric $\Lambda^N h$ on $\Lambda^N E$: if $v_1, \dots, v_N, w_1, \dots, w_N$ lie in E_{σ} , we let

$$(1.2) \quad \Lambda^N h_{\sigma}(v_1 \wedge \dots \wedge v_N, w_1 \wedge \dots \wedge w_N) = \det(h(v_i, w_j)), \quad 1 \leq i, j \leq N.$$

Then we let

$$(1.3) \quad \widehat{\deg}(E, h) = \widehat{\deg}(\Lambda^N E, \Lambda^N h).$$

Formulae (1.1) and (1.3) define $\widehat{\deg}(E, h)$ for any hermitian vector bundle (E, h) on S . If we allow E to have a nontrivial torsion subgroup E_{tors} of cardinality $\#(E_{\text{tors}})$, we may also define

$$(1.4) \quad \widehat{\deg}(E, h) = \log \#(E_{\text{tors}}) + \widehat{\deg}(E/E_{\text{tors}}, h).$$

1.1.3. Assuming again that E is torsion free, for any integer $i \leq N$ we let λ_i be the smallest real number λ such that there exist vectors $v_1, \dots, v_i$ in E which are linearly independent in $E \otimes_{\mathcal{O}_K} K$ and such that $\|v_{\alpha}\| \leq \lambda$ for all α , $1 \leq \alpha \leq i$. These numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ are called the *successive minima* of (E, h) . Let $\mu_i = -\log(\lambda_i)$, $1 \leq i \leq N$.

Denote by r_1 and r_2 be the number of real and complex embeddings of K , by $[K : \mathbb{Q}] = r_1 + 2r_2$ its degree over $\mathbb{Q}$, and by D_K the discriminant of K over $\mathbb{Q}$. If V_N is the volume of the standard unit ball in $\mathbb{R}^N$, we consider the following constant :

$$C(N, K) = N(r_1 + r_2) \log(2) + \frac{N}{2} \log |D_K| - r_1 \log V_N - r_2 \log V_{2N}.$$

THEOREM 1.2. *The following inequalities hold :*

$$\widehat{\deg}(E, h) - C(N, K) \leq [K : \mathbb{Q}](\mu_1 + \mu_2 + \dots + \mu_N) \leq \widehat{\deg}(E, h).$$

This result follows from [19, Theorem 6]. Since $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$, it implies that when $\widehat{\deg}(E, h)$ is bigger than $C(N, K)$ there is a nontrivial element v in E with $\|v\|_{\sigma} \leq 1$ for all $\sigma \in S(\mathbb{C})$. This is indeed a special case of Theorem 1.1 since the covolume V of E in the euclidean vector space $E \otimes_{\mathbb{Z}} \mathbb{R}$ is related to $\widehat{\deg}(E, h)$ by the formula

$$\widehat{\deg}(E, h) = -\log(V) + \frac{N}{2} \log |D_K|.$$

1.2. The analogy between number fields and function fields.

1.2.1. It has been known for some time [106] that Minkowski's Theorems 1.1 and 1.2 can be viewed as arithmetic analogues of the Riemann-Roch theorem on curves. In this analogy, the disjoint union of the scheme S and its set of complex points $S(\mathbb{C})$ is viewed as similar to a smooth complete curve C over the complex numbers (we do not try to give any topology to $S \coprod S(\mathbb{C})$). This analogy starts as follows :

Geometry	Arithmetic
$K =$ function field in one variable	$K =$ number field
$C =$ smooth complete curve over $\mathbb{C}$	$S \coprod S(\mathbb{C})$
$E =$ a vector bundle over C	$\overline{E} = (E, h)$ an hermitian vector bundle on S
$v \in H^0(C, E)$ a global section of E	$v \in E$ such that $\ v\ _\sigma \leq 1$ for all $\sigma \in S(\mathbb{C})$
$\deg(E) \in \mathbb{Z}$	$\widehat{\deg}(E, h) \in \mathbb{R}$.

Theorem 1.1 is then analogous to the statement that if a vector bundle E on a curve C of genus g has degree bigger than $\text{rk}(E)(g-1)$, then E has a non-trivial global section.

1.2.2. One way to make the analogy more complete is as follows. Given two real valued functions $f(\overline{E})$ and $g(\overline{E})$ on hermitian vector bundles, let us write

$$f(\overline{E}) \sim g(\overline{E})$$

when there exist two functions of N , $a(N)$ and $b(N)$, such that, given any number field K and any $\overline{E}$ on $S = \text{Spec}(\mathcal{O}_K)$, the inequality

$$|f(\overline{E}) - g(\overline{E})| \leq a(N) \log |D_K| + b(N) [K : \mathbb{Q}]$$

is satisfied. (For clarity, we shall not worry about the best values of $a(N)$ and $b(N)$, though it does play a role in applications.)

If $\overline{E}$ is an hermitian vector bundle over S , we let I be the set of indices i such that $\mu_i \geq 0$. Consider the following nonnegative real numbers :

$$h^0(S, \overline{E}) = [K : \mathbb{Q}] \cdot \sum_{i \in I} \mu_i$$

$$h^1(S, \overline{E}) = -[K : \mathbb{Q}] \sum_{i \notin I} \mu_i.$$

When $I = \emptyset$ we let $h^0(S, \overline{E}) = 0$ and when $I = \{1, \dots, N\}$ we let $h^1(S, \overline{E}) = 0$.

We can now rephrase Theorem 1.2 as a Riemann-Roch theorem :

$$(1.5) \quad h^0(S, \overline{E}) - h^1(S, \overline{E}) \sim \widehat{\deg}(\overline{E}).$$

1.2.3. A theorem of Mahler asserts that the successive minima λ_i^* of the dual lattice $E^* = \text{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$, equipped with the metric dual to h , are essentially the inverses of the successive minima of (E, h) . More precisely, if $K = \mathbb{Q}$ one knows that

$$1 \leq \lambda_1 \lambda_{N+1-i}^* \leq N$$

[6]. This implies, for any K , some kind of Serre's duality theorem :

THEOREM 1.3. *If $\overline{E}^*$ is the dual of $\overline{E}$, then*

$$h^1(S, \overline{E}) \sim h^0(S, \overline{E}^*).$$

1.2.4. Our definition of $h^0(S, \overline{E})$ is also in good agreement with the list of analogies given in 1.2.1 :

THEOREM 1.4.

$$h^0(S, \overline{E}) \sim \log \# \{v \in E \text{ such that } \|v\|_\sigma \leq 1 \text{ for all } \sigma \in S(\mathbb{C})\}.$$

Theorem 1.4 follows from [51, Theorem 2 and Proposition 6]. Note that more “transfer theorems”, e.g. on the successive minima of $\overline{E} \otimes \overline{F}$, can be written in this way, giving some arithmetic counterpart to Clifford's theorem for vector bundles on curves.

When E is a module of finite type over $\mathcal{O}_K$, with a metric as above and a non-trivial torsion subgroup E_{tors} , we define

$$h^0(S, \overline{E}) = \log \#(E_{\text{tors}}) + h^0(S, \overline{E/E_{\text{tors}}})$$

and

$$h^1(S, \overline{E}) = h^1(S, \overline{E/E_{\text{tors}}}).$$

Then (1.5), Theorem 1.3 and Theorem 1.4 remain valid.

1.2.5. Another way to develop the analogy between vector bundles on curves and the geometry of numbers is Stuhler's notion of a canonical polygon for $\overline{E}$. In the real plane consider the set Σ of all points $(\text{rank}(F), \widehat{\deg}(\overline{F}))$, where $F \subset E$ runs over all submodules of E , equipped with the induced metric. The canonical polygon of $\overline{E}$ is by definition the boundary of the convex hull of Σ (see [98] [56] [24]). If μ'_i is the i -th slope of this polygon, it can be shown that

$$\mu_i \sim \mu'_i \quad \text{for all } i \leq N.$$

In particular, $\overline{E}$ is called *semi-stable* when all its slopes are equal, i.e. when, for every $F \subset E$, the inequality

$$\text{rank}(E) \widehat{\deg}(\overline{F}) \leq \text{rank}(F) \widehat{\deg}(\overline{E})$$

holds. In terms of successive minima, this implies that

$$\mu_1 - \mu_N \sim 0.$$

1.3. Higher dimensions.

1.3.1. The analogy considered in the previous paragraph would be merely a curiosity if it did not extend to hermitian vector bundles $\overline{E}$ on varieties X of arbitrary dimension, thus enabling us to apply geometry of numbers to a wide class of problems. We proceed as follows. When X is a regular, projective and flat scheme over $\mathbf{Z}$ we view the map

$$f : X \coprod X(\mathbf{C}) \rightarrow S \coprod S(\mathbf{C}),$$

$S = \text{Spec}(\mathbf{Z})$, as similar to a flat morphism

$$\varphi : Y \rightarrow C,$$

where C is a smooth projective complex curve, and Y is a smooth projective complex variety of arbitrary dimension. An hermitian vector bundle $\overline{E}$ over X is then analogous to an algebraic vector bundle E on Y .

Once a Kähler metric on $X(\mathbf{C})$ has been chosen, the cohomology groups $H^q(X, E)$, $q \geq 0$, are equipped with the L^2 metric (see below 4.1.2 for the precise definition) and the hermitian modules $(H^q(X, E), h_{L^2})$ on S are viewed as analogous to the coherent sheaves $R^q \varphi_* E$ in the geometric set-up.

Now, when E is a vector bundle on Y , the coherent cohomology groups of the sheaf E on Y are the abutment of the Leray spectral sequence :

$$E_2^{pq} = H^p(C, R^q \varphi_* E) \Rightarrow H^{p+q}(Y, E),$$

which has only two columns since C is one-dimensional : $E_2^{pq} = 0$ when $p > 1$. Therefore the Betti numbers of E are

$$h^n(Y, E) = h^0(C, R^n \varphi_* E) + h^1(C, R^{n-1} \varphi_* E).$$

In the arithmetic set up it is then natural to define real Betti numbers by the formula :

$$(1.6) \quad h^n(X, \overline{E}) = h^0(S, (H^n(X, E), h_{L^2})) + h^1(S, (H^{n-1}(X, E), h_{L^2}))$$

(compare [52, formula (17)], where an analytic term is added to this number).

1.3.2. This discussion of the Leray spectral sequence, together with the results from geometry of numbers in Section 2, provide us with a method to phrase questions on hermitian vector bundles by analogy with classical geometry. Here is a sample of such problems and results :

- i) An arithmetic *section* of $\overline{E}$ is an element of L^2 -norm less than one in the lattice $H^0(X, E)$; by Minkowski's theorem, to prove the existence of nontrivial sections we need to compute the covolume of $H^0(X, E)$ in $H^0(X, E) \otimes_{\mathbf{Z}} \mathbf{R}$.
- ii) The arithmetic Riemann-Roch theorem gives a formula for the alternating sum $\sum_{q \geq 0} (-1)^q \widehat{\deg}(H^q(X, E), h_{L^2})$, see §4.3 ; when higher cohomology vanishes this will answer question i).

- iii) For an hermitian line bundle $\bar{L}$ on X to be called *ample* one requires that, when n is big enough, the lattice $H^0(X, L^{\otimes n})$ is spanned by sections of norm less than one. Since $H^k(X, L^{\otimes n}) = 0$ when $k > 0$ by the usual Serre's vanishing theorem, from (1.6) we see that ampleness implies $h^k(X, \bar{L}^{\otimes n}) = 0$ when $k > 0$ and n is big enough. Theorem 5.2 is a numerical criterion for arithmetic ampleness
- iv) By applying the usual Serre's duality and Theorem 1.3. one gets that, when $\bar{L}$ is ample in the sense of iii) and n is big enough, $h^k(X, \bar{L}^{\otimes -n}) = 0$ unless $k = \dim(X)$ (Theorem 5.4).
- v) The arithmetic Kodaira vanishing theorem asserts that, when $\bar{L}$ is ample, every small section in $H^1(X, L^{-1})$ vanishes. Indeed, since $H^0(X, L^{-1}) = 0$, we get from (1.6) that $h^1(X, \bar{L}^{-1}) = h^0(S, (H^1(X, L^{-1}), h_{L^2}))$. Such a vanishing is known in dimension two (Theorem 6.2).
- vi) The arithmetic theory of *stability* provides upper bounds for the ratio of successive minima in a given lattice of sections $H^0(X, E)$, see §1.2.5 and §6.3.

Before we can make more precise statements, we need to define numerical invariants for hermitian vector bundles. These *arithmetic intersection numbers* are presented in the next section.

2. Arithmetic intersection theory.

Twenty years ago, Arakelov developed an intersection theory on arithmetic surfaces [3] [4] [38] [69] [91]. For that purpose, he used Green functions for points on Riemann surfaces. These are asked to satisfy an "admissibility condition", a condition that is not required anymore in the work of Deligne [31]. Gillet and the author extended this theory to higher dimensions [48] [96] [21] [40] [25]. We describe in 2.1-2.3 below the main results of [48] and give some complements in section 2.4.

2.1. Green currents.

2.1.1. Let X be a smooth quasi-projective complex manifold. Let $A^{p,q}(X)$ be the space of smooth differential forms of type (p, q) on X and $A_c^{p,q}(X) \subset A^{p,q}(X)$ the subspace of forms with compact support. Recall that, when X is connected of dimension d , a *current* of type (p, q) is a linear map

$$S : A_c^{d-p, d-q}(X) \rightarrow \mathbb{C}$$

which is continuous for the Schwartz topology. We let $D^{p,q}(X)$ be the space of currents of type (p, q) . When X is not connected, define $D^{p,q}(X)$ as the direct sum of the $D^{p,q}(X_\alpha)$, where X_α runs over all components of X . Here are some examples :

1) If $Y \subset X$ is a closed irreducible subvariety of codimension p , we let $\delta_Y \in D^{p,p}(X)$ be the current obtained by integration on the smooth part $Y - Y^{\text{sing}}$ of Y :

$$\delta_Y(\omega) = \int_{Y - Y^{\text{sing}}} \omega.$$

This integral converges since, by Hironaka, we know that there exists a smooth projective manifold $\tilde{Y}$ and a morphism

$$\pi : \tilde{Y} \rightarrow Y$$

which is an isomorphism on a dense open set. Therefore $\delta_Y(\omega)$ is also equal to an integral on the compact manifold $\tilde{Y}$:

$$\delta_Y(\omega) = \int_{\tilde{Y}} \pi^*(\omega) .$$

2) More generally, if

$$Z = \sum_{\alpha} n_{\alpha} Y_{\alpha}$$

is a *cycle of codimension p* on X , i.e. a finite formal sum as above, where $n_{\alpha} \in \mathbf{Z}$ and Y_{α} are closed irreducible analytic subsets of codimension p in X , we put

$$\delta_Z = \sum_{\alpha} n_{\alpha} \delta_{Y_{\alpha}} .$$

3) There is an inclusion $A^{pq}(X) \subset D^{pq}(X)$, i.e. we may view any $\omega \in A^{pq}(X)$ as the current sending $\eta \in A_c^{d-p, d-q}(X)$ to the integral $\int_X \omega \wedge \eta$.

4) More generally, let $U \subset X$ be a dense open set and ω a smooth form on U . If ω extends to an L^1 -form on X , it defines a current $[\omega] \in D^{pq}(X)$ by the formula

$$[\omega](\eta) = \int_U \omega \wedge \eta .$$

2.1.2. Given S in $D^{pq}(X)$ we may define its derivatives $\partial S \in D^{p+1, q}(X)$ and $\bar{\partial} S \in D^{p, q+1}(X)$ by the formulas

$$\partial S(\eta) = (-1)^{p+q+1} S(\partial \eta)$$

and

$$\bar{\partial} S(\eta) = (-1)^{p+q+1} S(\bar{\partial} \eta) .$$

Let $d = \partial + \bar{\partial}$ and $d^c = (\partial - \bar{\partial})/(4\pi i)$, so that

$$dd^c = \frac{\bar{\partial}\partial}{2\pi i} .$$

DEFINITION. Let Z be a cycle of codimension p on X . A *Green current* for Z is a current $g \in D^{p-1, p-1}(X)$ such that

$$dd^c g + \delta_Z = \omega$$

for some smooth form $\omega \in A^{pp}(X) \subset D^{pp}(X)$.

2.1.3. For example, if $p = 1$, Z is a divisor, and there exists an holomorphic line bundle L on X together with a section s of L on some dense open set in X such that $Z = \text{div}(s)$ is the divisor of s .

Choose a C^∞ hermitian metric on L , i.e. a norm $\|\cdot\|$. Then the *Poincaré-Lelong equation* reads

$$dd^c [-\log \|s\|^2] + \delta_Z = c_1(L, \|\cdot\|) ,$$

where $[-\log \|s\|^2] \in D^{00}(X)$ is the current attached to the L^1 function $-\log \|s\|^2$ and $c_1(L, \|\cdot\|) \in A^{11}(X)$ is the *first Chern form*. In other words, $[-\log \|s\|^2]$ is a

Green current for $Z = \text{div}(s)$. When $p = 1$, all Green currents are obtained this way. When p is arbitrary we have :

PROPOSITION 2.1. a) *Any cycle has a Green current ;*

b) *If g_1 and g_2 are Green currents for the same cycle Z of codimension p , we may write*

$$g_1 - g_2 = \eta + \partial u + \bar{\partial} v$$

where $\eta \in A^{p-1,p-1}(X)$, $u \in D^{p-2,p-1}(X)$ and $v \in D^{p-1,p-2}(X)$.

Let us prove a) when X is projective (the general case follows ; see [47] for a proof of b)). The current $\delta_Z \in D^{pp}(X)$ is closed : $\partial(\delta_Z) = \bar{\partial}(\delta_Z) = 0$. Therefore, by the regularity of ∂ and $\bar{\partial}$, its cohomology class in $H^{pp}(X)$ can be represented by a closed smooth form. In other words, there exists a form $\omega \in A^{pp}(X)$ such that $\delta_Z - \omega$ is a boundary ($\delta_Z - \omega = dx$). By the dd^c -Lemma, we know that any current of type (p,p) which is a boundary lies in the image of dd^c . Therefore we can solve the equation

$$dd^c g = \omega - \delta_Z .$$

q.e.d.

2.2. Arithmetic Chow groups.

2.2.1. Let X be a regular quasi-projective flat scheme over $\mathbf{Z}$, $X(\mathbf{C})$ its set of complex points and $F_\infty : X(\mathbf{C}) \rightarrow X(\mathbf{C})$ the complex conjugation. Consider

$$D^{pp}(X) = \{S \in D^{pp}(X(\mathbf{C})) , S \text{ real} , F_\infty^*(S) = (-1)^p S\} ,$$

and

$$A^{pp}(X) = D^{pp}(X) \cap A^{pp}(X(\mathbf{C})) .$$

Let $Z = \sum_\alpha n_\alpha Y_\alpha$ be an *algebraic cycle* of codimension p on X ($n_\alpha \in \mathbf{Z}$, $Y_\alpha \subset X$ closed irreducible subsets of codimension p). A Green current for Z is, by definition, a Green current g for the associated cycle $\sum_\alpha n_\alpha Y_\alpha(\mathbf{C})$ on $X(\mathbf{C})$ such that g lies in $D^{pp}(X)$.

An *arithmetic cycle* is a pair (Z, g) where Z is an algebraic cycle on X and g a Green current for Z . For example, the pairs $(0, \partial u + \bar{\partial} v)$, where $u \in D^{p-2,p-1}(X, \mathbf{C})$, $v \in D^{p-1,p-2}(X(\mathbf{C}))$, and $\partial u + \bar{\partial} v \in D^{p-1,p-1}(X)$, are arithmetic cycles. Also, if $Y \subset X$ is irreducible of codimension $p-1$ and $f \in k(Y)^*$ a non zero rational function on Y , we get an arithmetic cycle $(\text{div}(f), -\log |f|^2)$, where $\text{div}(f)$ is the divisor of f (a cycle of codimension p on X) and $-\log |f|^2 \in D^{p-1,p-1}(X)$ maps ω to $-\int_{Y(\mathbf{C})} (\log |f|^2) \omega$ (this integral makes sense as can be seen by resolving the singularities of $Y(\mathbf{C})$).

The *arithmetic Chow group* $\widehat{\text{CH}}^p(X)$ is the group generated by arithmetic cycles of codimension p divided by the subgroup generated by those we just defined :

$$\widehat{\text{CH}}^p(X) = \frac{\{(Z, g) , \text{codim}(Z) = p\}}{\{(0, \partial u + \bar{\partial} v) ; (\text{div}(f), -\log |f|^2) , f \in k(Y)^* , \text{cod}(Y) = p-1\}} .$$

2.2.2. Let $\bar{L} = (L, h)$ be an hermitian line bundle on X , i.e. L is an algebraic line bundle on X and h is a C^∞ hermitian metric on the corresponding holomorphic line bundle $L_{\mathbb{C}}$ on $X(\mathbb{C})$. We assume that h is invariant by complex conjugation, i.e. the corresponding norm $\|\cdot\|$ is such that $\|F_\infty(u)\| = \|u\|$. Choose a rational section s of X (defined on a dense open subset of X). The pair $(\text{div}(s), [-\log \|s\|^2])$ is then an arithmetic cycle of codimension one (see 1.3 above). Its class

$$\widehat{c}_1(L, \|\cdot\|) \in \widehat{\text{CH}}^1(X)$$

is easily shown to be independent of the choice of the section s . We call it the *first Chern class* of $\bar{L} = (L, h)$. Any element of $\widehat{\text{CH}}^1(X)$ is of this form. More precisely, let $\widehat{\text{Pic}}(X)$ be the group of (isomorphism classes of) hermitian line bundles, with the tensor product as group law :

$$(L, h) \cdot (L', h') = (L \otimes L', h \otimes h')$$

where $\|s \otimes s'\| = \|s\| \cdot \|s'\|$.

PROPOSITION 2.2. *The first Chern class $\widehat{c}_1$ induces a group isomorphism*

$$\widehat{c}_1 : \widehat{\text{Pic}}(X) \xrightarrow{\sim} \widehat{\text{CH}}^1(X).$$

2.2.3. Let

$$\text{CH}^p(X) = \frac{\{Z, \text{cod}(Z) = p\}}{\langle \text{div}(f), f \in k(Y)^*, \text{cod}(Y) = p-1 \rangle}$$

be the algebraic Chow group of codimension p of X , and $\tilde{A}^{p-1, p-1}(X) = A^{p-1, p-1}(X) / \{\text{forms of type } \partial u + \bar{\partial} v\}$. From Proposition 1 one gets

PROPOSITION 2.3. *There are exact sequences*

$$\tilde{A}^{p-1, p-1}(X) \xrightarrow{a} \widehat{\text{CH}}^p(X) \xrightarrow{z} \text{CH}^p(X) \rightarrow 0$$

where $z(Z, g) = Z$ and $a(\eta) = (0, \eta)$.

The kernel of a is the image by Beilinson's regulator map of a K_1 -analogue of $\text{CH}^p(X)$ [48]. There is also a morphism

$$\omega : \widehat{\text{CH}}^p(X) \rightarrow A^{pp}(X)$$

which sends the class of (Z, g) to the form $\omega(Z, g) = dd^c g + \delta_Z$.

2.3. The intersection pairing.

2.3.1. **THEOREM 2.1.** *There exists an associative commutative graded bilinear pairing*

$$\widehat{\text{CH}}^p(X) \times \widehat{\text{CH}}^q(X) \rightarrow \widehat{\text{CH}}^{p+q}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

$p \geq 0, q \geq 0$, such that the maps z and ω are graded ring homomorphisms. This pairing takes values in $\widehat{\text{CH}}^{p+q}(X)$ in any of the following cases :

- a) p or $q \leq 1$;
- b) X is a smooth variety over $\text{Spec}(\mathcal{O}_K)$, K a number field ;
- c) any scheme Y of finite type over $\mathbf{Z}$ with $\dim(Y) \leq \dim(X)$ admits a resolution of its singularities.

To describe this *intersection pairing* let (Y, g_Y) and (Z, g_Z) be arithmetic cycles on X , of codimension p and q respectively. If $Y = \sum_{\alpha} n_{\alpha} Y_{\alpha}$ we let $|Y| = \bigcup_{\alpha} Y_{\alpha}$ be the support of Y . By Chow's moving lemma, we can assume that all components of $|Y| \cap |Z|$ which meet the generic fiber $X_{\mathbf{Q}} = X \times_{\text{Spec}(\mathbf{Z})} \text{Spec}(\mathbf{Q})$ have codimension $p + q$.

Under these assumptions, we define

$$(2.1) \quad (Y, g_Y) \cdot (Z, g_Z) = (Y \cap Z, g_Y * g_Z),$$

where $Y \cap Z$ is the intersection of the cycles Y and Z and $g_Y * g_Z$ a Green current for $Y \cap Z$ given by the formula

$$(2.2) \quad g_Y * g_Z = \delta_Y g_Z + g_Y \omega_Z,$$

where $\omega_Y = dd^c g_Y + \delta_Y$ and $\omega_Z = dd^c g_Z + \delta_Z$. The fact that $g_Y * g_Z$ is a Green current for $Y \cap Z$ is easy to check *formally* since δ_Y and ω_Z are closed :

$$\begin{aligned} dd^c(g_Y * g_Z) &= \delta_Y dd^c(g_Z) + dd^c(g_Y) \omega_Z \\ &= \delta_Y(\omega_Z - \delta_Z) + (\omega_Y - \delta_Y) \omega_Z \\ &= \omega_Y \omega_Z - \delta_Y \delta_Z = \omega_{Y \cap Z}. \end{aligned}$$

But there are serious difficulties to overcome in order to make sense of (2.1) and (2.2). First, the product of currents $\delta_Y g_Z$ is not defined in general. One can show that it exists when g_Z is the current associated to an L^1 form on $X(\mathbf{C}) - |Z(\mathbf{C})|$ which is of "log type" along $|Z(\mathbf{C})|$. We refer to [48] and [25] for this construction. Another approach to this product of Green currents is given by Burgos [27], who uses a description à la Dolbeault of the mixed Hodge structure of the cohomology of $X(\mathbf{C})$ with supports in $|Z(\mathbf{C})|$.

Another difficulty with this intersection pairing is the definition of the cycle $Y \cap Z$ in (2.1), since Chow's moving lemma does not hold for varieties over $\mathbf{Z}$. Using K -theory [46] [48] or operational Chow groups [60], it is possible to define $Y \cap Z$ in the Chow group with supports $\widehat{\text{CH}}_{|Y| \cap |Z|}^{p+q}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. This is why the pairing takes values in $\widehat{\text{CH}}^{p+q}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, except in cases a), b) and c), where this difficulty can be solved [48] [64].

2.3.2. THEOREM 2.2.

- i) For any morphism $f : X \rightarrow Y$ of regular, quasi-projective, flat schemes over $\mathbf{Z}$, there are pull-back morphisms of groups

$$f^* : \widehat{\text{CH}}^p(Y) \rightarrow \widehat{\text{CH}}^p(X)$$

compatible with a , z and ω , such that $f^*(xy) = f^*(x)f^*(y)$.

- ii) Furthermore, if f is proper and if it induces a smooth map $f : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$ (i.e. a submersion) on complex points, there is a direct image morphism

$$f_* : \widehat{\mathrm{CH}}^p(X) \rightarrow \widehat{\mathrm{CH}}^{p+\dim(Y)-\dim(X)}(Y)$$

compatible with a , z and ω , and such that :

$$f_*(x f^*(y)) = f_*(x) y.$$

We shall only define f_* in ii). It maps (Z, g) to $(f_*(Z), f_*(g))$ where, as indicated above, $f_*(g)(\omega) = g(f^*(\omega))$ and $f_*(Z)$ is defined as follows. By linearity we may assume that Z is an irreducible subset of X . Let $f(Z)$ be its image. If $f(Z)$ has the same dimension as Z , we let $f_*(Z) = n f(Z)$, where n is the degree of the extension of fields $k(Z)$ over $k(f(Z))$. If $f(Z)$ has a dimension different from $\dim(Z)$, we let $f_*(Z) = 0$. We ask f to be smooth on $X(\mathbb{C})$ because we get from $dd^c g + \delta_Z = \omega$ the equation

$$dd^c(f_* g) + \delta_{f_* Z} = f_* \omega,$$

and, when f is a submersion, $f_* \omega$ is C^∞ .

2.3.3. Assume $X = \mathrm{Spec}(\mathbb{Z})$. Then we can compute

$$\widehat{\mathrm{CH}}^0(\mathbb{Z}) = \mathrm{CH}^0(\mathbb{Z}) = \mathbb{Z},$$

$$\widehat{\mathrm{CH}}^p(\mathbb{Z}) = 0 \quad \text{if } p > 1,$$

and

$$\widehat{\mathrm{CH}}^1(\mathbb{Z}) = \widehat{\mathrm{Pic}}(\mathbb{Z}) = \mathbb{R}.$$

A real number $\lambda \in \mathbb{R}$ corresponds to the class of the cycle $(0, 2\lambda)$ on $\mathrm{Spec}(\mathbb{Z})$, and to the isomorphism class in $\widehat{\mathrm{Pic}}(\mathbb{Z})$ of an hermitian line bundle $\overline{L}$ such that any generator s of the rank one $\mathbb{Z}$ -module L has norm $\|s\| = \exp(-\lambda)$.

2.3.4. It follows from the previous discussion that if X is projective of relative dimension d over $\mathbb{Z}$ there exists a pairing

$$\widehat{\mathrm{CH}}^p(X) \times \widehat{\mathrm{CH}}^{d+1-p}(X) \rightarrow \mathbb{R}$$

mapping (x, y) to $f_*(xy) \in \widehat{\mathrm{CH}}^1(\mathbb{Z}) = \mathbb{R}$, where $f : X \rightarrow \mathrm{Spec}(\mathbb{Z})$ is the projection. In other words we obtain *arithmetic intersection numbers*. These are real numbers.

2.4. Variants and complements.

2.4.1. When $Y \subset X$ is an irreducible closed subscheme of Krull dimension $\dim(Y)$ which is proper over $\mathbb{Z}$ and $x \in \widehat{\mathrm{CH}}^{\dim(Y)}(X)$, we can “integrate” x on Y to get a real number $\widehat{\deg}(x \mid Y) \in \mathbb{R}$, which coincides with $f_*(x)$ in the situation

of 2.3.4 when $Y = X$ [25]. Namely, if g_Y is *any* Green current for Y , and if $y \in \widehat{\text{CH}}^{\dim(X)-\dim(Y)}(Y)$ is the class of (Y, g_Y) , we let

$$\widehat{\deg}(x | Y) = f_*(xy) - \frac{1}{2} \int_{X(\mathbf{C})} g_Y \omega(x).$$

2.4.2. If $\overline{L}$ is an hermitian line bundle on X and $Y \subset X$ is as above, the real number

$$(2.3) \quad h_{\overline{L}}(Y) = \widehat{\deg}(\widehat{c}_1(\overline{L})^{\dim(Y)} | Y),$$

is called the *height of Y* with respect to $\overline{L}$. It can be computed by induction, using the following rules :

- a) If Y is “vertical”, i.e. if Y is contained in the fiber of X over a prime integer p , then

$$(2.4) \quad h_{\overline{L}}(Y) = \deg_L(Y) \log(p),$$

where $\deg_L(Y) \in \mathbf{Z}$ is the usual algebraic degree of the variety Y over $\mathbf{F}_p$;

- b) If s is a rational section of L over Y , with divisor $\text{div}(s) = \sum_{\alpha} n_{\alpha} Z_{\alpha}$, the following formula holds

$$(2.5) \quad h_{\overline{L}}(Y) = \sum_{\alpha} n_{\alpha} h_{\overline{L}}(Z_{\alpha}) - \int_{Y(\mathbf{C})} \log \|s\| c_1(\overline{L}_{\mathbf{C}})^{\dim(Y_{\mathbf{C}})}.$$

According to Deligne [31] (and Elkik [34] in higher dimensions) one can also view $h_{\overline{L}}(Y)$ as the arithmetic degree of a canonical hermitian line bundle on $\text{Spec}(\mathbf{Z})$.

2.4.3. The idea of using Arakelov theory to define heights of varieties, as in (2.3) above, first appeared in the work of Faltings on diophantine approximation on abelian varieties [39]. It is both more precise and more intrinsic than the former definition by means of Chow coordinates due to Nesterenko [84] and Philippon [86]. We refer to [93], [25] and [60] for basic properties of these heights, and to [36] and [42] for effective proofs of the Faltings’ product theorem [39].

2.4.4. For example, if $\overline{\mathcal{O}(1)}$ is the standard line bundle on the projective space $\mathbf{P}^N$ over $\mathbf{Z}$, $N \geq 1$, using (2.5) and induction on N , one finds ([49], p.212; [25] Lemma 3.3.1)

$$(2.6) \quad h_{\overline{\mathcal{O}(1)}}(\mathbf{P}^N) = \frac{1}{2} \sum_{k=1}^N \sum_{m=1}^k \frac{1}{m}.$$

These rational numbers first appeared in the work of Stoll on Nevanlinna theory with values in $\mathbf{P}^N(\mathbf{C})$ [97]. We refer the reader to [102] and [92] for links between Nevanlinna theory and Arakelov theory.

2.4.5. When $X(\mathbf{C})$ is not compact, a better definition of $\widehat{\text{CH}}^p(X)$ consists in asking that the form ω in the definition of Green currents in 2.1.2 has log growth at infinity [27]. One can also lift the regularity assumption on X by assuming only that

$X(\mathbb{C})$ is smooth [52]. Furthermore, several attempts have been made to develop an *adelic intersection theory* for varieties over number fields, with no choice of a model over the integers [30] [108] [90] [13].

One major open problem is to obtain in the arithmetic context a theory of *improper intersections* which would extend the approach of Vogel [101] and Fulton [43] to arithmetic cycles which do not meet properly in the generic fiber $X_{\mathbb{Q}}$. This leads to interesting analytic questions [112].

2.4.6. More generally, one can expect that all results from geometric intersection theory and enumerative geometry (as in [43] say) will have analogues over $\mathbb{Z}$. A typical example is the arithmetic Bézout theorem proved in [14].

One of the many open questions along this line is the behaviour of $\widehat{\mathrm{CH}}^p$ under blow ups. More specifically, let X be a regular projective and flat scheme over $\mathbb{Z}$. Consider the blow up X' of X along a regular irreducible closed subscheme $Y \subset X$. Let $Y' \subset X'$ be the exceptional divisor, so that we get a cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ \pi_Y \downarrow & & \downarrow \pi \\ Y & \xrightarrow{i} & X. \end{array}$$

Question : Is the following sequence exact ($p \geq 0$) :

$$0 \rightarrow \widehat{\mathrm{CH}}^p(X) \xrightarrow{(i^*, \pi^*)} \widehat{\mathrm{CH}}^p(Y) \oplus \widehat{\mathrm{CH}}^p(X') \xrightarrow{\pi_Y^* - j^*} \widehat{\mathrm{CH}}^p(Y') \rightarrow 0?$$

(See [54] or [61] for the corresponding algebraic statement).

3. Chern classes.

Having defined arithmetic Chow groups $\widehat{\mathrm{CH}}^p(X)$ for varieties over $\mathbb{Z}$ in the previous section, we will see that hermitian vector bundles on X have characteristic classes lying in these groups. Their definition [49] involves secondary characteristic classes which were first introduced in complex geometry by Bott and Chern [26].

3.1. Bott-Chern secondary characteristic classes.

3.1.1. Let X be a smooth complex manifold. If E is a holomorphic vector bundle on X , an hermitian metric h on E is given by an hermitian scalar product

$$h : E_x \times E_x \rightarrow \mathbb{C}$$

on each fiber of E , which depends smoothly on x . We let $A^{p,q}(X, E) = A^{p,q}(X) \otimes_{C^\infty(X)} C^\infty(X, E)$ be the space of (p, q) differential forms with values in E . As is well-known [57], there exists a unique connection

$$\nabla : C^\infty(X, E) \rightarrow A^1(X, E) = A^{1,0}(X, E) \oplus A^{0,1}(X, E)$$

such that $\nabla^{01} = \bar{\partial}_E$ is the Cauchy-Riemann operator on E , and ∇ is unitary :

$$d h(s, t) = h(\nabla s, t) + h(s, \nabla t) , \quad s, t \in C^\infty(X, E) .$$

This connection can be extended to forms by using the Leibnitz rule :

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg(\omega)} \omega \otimes \nabla s .$$

The curvature is then defined as the square of ∇

$$\nabla^2 : C^\infty(X, E) \rightarrow A^2(X, E) .$$

It takes values in $A^{11}(X, E)$ and it is linear under the action of $C^\infty(X, \mathbb{C})$. So we can think of it as an element ∇^2 in $A^{11}(X, \text{End } E)$. Any symmetric power series ϕ in r variables, $r = \text{rank}(E)$, with real coefficients, defines invariant polynomial maps

$$\bigoplus_{p \geq 0} A^{p,p}(X, \text{End}(E)) \rightarrow \bigoplus_{p \geq 0} A^{p,p}(X) ,$$

hence characteristic forms by the formula

$$\phi(E, h) = \phi(-\nabla^2 / (2\pi i)) .$$

For instance, the *Chern character form* is

$$\text{ch}(E, h) = \text{tr} \exp(-\nabla^2 / (2\pi i)) \in \bigoplus_{p \geq 0} A^{pp}(X)$$

and the *Chern forms* are defined by

$$\det(\text{Id} - t \nabla^2 / (2\pi i)) = \sum_{p \geq 0} t^p c_p(E, h) , \quad c_p(E, h) \in A^{pp}(X) .$$

3.1.2. Let

$$\mathcal{E} : 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

be an exact sequence of holomorphic vector bundles. We endow S, E, Q with (arbitrary) hermitian metrics h', h, h'' . We let $\bar{\mathcal{E}} = (\mathcal{E}, h', h, h'')$. Recall from 2.2.3 the notation

$$\tilde{A}^{p-1, p-1}(X) = A^{p-1, p-1}(X) / (\text{Im } \partial + \text{Im } \bar{\partial}) .$$

We let $\tilde{A}(X) = \bigoplus_{p \geq 1} \tilde{A}^{p-1, p-1}(X)$.

THEOREM 3.1. *Given ϕ as above, there is a unique way to define classes*

$$\tilde{\phi}(\bar{\mathcal{E}}) \in \tilde{A}(X)$$

such that :

- i) $dd^c \tilde{\phi}(\bar{\mathcal{E}}) = \phi(S \oplus Q, h' \oplus h'') - \phi(E, h) ;$
- ii) *if $f : Y \rightarrow X$ is a map of complex manifolds, then*

$$f^* \tilde{\phi}(\bar{\mathcal{E}}) = \tilde{\phi}(f^* \bar{\mathcal{E}}) ;$$

- iii) if $E = S \oplus Q$, $h = h' \oplus h''$ and $\mathcal{E}$ is the obvious split exact sequence, then $\tilde{\phi}(\bar{\mathcal{E}}) = 0$.

These classes are called Bott-Chern secondary characteristic classes [26] [32] [10] [49]. For instance, when $Q = 0$ and $S = E$, we get a class $\tilde{c}_p(h', h) \in \tilde{A}^{p-1, p-1}(X)$ which solves the equation

$$dd^c(\tilde{c}_p(h', h)) = c_p(E, h') - c_p(E, h).$$

When $p = 1$ and $E = L$ is a line bundle, the smooth function $\tilde{c}_1(h', h) \in A^{0,0}(X)$ is given by the formula :

$$\tilde{c}_0(h', h) = \log(h/h').$$

3.2. Arithmetic characteristic classes.

Let X be a regular quasi-projective flat scheme over $\mathbf{Z}$. Recall from the introduction that an *hermitian vector bundle* $\bar{E} = (E, h)$ on X is defined as a pair consisting of :

- i) an algebraic vector bundle E on X , i.e. a locally free coherent $\mathcal{O}_X$ -module ;
- ii) an hermitian metric h on the associated holomorphic vector bundle $E_{\mathbf{C}}$ on $X(\mathbf{C})$, which is invariant under the action of complex conjugation : $h(F_{\infty} s, F_{\infty} t) = h(s, t)$.

THEOREM 3.2. [49] *Let ϕ be a symmetric power series as above with coefficients in a ring $R \subset \mathbf{R}$. There is a unique way to define characteristic classes*

$$\hat{\phi}(E, h) \in \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(X) \otimes_{\mathbf{Z}} R$$

such that the following holds :

- i) If (L, h) is an hermitian line bundle, let $\hat{c}_1(L, h) \in \widehat{\text{CH}}^1(X)$ be the class defined in 2.2.2 and $P(T) = \phi(T, 0, 0, \dots)$. Then

$$\hat{\phi}(L, h) = P(\hat{c}_1(L, h)) \in \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(X) \otimes_{\mathbf{Z}} R.$$

- ii) Given any map $f : Y \rightarrow X$ of arithmetic varieties

$$f^* \hat{\phi}(E, h) = \hat{\phi}(f^* E, f^* h).$$

- iii) $\omega(\hat{\phi}(E, h)) = \phi(E, h)$ in $\bigoplus_{p \geq 0} A^{pp}(X)$.

- iv) Let $\bar{\mathcal{E}} = (\mathcal{E}, h', h, h'')$ be an exact sequence of algebraic bundles on X , with arbitrary hermitian metrics. Then

$$\hat{\phi}(S \oplus Q, h' \oplus h'') - \hat{\phi}(E, h) = a(\tilde{\phi}(\bar{\mathcal{E}})).$$

- v) The class $\hat{\phi}$ is well behaved under tensor products. For instance the Chern character $\hat{\text{ch}}$ is multiplicative and, if $\bar{L}$ is an hermitian line bundle and $\bar{E}$ an hermitian vector bundle of rank r ,

$$\hat{c}_p(\bar{E} \otimes \bar{L}) = \sum_{i=0}^p \binom{r-i}{i} \hat{c}_i(\bar{E}) \hat{c}_1(\bar{L})^{p-i}$$

(see [49], §4.1, when ϕ is arbitrary).

We obtain this way Chern classes $\hat{c}_p(E, h) \in \widehat{CH}^p(X)$, $p \geq 0$, a Chern character class $\hat{ch}(E, h) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, and a Todd class $\widehat{Td}(E, h) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Theorem 3.2 can be proved by any of the classical methods : via Grassmannians [49], by Grothendieck's splitting principle [40], or via Segre classes [34], [52] Lemma 17. The new feature is Property iv). Notice that iii) and iv) are compatible with the definition of Bott-Chern secondary characteristic classes. Indeed, let $a : \tilde{A}^{p-1, p-1}(X) \rightarrow \widehat{CH}^p(X)$ be the map sending the class of η to the class of $(0, \eta)$ and let $\omega : \widehat{CH}^p \rightarrow A^{pp}(X)$ be the map sending (Z, g) to $dd^c g + \delta_Z = \omega(Z, g)$ (see 2.2.3). Clearly $\omega \circ a = dd^c$. So, if we apply ω to the equality iv) in Theorem 3.2 we get Property i) in Theorem 3.1.

3.3. Chern numbers.

3.3.1. Let $X = G_{m,n}$ be the Grassmannian manifold over $\mathbf{Z}$ such that, for any field F , its set of points $X(F)$ consists of all n -dimensional quotients of F^{m+n} . On $G_{m,n}$ there is a tautological exact sequence

$$(3.1) \quad 0 \rightarrow S \rightarrow \mathcal{O}_X^{m+n} \rightarrow Q \rightarrow 0$$

where S and Q are vector bundles of rank m and n respectively. We endow S and Q with the metrics h' and h'' induced from the standard metric h on $\mathcal{O}_X^{m+n}$:

$$h(z_1, \dots, z_{m+n} ; u_1, \dots, u_{m+n}) = \sum_i z_i \bar{u}_i .$$

Let $\hat{c}_p(\bar{S}) \in \widehat{CH}^p(X)$, $\hat{c}_p(\bar{Q}) \in \widehat{CH}^p(X)$ be the associated Chern classes, and $f : X \rightarrow \text{Spec}(\mathbf{Z})$ the projection map. A *Chern number* is by definition any real number of the form

$$f_* (\hat{c}_{p_1}(\bar{S}) \dots \hat{c}_{p_k}(\bar{S}) \hat{c}_{q_1}(\bar{Q}) \dots \hat{c}_{q_\ell}(\bar{Q})) \in \widehat{CH}^1(\mathbf{Z}) = \mathbf{R}$$

where $p_1 + \dots + p_k + q_1 + \dots + q_\ell = \dim(G_{m,n}) (= mn + 1)$. The following result is due to Maillot :

THEOREM 3.3. [73] *All Chern numbers are rational numbers.*

3.3.2. The main step in the proof of Theorem 3.3 is to show that the Bott-Chern secondary classes of the exact sequence (3.1) are linear combinations with rational coefficients of the invariant Chern forms of $\bar{S}_{\mathbf{C}}$ and $\bar{Q}_{\mathbf{C}}$ on $X(\mathbf{C})$.

One obtains explicit formulas for Chern numbers and the full *Schubert calculus* on Grassmannians can be developed in this arithmetic setting [73]. The next unsolved problem is to compute intersection numbers on *flag manifolds*.

3.3.3. If $\bar{Q}$ is the canonical quotient bundle on $\mathbf{P}^N$, its Chern classes $\hat{c}_p(\bar{Q}) \in \widehat{CH}^p(\mathbf{P}^N)$ can be described as follows. Let $(x_0, \dots, x_N)$ be the standard homogeneous coordinates on $\mathbf{P}^N$, and let $Z \subset \mathbf{P}^N$ be the linear subspace of equation

$$x_0 = x_1 = \dots = x_{p-1} = 0 .$$

Consider the functions

$$\begin{aligned}\tau &= \log(|x_0|^2 + \cdots + |x_N|^2) \\ \sigma &= \log(|x_0|^2 + \cdots + |x_{p-1}|^2),\end{aligned}$$

and the $(1, 1)$ forms $\alpha = dd^c \tau$ and $\beta = dd^c \sigma$ on $\mathbf{P}^N(\mathbf{C}) - Z(\mathbf{C})$. Then, the following form

$$\Lambda = (\tau - \sigma) \sum_{i=0}^{p-1} \alpha^i \beta^{p-1-i}$$

in $A^{p-1, p-1}(\mathbf{P}^N(\mathbf{C}) - Z(\mathbf{C}))$ was shown by Levine [72] to be L^1 on $\mathbf{P}^N(\mathbf{C})$ and to define a Green current $[\Lambda]$ for Z on $\mathbf{P}^N$. It can be proved that $\hat{c}_p(\overline{Q})$ is represented by the arithmetic cycle $(Z, [\Lambda])$ ([49] Theorem 9.2).

3.3.4. Formula (2.6) in 2.4.4 is an example illustrating Theorem 3. On the Grassmannian $X = G_{2,2}$ over $\mathbf{Z}$, the Chern classes $\hat{c}_p(\overline{Q})$ are zero when $p > 2$. Furthermore

$$\begin{aligned}f_* (\hat{c}_1(\overline{Q}) \hat{c}_2(\overline{Q})^2) &= \frac{5}{3}, \\ f_* (\hat{c}_1(\overline{Q})^3 \hat{c}_2(\overline{Q})) &= \frac{37}{12}, \\ (3.2) \quad f_* (\hat{c}_1(\overline{Q})^5) &= \frac{43}{6}.\end{aligned}$$

The last equality amounts to computing the height of $G_{2,2}$ in its Plücker embedding, where $G_{2,2} \subset \mathbf{P}^5$ is the quadric $x_1 x_2 - x_3 x_4 + x_5 x_6 = 0$. One proof of (3.2) [29] consists of using (2.5) and (2.6) to reduce the computation to the following integral :

$$(3.3) \quad \int_{S^{11}} \log |x_1 x_2 - x_3 x_4 + x_5 x_6| \, dv = - \left(1 + \frac{1}{3} + \frac{1}{5} \right),$$

where dv is the invariant measure of volume one on the sphere S^{11} with equation $\sum_{i=1}^6 |x_i|^2 = 1$. To prove (3.3) one can view the left hand side as the derivative at zero of a zeta function à la Igusa :

$$(3.4) \quad Z(s) = \int_{S^{11}} |x_1 x_2 - x_3 x_4 + x_5 x_6|^s \, dv = \frac{\Gamma(6)}{\Gamma(3)} \cdot \frac{\Gamma(1+s/2) \Gamma(3+s/2)}{\Gamma(6+s)}.$$

4. Riemann-Roch.

Recall that the Chern character class $\hat{ch}(E, h)$ of an hermitian vector bundle on a variety over $\mathbf{Z}$ is compatible with the pull-back maps f^* (Theorem 3.2.ii)). The arithmetic Riemann-Roch theorem describes the behaviour of $\hat{ch}$ under the direct image map f_* . This is the hardest result of the theory. It was first shown for arithmetic surfaces by Faltings [38], and then extended to families of curves by Deligne [31]. Moret-Bailly obtained a precise Noether formula [75]. In arbitrary dimension, Gillet and the author got an exact Grothendieck-Riemann-Roch theorem in degree one [52], using a long paper by Bismut and Lebeau [16] as well as several

other papers [50] [8] [11] [12] [7] . In [40], Faltings provides another approach, valid in all degrees, with short cuts to simplify part of the analysis involved. The recent work of Bismut [9] makes the proof of [52] valid also in arbitrary degree.

Here we shall mainly focus on the arithmetic Riemann-Roch theorem in degree one, which computes the first Chern class of an hermitian line bundle called the determinant of cohomology.

4.1. The determinant of cohomology.

4.1.1. Let X and Y be regular quasi-projective flat schemes over $\mathbf{Z}$, $f : X \rightarrow Y$ a proper map from X to Y , and E an algebraic vector bundle on X . We may then define an algebraic line bundle $\lambda(E)$ on Y , called the *determinant of cohomology*. It is defined as follows [65]. Let $R^q f_* E$, $q \geq 0$, be the sheaf on Y attached to the presheaf

$$U \mapsto H^q(f^{-1}(U), E)$$

(coherent cohomology of $f^{-1}(U)$ with coefficients in the sheaf E). Its determinant $\det R^q f_* E$ is a line bundle on Y and, by definition,

$$\lambda(E) = \bigotimes_{q \geq 0} (\det R^q f_* E)^{(-1)^q}$$

where we denote by L^{-1} the dual of any line bundle L on Y . If $y \in Y$ is a point and if $X_y = f^{-1}(y)$ is the fiber of f over y there is a canonical isomorphism of $k(y)$ -vector spaces

$$\lambda(E)_y \simeq \bigotimes_{q \geq 0} \det H^q(X_y, E)^{(-1)^q}.$$

4.1.2. We want to define a metric on $\lambda(E)$. The holomorphic line bundle $\lambda(E)_{\mathbf{C}}$ attached to $\lambda(E)$ has fibers

$$\lambda(E)_y \simeq \bigotimes_{q \geq 0} \det H^q(X_y, E_{\mathbf{C}})^{(-1)^q},$$

for any complex point $y \in Y(\mathbf{C})$, with $X_y = f^{-1}(y)$ the fiber at y of $f_{\mathbf{C}} : X(\mathbf{C}) \xrightarrow{f} Y(\mathbf{C})$. We assume that $f_{\mathbf{C}}$ is smooth on $X(\mathbf{C})$. Therefore X_y is a smooth compact complex manifold, of dimension d say.

The cohomology groups $H^q(X_y, E_{\mathbf{C}})$ can be computed using the Dolbeault resolution

$$C^\infty(X_y, E_{\mathbf{C}}) \xrightarrow{\bar{\partial}} A^{01}(X_y, E_{\mathbf{C}}) \xrightarrow{\bar{\partial}} A^{02}(X_y, E_{\mathbf{C}}) \xrightarrow{\bar{\partial}} \dots$$

Assume we are given an hermitian metric h_f on the relative tangent bundle $Tf_{\mathbf{C}}$, i.e. an hermitian metric h_{X_y} on TX_y which depends smoothly on $y \in Y(\mathbf{C})$. To this metric is attached a Kähler form $\omega_0 \in A^{11}(X_y)$ by the formula

$$\omega_0 = \frac{i}{2\pi} \sum_{\alpha, \beta} h_{X_y} \left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta} \right) dz_\alpha d\bar{z}_\beta,$$

for any choice of an holomorphic chart (z_α) on X_y . We shall assume that h_{X_y} is Kähler, i.e. $d\omega_0 = 0$.

The space of forms of type $(0, q)$ with coefficients in $E_{\mathbf{C}}$ consists of C^∞ sections of $\Lambda^q(\overline{TX}_y^*) \otimes E_{\mathbf{C}}$, where TX_y^* , the dual of TX_y , are holomorphic differentials, and $\overline{TX}_y^*$ are the antiholomorphic differentials :

$$A^{0q}(X_y, E_{\mathbf{C}}) = C^\infty(X_y, \Lambda^q(\overline{TX}_y^*) \otimes E_{\mathbf{C}}).$$

Since we have metrics h_y on TX_y and h on $E_{\mathbf{C}}$ we get a (pointwise) scalar product $\langle \cdot, \cdot \rangle$ on each fiber of $\Lambda^q(\overline{TX}_y^*) \otimes E_{\mathbf{C}}$. The L^2 -metric on $A^{0q}(X_y, E_{\mathbf{C}})$ is defined by the formula :

$$h_{L^2}(s, t) = \int_{X_y} \langle s(x), t(x) \rangle \frac{\omega_0^d}{d!},$$

where $s, t \in A^{0q}(X_y, E_{\mathbf{C}})$.

The Cauchy-Riemann operator

$$\bar{\partial} = \bar{\partial}_{E_{\mathbf{C}}} : A^{0q}(X_y, E_{\mathbf{C}}) \rightarrow A^{0, q+1}(X_y, E_{\mathbf{C}})$$

has an adjoint $\bar{\partial}^* : h_{L^2}(s, \bar{\partial}t) = h_{L^2}(\bar{\partial}^*(s), t)$, and the Laplace operator is by definition

$$\Delta_q = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} \quad \text{on} \quad A^{0q}(X_y, E_{\mathbf{C}}).$$

There is a canonical isomorphism between the cohomology and harmonic forms :

$$H^q(X_y, E_{\mathbf{C}}) \simeq \text{Ker}(\Delta_q).$$

Therefore $H^q(X_y, E_{\mathbf{C}})$ inherits an L^2 -metric h_{L^2} , and so does the one dimensional vector space

$$\lambda(E)_y = \bigoplus_{q \geq 0} \Lambda^{\max} H^q(X_y, E_{\mathbf{C}})^{(-1)^q}.$$

4.1.3. In general, when y varies, this metric h_{L^2} on $\lambda(E)_y$ does *not* depend smoothly on y . For that reason, Quillen defined another metric as follows [87]. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$ be the positive eigenvalues of Δ_q , counted with multiplicities, and

$$(4.1) \quad \zeta_q(s) = \sum_{n \geq 1} \lambda_n^{-s}$$

the zeta function of Δ_q . It converges absolutely when $\text{Re}(s) > \dim_{\mathbf{C}}(X_y)$, and extends to a meromorphic function in $s \in \mathbf{C}$, which has no pole at $s = 0$.

The Ray-Singer *analytic torsion* of $(E_{\mathbf{C}}, h)$ is defined as

$$T(E_{\mathbf{C}}, h) = \exp \left(\frac{1}{2} \sum (-1)^{q+1} q \zeta_q'(0) \right),$$

and Quillen considered the metric

$$h_Q = h_{L^2} \cdot T(E_{\mathbf{C}}, h)^2$$

on $\lambda(E)_y$. It turns out that this metric varies smoothly with y , thus defining an hermitian metric h_Q on the line bundle $\lambda(E_{\mathbf{C}})$ [87], [10].

4.2. A new characteristic class.

Let $\zeta(s)$ be the Riemann zeta function, $\zeta'(s)$ its derivative, and $R(x)$ the formal power series

$$R(x) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} \left(2\zeta'(-m) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{m} \right) \zeta(-m) \right) \frac{x^m}{m!}.$$

For any holomorphic vector bundle E on a complex manifold X we define a characteristic class $R(E) \in \bigoplus_{p \geq 0} H^{pp}(X)$ by the conditions :

- i) $R(f^* E) = f^* R(E)$ for any map $f : Y \rightarrow X$ of complex manifolds ;
- ii) $R(E) = R(S) + R(Q)$ for any exact sequence $0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$;
- iii) If L is a line bundle, $R(L) = R(x)$, where $x = c_1(L) \in H^{1,1}(X)$ is the first Chern class of L .

4.3. An arithmetic Riemann-Roch theorem.

4.3.1. Let f be a projective morphism between regular, quasi-projective, flat schemes over $\mathbb{Z}$. Assume that f induces a smooth submersion $f_{\mathbb{C}} : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$. Choose an hermitian metric h_f on the relative tangent bundle $Tf_{\mathbb{C}}$ which, as in 4.1.2, induces a Kähler metric on each fiber X_y , $y \in Y(\mathbb{C})$. We can then define a Todd class

$$\widehat{\text{Td}}(f, h_f) \in \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

as follows. Consider a factorization of f

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ f \searrow & & \swarrow g \\ & Y & \end{array}$$

where $g : P \rightarrow Y$ is a projective bundle and i is a closed regular immersion. Choose arbitrary hermitian metrics on the normal bundle N of X in P and on the tangent bundle Tg to the smooth map g . On $X(\mathbb{C})$, there is an exact sequence of holomorphic vector bundles

$$\mathcal{E} : 0 \rightarrow Tf_{\mathbb{C}} \rightarrow i^* Tg_{\mathbb{C}} \rightarrow N_{\mathbb{C}} \rightarrow 0,$$

hence a Bott-Chern secondary characteristic class

$$\widetilde{\text{Td}}(\overline{\mathcal{E}}) \in \widetilde{A}(X)$$

such that

$$dd^c(\widetilde{\text{Td}}(\overline{\mathcal{E}})) = \text{Td}(\overline{Tf}_{\mathbb{C}}) \text{Td}(\overline{N}_{\mathbb{C}}) - \text{Td}(i^* \overline{Tg}_{\mathbb{C}})$$

(Theorem 3.1). It can be shown that

$$\widehat{\text{Td}}(f, h_f) = \widehat{\text{Td}}(i^* \overline{Tg}) \widehat{\text{Td}}(\overline{N}^{-1}) - a(\widetilde{\text{Td}}(\overline{\mathcal{E}}) \text{Td}(\overline{N}_{\mathbb{C}})^{-1})$$

does not depend on the choice of the factorization of f .

4.3.2. Given any hermitian vector bundle (E, h) on X , we let

$$\widehat{\text{ch}}(E, h) \in \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

be its Chern character class (Theorem 3.2).

Also we let $R(f_{\mathbf{C}}) \in \bigoplus_{p \geq 0} H^{pp}(X)$ be the R -class of $Tf_{\mathbf{C}}$ and $a(R(f_{\mathbf{C}}))$ its image by the composite of $\bigoplus_{p \geq 0} H^{pp}(X) \rightarrow \tilde{A}(X)$ with the map

$$a : \tilde{A}(X) \rightarrow \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(X)$$

defined in 2.2.3. Finally, given $\alpha \in \bigoplus_{p \geq 0} \widehat{\text{CH}}^p(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$, denote by $\alpha^{(1)} \in \widehat{\text{CH}}^1(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ its component of degree one.

We can now state an arithmetic Riemann-Roch theorem [52] :

THEOREM 4.1. *The following equality holds in $\widehat{\text{CH}}^1(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$:*

$$\widehat{c}_1(\lambda(E), h_Q) = f_* \left[\widehat{\text{ch}}(E, h) \widehat{\text{Td}}(f, h_f) (1 - a(R(f_{\mathbf{C}}))) \right]^{(1)}.$$

4.3.3. Theorem 4.1 can be made more explicit when $Y = \text{Spec}(\mathbf{Z})$, in which case it is an equality of real numbers. First

$$\chi_Q(E, h) = \widehat{c}_1(\lambda(E), h_Q) \in \widehat{\text{CH}}^1(\mathbf{Z}) = \mathbf{R}$$

can also be written

$$\chi_Q(E, h) = \sum_{q \geq 0} (-1)^q \left(\widehat{\deg} H^q(X, E)_{L^2} + \frac{q}{2} \zeta'_q(0) \right),$$

where $\widehat{\deg} H^q(X, E)_{L^2}$ is the degree of the finitely generated $\mathbf{Z}$ -module $H^q(X, E)$ equipped with the L^2 -metric on $H^q(X, E) \otimes_{\mathbf{Z}} \mathbf{C} = H^q(X(\mathbf{C}), E_{\mathbf{C}})$ (see 1.1.2). Then Theorem 4.1 says that

$$(4.2) \quad \chi_Q(E, h) = f_* (\widehat{\text{ch}}(E, h) \widehat{\text{Td}}(f, h_f))^{(1)} - \frac{1}{2} \int_{X(\mathbf{C})} \text{ch}(E_{\mathbf{C}}) \text{Td}(f_{\mathbf{C}}) R(f_{\mathbf{C}}),$$

where $\text{ch}(E_{\mathbf{C}})$ and $\text{Td}(f_{\mathbf{C}})$ are the usual Chern character and Todd classes.

4.4. Variants and complements.

4.4.1. Theorem 7 in [52] computes $\widehat{c}_1(\lambda(E), h_Q)$ for any projective map $f : X \rightarrow Y$ of quasi-projective flat schemes over $\mathbf{Z}$, when $X(\mathbf{C})$ and $Y(\mathbf{C})$ are smooth and $f_{\mathbf{C}}$ is a submersion and either f is a local complete intersection morphism or Y is regular.

4.4.2. Given X as above, consider the group $\widehat{K}_0(X)$ generated by pairs $(\overline{E}, \eta)$ where $\overline{E}$ is an hermitian vector bundle on X and $\eta \in \tilde{A}(X)$, submitted to the relations

$$(\overline{S}, \eta') + (\overline{Q}, \eta'') = (\overline{E}, \eta' + \eta'' + \tilde{\text{ch}}(\overline{\mathcal{E}}))$$

for any exact sequence

$$\mathcal{E} : 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

on X , with arbitrary metrics on S, E, Q .

Under the hypotheses of Theorem 4.1 there exists a direct image morphism

$$f_* : \widehat{K}_0(X) \rightarrow \widehat{K}_0(Y)$$

(see [50] [15] [40]) such that $\det(f_*(\overline{E})) \in \widehat{\text{Pic}}(Y)$ is the class of $(\lambda(E), h_Q)$. In higher degrees, the arithmetic Riemann-Roch theorem reads

$$\widehat{\text{ch}}(f_*(x)) = f_* \left(\widehat{\text{ch}}(x) \widehat{\text{Td}}(f, h_f)(1 - a(R(f\mathbf{c}))) \right)$$

for any x in $\widehat{K}_0(X)$ [40].

4.4.3. In [89] Roessler gives an arithmetic Riemann-Roch theorem for Adams operations. More extensions of the arithmetic Riemann-Roch theorem can be thought of, e.g. involving group actions or higher K -groups $\widehat{K}_m(X)$.

For families of curves, Theorem 4.1 computes exactly the Quillen norm of the Mumford's isomorphism

$$\lambda(\omega^{\otimes 2}) \simeq \lambda(\omega)^{\otimes 12},$$

where ω is the relative dualizing sheaf of the universal curve over $\mathbf{Z}$ [50, Prop. 1.5.2]. Bost used it to compare several notions of heights for abelian varieties [23]. We shall see more applications in the next sections. Enumerative formulae as in [5, Chapter VIII], or on moduli spaces, like Verlinde's formula, probably have arithmetic analogues, for which this theorem could be used.

4.4.5. A surprising feature is the appearance of the R -class in Theorem 4.1. This suggests that the occurrence of values of the Riemann zeta function in the classical Todd series :

$$\text{Td}(x) = \frac{x}{1 - e^{-x}} = 1 - \sum_{n \geq 1} \zeta(1 - n) \frac{x^n}{(n - 1)!}$$

has some arithmetic meaning.

Actually, three different kinds of zeta functions are involved in formula (4.2) : Riemann zeta function, the zeta function of Laplace operators (4.1), and, in some cases, Igusa zeta functions (3.4). An optimistic guess would be that these are part of some equality of zeta functions, from which (4.2) follows by taking derivatives at the origin.

5. Ampleness.

We shall now describe properties of hermitian vector bundles on arithmetic varieties which are parallel to those in algebraic geometry. First, we discuss the existence of small sections for powers of a given hermitian line bundle (5.1), and the arithmetic analogue of the Hilbert-Samuel polynomial. We then give a numerical criterion à la Nakai-Moishezon for the existence of a basis made of small global sections (Theorem 5.2). By duality, this implies an analogue of a vanishing theorem of Serre (Theorem 5.4). With the notion of ample hermitian line bundle available, it makes sense to ask for an arithmetic version of Grothendieck's standard conjectures (Conjecture 5.5). These hard questions led to several interesting recent developments.

5.1. Effectivity.

5.1.1. Let X be a regular projective flat scheme over $\mathbf{Z}$, and $\overline{L}$ an hermitian line bundle on X for which we make the following assumptions :

- 1) L is ample on X ;
- 2) the first Chern form $c_1(\overline{L}_{\mathbf{C}})$ is nonnegative on $X(\mathbf{C})$.

Let $\overline{E}$ be an arbitrary hermitian vector bundle of rank r over X and $Y \subset X$ a closed irreducible subscheme in X . Recall that we defined in 2.4.2 the height of Y with respect to $\overline{L}$, which is the real number

$$h_{\overline{L}}(Y) = \widehat{\deg}(\widehat{c}_1(\overline{L})^{\dim(Y)} \mid Y).$$

THEOREM 5.1. [47] [109] *Assume that $h_{\overline{L}}(Y)$ is positive. Then there exists n_0 such that, for all $n \geq n_0$, $E \otimes L^{\otimes n}$ has a nontrivial section $s \in H^0(Y, E \otimes L^{\otimes n})$ of sup norm bounded by one :*

$$\sup_{x \in Y(\mathbf{C})} \|s(x)\| \leq 1.$$

5.1.2. Let V be the covolume of $H^0(Y, E \otimes L^{\otimes n})$ in $H^0(Y, E \otimes L^{\otimes n}) \otimes_{\mathbf{Z}} \mathbf{R}$ for the Haar measure giving volume one to the unit ball of the sup norm. Theorem 5.1 follows from Minkowski's Theorem 1.1 and the following formula à la Hilbert-Samuel for the asymptotic value of V as n goes to infinity :

$$(5.1) \quad -\log V = rh_{\overline{L}}(Y) \frac{n^{\dim(Y)}}{\dim(Y)!} + O(n^{\dim(Y)-1} \log n),$$

where $\dim(Y)$ is the (absolute) Krull dimension of Y . When $Y = X$, (5.1) can be derived from the Riemann-Roch formula (4.2), an asymptotic evaluation of the analytic torsion $T(\overline{E}_{\mathbf{C}} \otimes \overline{L}_{\mathbf{C}}^{\otimes n})$ due to Bismut and Vasserot [17], and a comparison of the sup and L^2 norms due to Gromov ; see [47] and [52] for more details. In [109], Zhang deduced (5.1) in general from the case $Y = X$.

5.1.3. Notice that the proof of (5.1) uses Riemann-Roch only asymptotically as n goes to infinity, and does not need the precise equality of Theorem 4.1 [47] [96]. When $E = \mathcal{O}_X$ and $Y = X$, Abbès and Bouche gave recently a more direct proof of (5.1) which does not use Riemann-Roch [1].

Note also that (5.1) is a special case of a general theorem in the adelic setting due to Lau, Rumely and Varley [71]. Let K be a number field. For any place v of K , denote by K_v its completion, and by $\mathbb{C}_v$ the completion of the algebraic closure of K_v . Let X be a projective variety over K and L an ample line bundle on X . Assume that, given any place v and any point x in $X(\mathbb{C}_v)$, the stalk $L_x \simeq \mathbb{C}_v$ is equipped with a norm $\|\cdot\|_v$, which is invariant under $\text{Gal}(\mathbb{C}_v/K_v)$. Under a weak boundedness assumption on these norms, it is shown in [71] that the covolume of $H^0(X, L^{\otimes n})$ in its adelic completion $H^0(X, L^{\otimes n}) \otimes_K \mathbf{A}$ grows as $\exp(-A n^{\dim(X)+1})$ as n goes to infinity.

5.2. Ampleness.

5.2.1. We keep the notation of 5.1.1. Using Theorem 5.1., geometry of numbers, L^2 -estimates and a clever induction on dimensions, Zhang proved the following :

THEOREM 5.2. [109] *Assume that, for all irreducible closed subsets $Y' \subset Y$, the inequality $h_{\overline{L}}(Y') > 0$ holds. Then there exists n_0 such that, for all $n \geq n_0$, the lattice $H^0(Y, E \otimes L^{\otimes n})$ has a basis made of sections s such that*

$$\sup_{x \in Y(\mathbb{C})} \|s(x)\| \leq 1.$$

5.2.2. We shall say that $\overline{L}$ is *ample* on Y when the hypothesis of Theorem 5.2 holds. We view this result as an arithmetic analogue of the criterion of Nakai-Moishezon. Note the following fact :

PROPOSITION 5.3. *Given any $\overline{L} = (L, h)$ as in 5.1.1, there exists a positive real number $t_0 > 0$ such that, if $t < t_0$, the hermitian line bundle $(L, t h)$ is ample on X .*

We refer to [109, Theorem 5.2], for a stronger statement and to [110] for an adelic variant of Theorem 5.2.

5.2.3. From Theorem 5.2, Serre's duality and Mahler's Theorem 1.3 one gets the following vanishing theorem. Fix a Kähler metric on $X(\mathbb{C})$, invariant under complex conjugation. Let d be the relative dimension of X over $\mathbb{Z}$.

THEOREM 5.4. [44] *When n is big enough, any cohomology class in $H^d(X, E \otimes L^{\otimes n})$ of L^2 -norm less than one vanishes.*

There is probably also a good theory of ample hermitian vector bundles of higher rank.

5.3. Standard conjectures.

5.3.1. Let X be a regular, projective, flat scheme over $\mathbb{Z}$ and $\overline{H}$ an ample hermitian line bundle on X , in the sense of 5.2. For any integer $p \geq 0$, denote by $\widehat{\text{CH}}^p(X)_{\mathbb{R}}$ the real vector space generated by pairs (Z, g) where Z is a codimension p algebraic cycle with *real* coefficients on X and g a Green current for Z , with the

same relations as those defining $\widehat{\mathrm{CH}}^p(X)$ (cf. 2.2.1). The product by the first Chern class $\widehat{c}_1(\overline{H}) \in \widehat{\mathrm{CH}}^1(X)$ defines a Lefschetz operator

$$L : \widehat{\mathrm{CH}}^p(X)_{\mathbb{R}} \rightarrow \widehat{\mathrm{CH}}^{p+1}(X)_{\mathbb{R}}$$

for all $p \geq 0$. Let $d = \dim(X/\mathbb{Z})$ be the relative dimension of X over $\mathbb{Z}$. Recall from 2.3.4 and 2.4.1 the existence of a degree map

$$f_* : \widehat{\mathrm{CH}}^{d+1}(X)_{\mathbb{R}} \rightarrow \mathbb{R}.$$

CONJECTURE 5.5. *Assume that $2p \leq d+1$.*

i) *(Hard Lefschetz) The map*

$$L^{d+1-2p} : \widehat{\mathrm{CH}}^p(X)_{\mathbb{R}} \rightarrow \widehat{\mathrm{CH}}^{d+1-p}(X)_{\mathbb{R}}$$

is an isomorphism ;

ii) *(Hodge index) If $x \in \widehat{\mathrm{CH}}^p(X)_{\mathbb{R}}$ is such that $x \neq 0$ and $L^{d+2-p}(x) = 0$, then*

$$(-1)^p f_*(x L^{d+1-2p}(x)) > 0.$$

Notice an immediate consequence of this conjecture : the intersection pairing

$$\widehat{\mathrm{CH}}^p(X)_{\mathbb{R}} \otimes \widehat{\mathrm{CH}}^{d+1-p}(X)_{\mathbb{R}} \rightarrow \mathbb{R}$$

is nondegenerate.

5.3.2. A weaker form of Conjecture 5.5 is proposed in [53] : given any ample line bundle H on X , there exists a positive metric on H such that i) and ii) hold true.

When $d = 1$, Conjecture 5.5 is a consequence of the Hodge index theorem for arithmetic surfaces due to Hiriljac [62] and Faltings [38]. This was extended to higher dimensions by Moriwaki, who proved Conjecture 5.5 ii) when $p = 1$ [79, Theorem 1.1].

Künneman proved general facts on Conjecture 5.5 [66] [67]. In [68] he combines the theory of intermediate Jacobians, Fourier transforms and the arithmetic intersection theory to obtain statements i) and ii) for the group of algebraically trivial algebraic cycles on any abelian variety over a number field, modulo the incidence equivalence relation.

When X is a Grassmannian manifold and H is any ample line bundle on X , Künneman and Maillot proved that H has some positive metric for which Conjecture 5.5 holds (unpublished). A more precise statement seems difficult to obtain : even in this simple case, it is not easy to describe the cone of ample hermitian line bundles on X .

Recently Zhang extended the work of Gross and Zagier on Heegner points to modular forms f of weight $2k$, with k bigger than one [112]. He showed that Conjecture 5.5 ii) (and some additional assumption), when applied to Heegner

cycles, says that the derivative of the L-function of f at $s = k$ is nonnegative, as is also predicted by the generalized Riemann hypothesis.

5.3.3. The original standard conjectures of Grothendieck [58] deal with the group $A^p(Y)$ of codimension p algebraic cycles modulo homological equivalence on a smooth projective variety Y over an algebraically closed field k . Lefschetz pencils are expected to play a role there. Note that, when such a Lefschetz pencil is given, the variety Y (or rather its blow up along the base of the pencil) appears as a flat family $Y \rightarrow \mathbf{P}^1$, with finitely many singular fibers. Let $\mathcal{M}$ be the category of all smooth varieties $Y' \rightarrow Y$ obtained by a finite sequence of blow-ups with smooth centers contained in one of these singular fibers. It is shown in [13] that, at least when $\text{char}(k) = 0$, the groups

$$\widehat{A}^p(Y) = \varinjlim_{\mathcal{M}} A^p(Y'), \quad p \geq 0,$$

enjoy properties very similar to the groups $\widehat{\text{CH}}^p(X)$ in the arithmetic case, thus making the analogy between function fields and number fields (see 1.3.3) more complete [13] [14]. It is still an open problem to formulate and to prove a Grothendieck-Riemann-Roch theorem in that context.

6. Stability.

Several attempts were made in recent years to apply the techniques of Mumford's geometric invariant theory in the context of Arakelov theory. Semi-stable hermitian vector bundles satisfy an inequality à la Bogomolov-Gieseker (Theorem 6.1), which implies a Kodaira vanishing theorem on arithmetic surfaces (Theorem 6.2). Semi-stable projective varieties have their height bounded below (Theorem 6.3) and, in some cases, more information on the successive minima of lattices of global sections can be obtained (Theorem 6.4). We conclude with a short summary of the number theoretic consequences of Arakelov theory.

6.1. Semi-stable hermitian vector bundles.

6.1.1. Let $f : X \rightarrow \text{Spec}(\mathbf{Z})$ be a regular projective flat scheme over $\mathbf{Z}$, $\overline{H}$ an ample hermitian line bundle over X , in the sense of 5.2.2, and $\overline{E} = (E, h)$ an hermitian vector bundle of rank $r \geq 1$ over X . Consider the restriction $E_{\overline{\mathbf{Q}}}$ of E to the geometric generic fiber $X_{\overline{\mathbf{Q}}} = X \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ of X over $\text{Spec}(\mathbf{Z})$. Assume that $E_{\overline{\mathbf{Q}}}$ is $H_{\overline{\mathbf{Q}}}$ -semi-stable, i.e. that, for any subsheaf F of $E_{\overline{\mathbf{Q}}}$ of positive rank, the following inequality holds

$$\frac{c_1(F) \cdot c_1(H_{\overline{\mathbf{Q}}})^{d-1}}{\text{rk}(F)} \leq \frac{c_1(E_{\overline{\mathbf{Q}}}) \cdot c_1(H_{\overline{\mathbf{Q}}})^{d-1}}{\text{rk}(E)},$$

where $d = \dim(X/\mathbf{Z})$ and $c_1(F) \cdot c_1(H_{\overline{\mathbf{Q}}})^{d-1} \in \mathbf{Z}$ is the usual degree of F over $X_{\overline{\mathbf{Q}}}$. The following result is due to Miyaoka [74] when $d = 1$ and the metric h on E is Einstein-hermitian, and to Moriwaki [77] in general :

THEOREM 6.1. *Under the above assumptions, the following inequality of real numbers holds :*

$$(r-1) f_*(\widehat{c}_1(\overline{E}))^2 \widehat{c}_1(\overline{H})^{d-1} \leq 2r f_*(\widehat{c}_2(\overline{E})) \widehat{c}_1(\overline{H})^{d-1}.$$

6.1.2. In his proof of Theorem 6.1, in the case $d = 1$ and h is Einstein-hermitian, Miyaoka applied the arithmetic Riemann-Roch formula (4.2) to $F_r = S^r(E) \otimes \det(E)^{-n}$, where S^k is the k -th symmetric power. When n goes to infinity, this formula implies that the leading coefficient of $\chi_Q(\overline{F}_n)$ is proportional to the degree of $\widehat{c}_2(\overline{E}) - \frac{r-1}{2r} \widehat{c}_1(\overline{E})^2$. On the other hand, a result of Selberg shows that the analytic torsion $T(\overline{F}_{n,\mathbf{C}})$ does not grow very fast with n , and a direct argument evaluates the growth of $\deg H^q(X, \overline{F}_n)$, $q = 0, 1$, thus proving the desired inequality (see [94, Theorem 1] for details).

Moriwaki gets the same result by studying the ampleness of an appropriate line bundle on the projective space $\mathbf{P}(E)$ (a variant of [17] is needed to keep track of analytic torsion). When $d = 1$ and h is arbitrary, he uses the theorem of Narasimhan and Seshadri [83] as well as a functional of Donaldson [32] to deduce Theorem 6.1 from the case where h is Einstein-hermitian. When $d > 1$, Zhang's Theorem 5.2 makes it possible to proceed by induction on d (note that the restriction of $E_{\overline{\mathbf{Q}}}$ to a general hyperplane section is still semi-stable).

In [78], Moriwaki proves an analogue for arithmetic surfaces of the Bogomolov's unstability theorem.

6.2. Vanishing.

6.2.1. We keep the notation of 6.1.1. Let $e \in H^1(X, L^{-1})$ be an extension class of L by the trivial line bundle on X . On $X(\mathbf{C})$ consider the Kähler metric such that its associated Kähler form ω_0 is proportional to $c_1(\overline{L}_{\mathbf{C}})$ and each component of $X(\mathbf{C})$ has volume one. Denote by $\|e\|$ the L^2 -norm of e for this choice of metrics on $L_{\mathbf{C}}$ and $X(\mathbf{C})$.

THEOREM 6.2. *Assume $d = 1$. If $e \in H^1(X, L^{-1})$ has infinite order, the following inequality holds :*

$$\|e\| \geq \exp(-1).$$

.

6.2.2. Theorem 6.2 is proved in [94] by adapting an argument of Mumford [88]. If E is an extension of L by $\mathcal{O}_X$ classified by e , one distinguishes two cases according to whether or not $E_{\mathbf{C}}$ has an Einstein-hermitian metric. If it does, the inequality of Theorem 6.1 is shown to imply the result. If it does not, one uses the Hodge index theorem on arithmetic surfaces (see 5.3.2).

6.2.3. Assume that X is a semi-stable arithmetic surface over $\text{Spec}(\mathcal{O}_K)$, where K is a number field, and let ω_X be its relative dualizing sheaf. Using Serre's duality and the result in §1.2.3 above, one can derive from Theorem 6.2 an explicit integer n_0 such that, if $n \geq n_0$, the lattice of global sections of $\omega_X^{\otimes n}$ has a basis made of sections of L^2 -norm less than one ([95], Theorem 5).

6.2.4. One may ask if Theorem 6.2 remains true when $d > 1$ (probably with a different lower bound for $\|e\|$), since the Kodaira vanishing theorem holds in arbitrary dimension. Nakashima and Takeda proved this in some cases [82].

In general, this might be a difficult problem because, in the geometric case, the proof of vanishing theorems in higher dimension requires the use of Hodge cohomology (as in [35] for instance), and today there is no such cohomology in the arithmetic set-up.

6.3. Stability of projective varieties.

6.3.1. Let K be a number field, $\overline{K}$ an algebraic closure of K , (E, h) an hermitian vector bundle of rank N on $\text{Spec}(\mathcal{O}_K)$, $\mathbf{P}(E^*)$ the projective space representing lines in the dual module $E^* = \text{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$, and $\mathbf{P}(E_K^*)$ its generic fiber. The canonical line bundle $\mathcal{O}(1)$ on $\mathbf{P}(E^*)$, is a quotient of the constant bundle E , therefore h induces a metric on $\mathcal{O}(1)$.

Given any projective variety $X_K \subset \mathbf{P}(E_K^*)$ of dimension n and degree d , we let X be the Zariski closure of X_K in $\mathbf{P}(E^*)$, and we denote by $h(X_K)$ the height of X , defined as in 2.4.2 :

$$h(X_K) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}(1)})^{n+1} | X).$$

The projective variety $X_{\overline{K}} = X_K \otimes_{\overline{K}}$ is called *Chow semi-stable* when the corresponding Chow point in $\mathbf{P}((S^d E_{\overline{K}}^*)^{\otimes n+1})$ is semi-stable under the action of $\text{SL}_N(\overline{K})$ [81].

THEOREM 6.3. [22] [95] [111] *If $X_{\overline{K}}$ is Chow semi-stable the following inequality holds :*

$$\frac{h(X_K)}{d(n+1)} - \frac{\widehat{\deg}(E, h)}{N} \geq -\frac{[K : \mathbb{Q}]}{2} \sum_{k=1}^{N-1} \sum_{m=1}^k \frac{1}{m}.$$

6.3.2. One method to prove Theorem 6.3 ([22, Theorem 2]; [111, Theorem 4.4] gives the best constant) consists in choosing elements $v_1, \dots, v_N$ in E which form a basis in E_K and such that $\|v_i\| = \lambda_i$, $i = 1, \dots, N$, are the successive minima of (E, h) (as in 1.1.3). By Mumford's numerical criterion for stability [81, Theorem 2.9], when m is big enough one can choose a basis of $H^0(X_K, \mathcal{O}(m))$ made of monomials $v_{i_1} \otimes \dots \otimes v_{i_m}$ of small norm. Then, by applying Theorem 1.2 and the asymptotic estimate (5.1) one gets a lower bound for $h(X_K)$.

When Mumford's criterion has been checked effectively, this proof gives more information : it provides an upper bound for the difference $\mu_1 - \mu_N$, where $\mu_i = -\log(\lambda_i)$. In other words, the semi-stability of the geometric generic fiber $X_{\overline{K}}$ implies the arithmetic semi-stability, in the sense of 1.2.5, of a lattice of global sections on X . For example, Gieseker's proof of the stability of surfaces of general type [45] implies the following :

THEOREM 6.4. [95, Theorem 3] *Let Y be a smooth surface of general type over K and L a high enough power of the canonical sheaf on Y . For any choice of an hermitian vector bundle (E, h) on $\text{Spec}(\mathcal{O}_K)$ such that $E_K = H^0(Y, L)$, the image X_K of Y in $\mathbf{P}(H^0(Y, L)^*)$ satisfies the following inequality*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} - 3 \frac{\mu_1 + \dots + \mu_N}{N} \geq \frac{d}{N} (\mu_1 - \mu_N).$$

6.3.3. When X_K is a smooth projective curve, [95, Theorem 4], or when X_K is an abelian variety [24], one can also prove the semi-stability of lattices of global

sections on X . It would be valuable to get more results of this kind, e.g. by checking effectively Mumford's numerical criterion for the projective stability of $X_{\overline{K}}$.

6.4. Applications to number theory, a short survey.

6.4.1. The proof by Faltings of Mordell's conjecture [37] made little use of Arakelov theory, except in the definition of the height of an abelian variety. Vojta's second proof of that result [104], on the other hand, relied on the arithmetic Riemann-Roch formula (4.2) to obtain a bounded section $s \neq 0$ of a big power of a line bundle on a regular model W over $\mathrm{Spec}(\mathcal{O}_K)$ of the product $C \times C$ of two copies of a given curve C over K , with genus ≥ 2 . If P and Q are two points in $C(K)$, Vojta considers the restriction of the section s , and of some partial derivatives of s , to the Zariski closure of (P, Q) inside W . Unless the heights of P and Q on C are comparable in size, these restrictions are so small that they have to vanish, i.e. s vanishes to high order at the point (P, Q) . By Dyson's lemma on $C \times C$ [103] this would force s to vanish identically on W . As a consequence, given P in $C(K)$, the height of Q must be bounded by a constant multiple of the height of P , and hence $C(K)$ is finite. Note that this proof is ineffective, since it does not give any explicit bound for the height of P .

This scheme of proof is inspired by the general method of diophantine approximation. It was simplified and generalized by Faltings [39], Vojta, and McQuillan to prove a conjecture of Lang on rational points on (semi-)abelian varieties. In [39] the arithmetic Riemann-Roch theorem is replaced by simpler arguments, but the Arakelov theory of projective heights, as in 6.4.2, is of some use; see also [105], [41] and [42]. Bombieri simplified further Vojta's proof of Mordell's conjecture in [18], where no Arakelov theory is needed.

Note that these works make an essential use of the Mordell-Weil theorem, which states that the group of rational points of any abelian variety over K is finitely generated. This is one more reason which renders ineffective this method, and makes it hard to extend to other varieties than curves and abelian varieties.

6.4.2. The work of Masser and Wüstholz on isogenies of abelian varieties is a striking development of Baker's method in transcendental number theory. Bost showed in [24] how to use Arakelov theory to improve their results and to give a more conceptual approach to the method; [24] uses both the arithmetic Riemann-Roch Theorem 4.1 and the arithmetic stability theory (1.2.5 and 6.3.3).

6.4.3. Let X_K be a smooth projective curve over a number field K , and X a semi-stable model of X_K over $S = \mathrm{Spec}(\mathcal{O}_K)$. We endow the relative dualizing sheaf ω_X with its Arakelov metric, which is the unique metric for which the "adjunction formula" holds [3] [69]. Much effort has been devoted to compute the following number

$$\omega^2 = \widehat{\deg}(\widehat{c}_1(\overline{\omega}_X)^2 | X) / [K : \mathbb{Q}].$$

This number depends only on $X_{\overline{K}}$ and it is known that $\omega^2 \geq 0$ [38]. Assuming $\omega^2 > 0$, Szpiro [99] and Zhang [107] proved a conjecture of Bogomolov on points of small Néron-Tate height in the Jacobian of X_K . Several *lower bounds* for ω^2 are known today [107] [28] [22] [80], e.g. $\omega^2 > 0$ when X is not smooth over S .

Upper bounds for ω^2 are known in genus two [20] and more generally for hyperelliptic curves [63] (see also [2] for the case of modular curves). Finding good

general upper bounds for ω^2 is highly desirable since it has been shown that this would imply an effective version of Mordell's conjecture [85] [76], namely : given any smooth projective curve C of genus ≥ 2 over $\overline{\mathbb{Q}}$ there are positive constants α and β such that, if $K \subset \overline{\mathbb{Q}}$ is a number field and $P \in C(K)$ is a rational point, its canonical height $h_{\overline{\mathbb{Q}}}(P)$ satisfies the following inequality :

$$(6.1) \quad h_{\overline{\mathbb{Q}}}(P) \leq \alpha \log |D_K| + \beta[K : \mathbb{Q}].$$

If (6.1) held true for a single curve C as above, it would imply the famous *abc*-conjecture : given three integers a, b, c such that $a + b = c$, then

$$(6.2) \quad \sup(|a|, |b|, |c|) \leq A(abc)^B.$$

The value of the constant A would depend on C , α , and β , when B would depend only on C and α (one expects that, given any $\varepsilon > 0$, (6.2) is true with $B = 1 + \varepsilon$ and some constant $A = A(\varepsilon)$) [76][33]. The *abc*-conjecture itself has many beautiful consequences [102] [100] [33] [55].

6.4.3. For X_K of higher dimension, Lang [70] and Vojta [102] have proposed very strong conjectures, among which the following one : if X_K is a surface of general type, its set of rational points is not Zariski dense in X_K . Whether a statement like Theorem 6.4 has anything to do with such a problem remains to be seen.

References

- [1] Abbès A., Bouche T. : Théorème de Hilbert-Samuel "arithmétique", *Ann. Inst. Fourier, Grenoble* **45** (1995), 375-401.
- [2] Abbès A., Ullmo U. : Self-intersection du dualisant relatif des courbes modulaires $X_0(N)$, *preprint Orsay* (1994), 83 pp.
- [3] Arakelov S.J. : Intersection theory of divisors on an arithmetic surface, *Math. USSR Izvestiya* **8** (1974), 1167-1180.
- [4] Arakelov S.J. : Theory of intersections on an arithmetic surface, *Proc. Int. Congr. of Mathematicians* **1** Vancouver (1975), 405-408.
- [5] Arbarello E., Cornalba M., Griffiths P.A., Harris J. : Geometry of Algebraic Curves, Vol.I, *Grundlehren* **267** (1985), Springer-Verlag.
- [6] Banaszczyk W. : New bounds in some transference theorems in the geometry of numbers, *Math. Annalen* **296** (1993), 625-635.
- [7] Bismut J.-M. : Koszul complexes, harmonic oscillators and the Todd class, *Journal of the AMS* **3** (1990), 159-256.
- [8] Bismut J.-M. : Superconnection currents and complex immersions, *Invent. Math.* **99** 1 (1990), 59-113.
- [9] Bismut J.-M. : Familles d'immersions et formes de torsion analytique en degré supérieur, *C.R. Acad. Sci. Paris* **320** I (1995), 969-969.
- [10] Bismut J.-M., Gillet H., Soulé C. : Analytic torsion and holomorphic determinant bundles I, II, III, *Comm. Math. Physics* **115** (1988), 49-78, 79-126, 301-351.
- [11] Bismut J.-M., Gillet H., Soulé C. : Bott-Chern currents and complex immersions, *Duke Journal* **60** (1990), 255-284.

- [12] Bismut J.-M., Gillet H., Soulé C. : Complex immersions and Arakelov geometry, *Grothendieck Festschrift I* (1990), Birkhäuser, 249-331.
- [13] Bloch S., Gillet H., Soulé C. : Non archimedean Arakelov theory, *Journal of Alg. Geometry* **5** (1995), 427-485.
- [14] Bloch S., Gillet H., Soulé C. : Algebraic cycles on degenerate fibers, *preprint IHES* (1995), 25 pages.
- [15] Bismut J.-M., Köhler K. : Higher analytic torsion forms for direct images and anomaly formulas, *Journal of Alg. Geometry* **1** (1992), 647-684.
- [16] Bismut J.-M., Lebeau G. : Complex immersions and Quillen metrics, *Publ. Math. IHES* **74** (1991), 1-298.
- [17] Bismut J.-M., Vasserot E. : The asymptotic of the Ray-Singer analytic torsion associated with high powers of a positive line bundle, *Commun. Math. Phys.* **125** (1989), 355-367.
- [18] Bombieri E. : The Mordell conjecture revisited, *Ann. Sc. Norm. Sup. Pisa, Cl. Sci.* IV **17** (1990), 615-640.
- [19] Bombieri E., Vaaler J. : On Siegel's lemma, *Invent. Math.* **73** (1983), 11-32.
- [20] Bost J.-B. : Fonctions de Green-Arakelov, fonctions θ et courbes de genre 2, *C.R. Acad. Sci. Paris* **305 I** (1987), 643-646.
- [21] Bost J.-B. : Théorie de l'intersection et théorème de Riemann-Roch arithmétiques, Séminaire Bourbaki n^o731, 1990/1991, *Astérisque* **201-203** (1991), 43-88.
- [22] Bost J.-B. : Semi-stability and heights of cycles, *Invent. Math.* **118** (1994), 223-253.
- [23] Bost J.-B. : Intrinsic heights of stable varieties - Application to abelian varieties, *preprint IHES* (1994), 53 pp.
- [24] Bost J.-B. : Périodes et isogénies des variétés abéliennes sur les corps de nombres [d'après D. Masser et G. Wüstholz], *Séminaire Bourbaki* n^o795 (1995), 43 pp.
- [25] Bost J.-B., Gillet H., Soulé C. : Heights of projective varieties and positive Green forms, *Journal of the AMS* **7** (1994), 903-1027.
- [26] Bott R., Chern S.S. : Hermitian vector bundles and the equidistribution of zeroes of their holomorphic sections, *Acta Math.* **114** (1968), 71-112.
- [27] Burgos J.I. : Green forms and their product, *Duke Math. Journal* **75** (1994), 529-574.
- [28] Burnol J.-F. : Weierstrass points on arithmetic surfaces, *Invent. Math.* **107** (1992), 421-432.
- [29] Cassaigne J., Maillot V. : Hauteurs des hypersurfaces et fonctions zêta d'Igusa, *preprint LMENS* **94-5** (1994), 24 pp.
- [30] Chinburg T., Rumely R. : The capacity pairing, *J. Reine Angew. Math.* **434** (1993), 1-44.
- [31] Deligne P. : Le déterminant de la cohomologie, Current Trends in Arithmetical Algebraic Geometry, *Contemporary Math.* **67** (1987), 93-178.
- [32] Donaldson S.K. : Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London. Math. Soc.* **3** **50** (1985), 1-26.
- [33] Elkies N.D. : ABC implies Mordell, *Int. Math. Research Notices* **7** (1991), 99-109.

- [34] Elkik R. : Métriques sur les fibrés d'intersection, *Duke Math. Journal* **61** (1990), 303-328.
- [35] Esnault H., Viehweg E. : Lectures on Vanishing Theorems, *DMV Seminar Band 20* (1992), Birkhäuser.
- [36] Evertse J.-H. : An explicit version of Faltings' Product Theorem and an improvement of Roth's lemma, *Acta Arithmetica* **73** (1995), 215-248.
- [37] Faltings G. : Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), 349-366.
- [38] Faltings G. : Calculus on arithmetic surfaces, *Ann. of Math.* **119** (1984), 387-424.
- [39] Faltings G. : Diophantine approximation on Abelian varieties, *Ann. of Math.* **133** (1991), 549-576.
- [40] Faltings G. : Lectures on the arithmetic Riemann-Roch theorem, notes by S. Zhang, *Annals of Math. Study* **127** (1992).
- [41] Faltings G., Wüstholz G. : Diophantine approximations on projective spaces, *Invent. Math.* **116** (1994), 109-138.
- [42] Ferretti R. : An effective version of Faltings' product theorem, *preprint ETH* (1994), 27 pp.
- [43] Fulton W. : *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, Band 2, Springer-Verlag, Berlin-Heidelberg-New York (1984).
- [44] Gasbarri C. : Le théorème d'annulation de Serre pour les variétés arithmétiques, *preprint Orsay* (1995), 16 pp.
- [45] Gieseker D. : Global moduli for surfaces of general type, *Inventiones Math.* **43** (1977), 233-282.
- [46] Gillet H., Soulé C. : Intersection theory using Adams operations, *Invent. Math.* **90** (1987), 243-277.
- [47] Gillet H., Soulé C. : Amplitude arithmétique, *C. R. Acad. Sci. Paris* **307** Série I (1988), 887-890.
- [48] Gillet H., Soulé C. : Arithmetic intersection theory, *Publ. Math. IHES* **72** (1990), 94-174.
- [49] Gillet H., Soulé C. : Characteristic classes for algebraic vector bundles with Hermitian metric, I, II, *Annals of Math.* **131** (1990), 163-203.
- [50] Gillet H., Soulé C. : Analytic torsion and the arithmetic Todd genus, with an Appendix by D. Zagier, *Topology* **30** 1 (1991), 21-54.
- [51] Gillet H., Soulé C. : On the number of lattice point in convex symmetric bodies and their duals, *Israel Journal of Math.* **74** (1991), 347-357.
- [52] Gillet H., Soulé C. : An arithmetic Riemann-Roch Theorem, *Invent. Math.* **110** (1992), 473-543.
- [53] Gillet H., Soulé C. : Arithmetic analogs of the standard conjectures, in Jannsen U., Kleiman S.L., Serre J.-P. : Proceedings of the conference on motives, Seattle 1991, *Proc. Symp. in Pure Maths.* **55** (1994), 129-140.
- [54] Gillet H., Soulé C. : Descent, motives and K-theory, *J. Reine Angew. Math.*, to appear.

- [55] Granville A., Stark H.M. : ABC implies no “Siegel zeroes”, *preprint U. of Georgia* (1994), 6 pp.
- [56] Grayson D.R. : Reduction theory using semistability, *Comment. Math. Helvetici* **59** (1984), 600-634.
- [57] Griffiths P., Harris J. : *Principles of Algebraic Geometry*, John Wiley and Sons (1978).
- [58] Grothendieck A. : Standard conjectures on algebraic cycles, in *Algebraic Geometry*, Bombay 1968, Oxford University Press, Oxford (1969), 193-199.
- [59] Gruber P.M., Lekkerkerker : *Geometry of numbers*, North-Holland (1987).
- [60] Gubler W. : Höhentheorie , *Math. Ann.* **298** (1994), 427-456.
- [61] Guillen F., Navarro Aznar V. : Un critère d’extension d’un foncteur défini sur les schémas lisses , *Preprint Barcelona* (1995), 37 pp.
- [62] Hriljac P. : Heights and Arakelov’s Intersection Theory, *Am. J. Math.* **107** (1985), 23-38.
- [63] Kausz I. : Eine Abschätzung der Selbstschrittzahl des kanonischen Divisors auf arithmetischen Flächen mit hyperelliptischer generischer Faser, *preprint U. Köln* (1995), 80 pp.
- [64] Kleiman S.L., Thorup A. : Intersection theory and enumerative geometry : a decade in review §3, *Proc. of Symp. in Pure Math.* **46** (1987), 332-338.
- [65] Knudsen F.F., Mumford D. : The projectivity of the moduli space of stable curves I : Preliminaries on “det” and “div”, *Math. Scand.* **39** (1976), 19-55.
- [66] Künneman K. : Arakelov Chow groups of abelian schemes, arithmetic Fourier transform, and analogues of the standard conjectures of Lefschetz type, *Math. Annalen* **300** (1994), 365-392.
- [67] Künneman K. : Some remarks on the arithmetic Hodge index theorem, *Compositio Math.* **99** (1995), 109-128.
- [68] Künneman K. : Higher Picard varieties and the height pairing, *preprint Münster* (1995), 14 pp.
- [69] Lang S. : *Introduction to Arakelov Theory*, Springer-Verlag, Berlin-Heidelberg-New York (1988).
- [70] Lang S. : Hyperbolic and diophantine analysis, *Bull. AMS* **14** n°2 (1986), 159-205.
- [71] Lau C.F., Rumely R., Varley R. : Existence of the Sectional Capacity, *preprint U. of Georgia* (1993), 84 pp.
- [72] Levine H. : A theorem on holomorphic mappings into complex projective space, *Ann. of Math.* **71** (1960), 529-535.
- [73] Maillot V. : Un calcul de Schubert arithmétique, *Duke Math. J.* **80** (1995), 195-221.
- [74] Miyaoka Y. : Bogomolov inequality on arithmetic surfaces, talk at the Oberwolfach conference on “Arithmetical Algebraic Geometry”, G. Harder and N. Katz org. (1988)
- [75] Moret-Bailly L. : La formule de Noëther pour les surfaces arithmétiques, *Invent. Math.* **98** (1989), 491-498.
- [76] Moret-Bailly L. : Hauteurs et classes de Chern sur les surfaces arithmétiques, in [95], 37-58.

- [77] Moriwaki A. : Inequality of Bogomolov-Gieseker's type on arithmetic surfaces, *Duke Math. Journal* **74** (1994), 713-761.
- [78] Moriwaki A. : Bogomolov unstability on arithmetic surfaces, *Mathematical Research Letters* **1** (1995), 601-611.
- [79] Moriwaki A. : Hodge index theorem for arithmetic cycles of codimension one , *preprint UCLA* (1994), 9 pp.
- [80] Moriwaki A. : Self-intersection of dualizing sheaves of arithmetic surfaces with reducible fibers, *preprint UCLA* (1995), 8 pp.
- [81] Mumford D. : Stability of projective varieties, *l'Ens. Math.* **23** (1977), 39-110.
- [82] Nakashima T., Takeda Y. : A vanishing theorem for vector bundles on arithmetic varieties, *Preprint Tokyo Metropolitan U.* (1994), 14 pp.
- [83] Narasimhan M.S., Seshadri C.S. : Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540-567.
- [84] Nesterenko Yu.V. : Estimates for the orders of zeros of functions of a certain class and their applications in the theory of transcendental numbers, *Izv. Akad. Nauk. USSR ser. Math.* **41** (1977) = *Math. USSR Izv.* **11** (1977), 239-270.
- [85] Parshin, A.N. : The Bogomolov-Yau-Miyaoka inequality for arithmetical surfaces and its applications, in *Séminaire de théorie des nombres de Paris*, *Progr. Maths.* **75** (1988), Basel Boston Birkhäuser, 299-312.
- [86] Philippon P. : Critères pour l'indépendance algébrique, *Publ. Math. IHES* **64** (1986), 5-52.
- [87] Quillen D. : Determinants of Cauchy-Riemann operators over a Riemann surface, *Funct. Anal. Appl.* (1985), 31-34.
- [88] Reid M. : Bogomolov's theorem $c_1^2 \leq 4c_2$, *International Symposium on Algebraic Geometry*, Tokyo (1977), 623-642. p
- [89] Roessler D. : Bogomolov's theorem $c_1^2 \leq 4c_2$, *International Symposium on Algebraic Geometry*, Tokyo (1977), 623-642.
- [90] Rumely R. : An intersection pairing for curves, with analytic contributions from nonarchimedean places, *preprint U. of Georgia* (1992), 25 pp.
- [91] Soulé C. : Géométrie d'Arakelov des surfaces arithmétiques, *Séminaire Bourbaki* n°713, (1988/1989), *Astérisque* **177-178** (1989), 327-344.
- [92] Soulé C. : Théorie de Nevanlinna et théorie d'Arakelov, *Astérisque* **183** (1990), 127-135.
- [93] Soulé C. : Géométrie d'Arakelov et théorie des nombres transcendants, in *Journées arithmétiques de Luminy*, 17-21 juillet 1989, *Astérisque* **198-200** (1991), 355-371.
- [94] Soulé C. : A vanishing theorem on arithmetic surfaces, *Invent. Math.* **116** (1994), 577-599.
- [95] Soulé C. : Successive minima on arithmetic varieties, *Compositio Math.* **96** (1995), 85-98.
- [96] Soulé C., Abramovich D., Burnol J.-F., Kramer J. : *Lectures on Arakelov geometry*, Cambridge studies in advanced mathematics **33**, Cambridge University press (1992).
- [97] Stoll W. : About the value distribution of holomorphic maps into projective space, *Acta Math.* **123** (1969), 83-114.

- [98] Stuhler U. : Eine Bemerkung zur Reduktionstheorie quadratischen Formen, *Archiv. der Math.* **27** (1976), 604-610.
- [99] Szpiro L. : Small points and torsion points, *Contemporary Math* **58** (1986), 251-260.
- [100] Szpiro L. : Séminaire sur les pinceaux de courbes elliptiques (à la recherche de "Mordell effectif"), *Astérisque* **183** (1990).
- [101] Vogel W. : *Lectures on Results on Bézout's Theorem*, Tata Institute of Fundamental Research, Bombay (1984).
- [102] Vojta P. : *Diophantine Approximations and Value Distribution Theory*, Lecture Notes in Mathematics **1239**, Springer-Verlag (Berlin-Heidelberg) (1987).
- [103] Vojta P. : Dyson's lemma for products of two curves of arbitrary genus, *Invent. Math.* **98** (1989), 107-113.
- [104] Vojta P. : Siegel's Theorem in the compact case, *Ann. of Math.* **133** (1991), 509-548.
- [105] Vojta P. : Applications of arithmetic algebraic geometry to diophantine approximation, in Arithmetic Algebraic Geometry, CIME Lectures 1991, *Lecture Notes in Math.* **1553** (1993), 164-208.
- [106] Weil A. : Sur l'analogie entre les corps de nombres algébrique et les corps de fonctions algébriques, [1939a] in *Œuvres Scient. I* (1980), 236-240 (Springer-Verlag).
- [107] Zhang S. : Positive line bundles on arithmetic surfaces, *Ann. of Math.* **136** (1992), 569-587.
- [108] Zhang S. : Admissible pairing on a curve, *Invent. Math.* **112** (1993), 171-193.
- [109] Zhang S. : Positive line bundles on arithmetic varieties, *Journal of the AMS* **8**(1995) 187-222.
- [110] Zhang S. : Small points and adelic metrics, *Journal of Alg. Geometry* (1995), 281-300.
- [111] Zhang S. : Heights and reductions of semi-stable varieties, *Compositio Math.*, to appear.
- [112] Zhang S. : Heights of Heegner cycles and derivatives of L-series, *Preprint Princeton U.*, (1995), 65pp.