Representations of O(N) Spin Models
by Self-Avoiding Random Walks

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Abstract

We develop a method by which we analyze classical lattice spin systems
by series representations in terms of self-avoiding random walks. Using the
method, we get new upper bounds of critical temperatures of the O(N) sym-
matic classical Heisenberg models.

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I. INTRODUCTION

Based on the idea of Symanzik [16], the authors of [4,3,8] formulated the random walk representations of classical lattice spin systems and used them to derive various correlation inequalities and bounds for the critical inverse temperatures $\beta_c$. We tried to combine the idea of renormalization group with the random walk representations, and succeeded in the first step of transformations of block spin type. Namely we could renormalize the contribution of the smallest loops in the expansions as the changes of the single spin distributions and obtain an improvement of $\beta_c$ for the $O(N)$ Heisenberg models [9,10].

The purpose of this paper is to give series representations of correlation functions of classical lattice spin systems in terms of self-avoiding random walks. Applying the transformations of [9] to the representations repeatedly, we obtain new lower bounds of $\beta_c$ of $O(N)$ Heisenberg models which are the most accurate among the theoretical values so far obtained. In fact, we recover $\beta_c = \infty$ for every $N$ on one dimensional lattice, and we expect that this provides us with new methods to solve the long standing conjecture of non-existence of phase transition in the two dimensional Heisenberg models.

II. SPIN MODELS AND SELF-AVOIDING WALKS

Let $\Lambda$ be a $\nu$ dimensional lattice, i.e., a finite subset of $\mathbb{Z}^\nu$. We consider classical $O(N)$ symmetric classical Heisenberg models ($N$-vector models) on $\Lambda$ with free boundary condition. Its partition function is

$$Z = \int_{\mathbb{R}^{N|\Lambda|}} \exp \left( \sum_{j,k \in \Lambda} J_{jk} S_j \cdot S_k / 2 \right) \prod_{j \in \Lambda} \frac{\delta(S_j^2 - 1) dS_j}{(2\pi)^{/N/2}}, \quad (2.1)$$

where

$$J_{jk} = \begin{cases} \beta & \text{if } |j - k| = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

for $j, k \in \Lambda$ and for the inverse temperature $\beta > 0$. We adopt the convention $|j| = \sum_{\mu=1}^{\nu} |j_\mu|$ for the norm of $j \in \Lambda$ in this paper.
Let $\Gamma_{\lambda}$ be the contour given by the map

$$
t \rightarrow \begin{cases} 
t \lambda e^{-i\pi/8} & (-\infty < t \leq -1) \\
\lambda e^{i(5t-4)\pi/8} & (-1 \leq t \leq 1) \\
 t \lambda e^{i\pi/8} & (1 \leq t < \infty)
\end{cases}
$$

for $\lambda > 0$. Then we get the representations:

**Lemma 1**

$$
Z = \int_{\Gamma_{\lambda}} \det^{-N/2}(2iA - J) \prod_{j \in A} \frac{e^{iza_j} da_j}{2\pi},
$$

$$
\int_{\mathbb{R}^{N|A|}} S_j^{(1)} S_m^{(1)} \exp \left( \frac{\delta(S_j^2 - 1)dS_j}{(2\pi)^{N/2}} \right) = \int_{\Gamma_{\lambda}} (2iA - J)^{-1m} \det^{-N/2}(2iA - J) \prod_{j \in A} \frac{e^{iza_j} da_j}{2\pi}.
$$

Here, $l, m \in A, S_j = (S_j^{(1)}, \ldots, S_j^{(N)}) \in \mathbb{R}^N$ and $A$ denotes the diagonal matrix given by $A_{jk} = a_j \delta_{jk}$ ($j, k \in A$).

**Proof.** After approximating $\delta(S^2 - 1)$ by the gaussian function, we perform the Fourier transformations by the formula

$$(2\pi \epsilon^2)^{-1/2} \exp \left( - \frac{(S^2 - 1)^2}{2\epsilon^2} \right) = \int_{-i\lambda + \mathbb{R}} \exp \left( -ia(S^2 - 1) - \frac{\epsilon^2 a^2}{2} \right) \frac{da}{2\pi}.
$$

Then the lemma follows from Fubini's theorem and the integration with respect to $S_j$'s, followed by the replacement of the contour $-i\lambda + \mathbb{R}$ by $\Gamma_{\lambda}$. \(\square\)

Note that the representation of Lemma 1 is valid for all $\lambda > 0$. We set $\lambda$ large in the following sections. Now, we develop self-avoiding random walk representations for $<S_i^{(o)} S_m^{(o)}>$. We regard the matrices $A$ and $J$ as the operators acting on the linear space $\mathbb{C}^A$ of all the $\mathbb{C}$-valued mappings defined on $A$. The set of mappings

$$
e_k : \Lambda \ni j \mapsto \delta_{jk} \in \mathbb{C} \quad (k \in A)
$$

forms a basis of the space. Let $(\cdot, \cdot)$ be the bilinear form on $\mathbb{C}^A$ defined by

$$(\sum_{j \in A} z_j e_j, \sum_{k \in A} w_k e_k) = \sum_{j \in A} z_j w_j.
$$
Then \( \{ e_k \}_{k \in \Lambda} \) is the orthonormal basis with respect to \( \langle \cdot, \cdot \rangle \) defined in the obvious way. The operators \( A \) and \( J \) are defined by

\[
( e_j, A e_k ) = A_{jk} = a_k \delta_{jk} \quad \text{(2.9)}
\]

\[
( e_j, J e_k ) = J_{jk} = \beta \delta_{|j-k|,1}. \quad \text{(2.10)}
\]

Let \( \omega \) be a self-avoiding walk starting from \( l \) and ending at \( m \). That is, let \( \omega \) be a set of ordered pairs

\[
\{ (\omega(n-1), \omega(n)) \in \Lambda^2 | n = 1, \cdots, \|\omega\| \}\quad \text{(2.11)}
\]

satisfying

\[
\omega(0) = l, \quad \omega(\|\omega\|) = m,
\]

\[
|\omega(n-1) - \omega(n)| = 1 \quad (n = 1, \cdots, \|\omega\|)
\]

\[
\omega(n) \neq \omega(n') \quad (n \neq n'),
\]

where \( \|\omega\| \in \mathbb{N} \) is called the number of steps of the walk \( \omega \). Let \( Q_\omega \) be the orthonormal projection to the subspace spanned by \( \{ e_{\omega(0)}, \cdots, e_{\omega(\|\omega\|)} \} \):

\[
Q_\omega(\sum_{j \in \Lambda} z_j e_j) = \sum_{n=0}^{\|\omega\|} z_{\omega(n)} e_{\omega(n)}. \quad \text{(2.12)}
\]

We set \( P_\omega = I_d - Q_\omega \). Now we have the following representation of the correlation function of the \( O(N) \) Heisenberg models in terms of the self-avoiding random walk.

**Theorem 1**

\[
< S^{(1)}_l S^{(1)}_m >= \sum_{\omega: l \rightarrow m} \beta^{\|\omega\|} Z(\omega)/Z \quad \text{(2.13)}
\]

Here, the summation is taken over all self-avoiding nearest neighbor walks \( \omega \) on \( \Lambda \) starting from \( l \) and ending at \( m \). The weight \( Z(\omega) \) is given by

\[
Z(\omega) = \int_{\Lambda} \det \left( P_\omega(2iA - J) P_\omega \right) \det^{-\left( \frac{N+2}{2} \right)}(2iA - J) \left( \prod_{j \in \Lambda} \frac{e^{i \alpha_j} - \alpha_j}{2 \pi} \right), \quad \text{(2.14)}
\]

where \( \det \left( P_\omega(2iA - J) P_\omega \right) \) is the determinant of \( P_\omega(2iA - J) P_\omega \) as the operator acting on the space \( P_\omega \mathbb{C}^\Lambda \), i.e., the corresponding minor determinant of \( 2iA - J \).
Remark 1. We frequently deal with the operators of type \( \hat{T} = PTP \) in the sequel as well as in the theorem, where \( T \) is an operator on \( C^\Lambda \) and \( P \) is an orthonormal projection like \( P_\omega \) or \( Q_\omega \). By \( \det \hat{T} \), we always mean the determinant of \( \hat{T} \) which is regarded as the operator acting on \( P C^\Lambda \) as in the theorem. The operator which acts as the inverse of \( \hat{T} \) on \( P C^\Lambda \) and 0 on \((I_d - P)C^\Lambda \) is denoted by \( \hat{T}^{-1} \), i.e., \( \hat{T}^{-1} \) satisfies

\[
\hat{T}^{-1}\hat{T} = \hat{T}\hat{T}^{-1} = P, \quad (I_d - P)\hat{T}^{-1} = \hat{T}^{-1}(I_d - P) = 0. \tag{2.15}
\]

For the norm of operators on \( C^\Lambda \), we use

\[
\|T\| = \sup_{j \in \Lambda} \sum_{k \in \Lambda} |(e_j, T e_k)|, \tag{2.16}
\]

which is useful to check the convergence of series defining operators like \( \hat{T}^{-1} \).

Proof. Let \( D(l_1, \ldots, l_n; m_1, \ldots, m_n) \) be the minor determinant made by eliminating the \( l_1, \ldots, l_n \)-th rows and \( m_1, \ldots, m_n \)-th columns from the matrix \( 2iA - J \). In order to define determinants of operators on \( C^\Lambda \), we number all \( j \in \Lambda \) by \( \{1, 2, \ldots, |\Lambda|\} \). Let \( N_j \) be the number of \( j \). If \( l = m \), the representation of the theorem is obvious from Lemma 1. So, we assume \( l \neq m \). Applying the Laplace expansion along the \( l \)-th column to \( D(l; m) \), we have

\[
(2iA - J)_{lm}^{-1} = \epsilon_l \epsilon_m D(l; m)/\det(2iA - J)
\]

\[
= \sum_k \epsilon_l \epsilon_m \epsilon_l \epsilon_k \epsilon_l \epsilon_m (-\beta) D(l, k; m, l)/\det(2iA - J),
\]

where \( \epsilon_l = (-1)^{N_l} \) and \( \epsilon_k = 1 \) if \( N_l > N_k \), \(-1 \) if \( N_l < N_k \). The summation is taken over all \( k \in \Lambda \) satisfying \( |k - l| = 1 \), because of (2.2). Except for the term \( k = m \), we apply the Laplace expansion along the \( k \)-th column to \( D(l, k; m, l) \). We repeat the procedure until no non-zero terms remain except for the terms of type \( \beta^{n+1} D(l, k_1, \ldots, k_n; m, l, k_1, \ldots, k_n) \).

Note that each of these terms has the sign plus. Since the lattice \( \Lambda \) is finite, the procedure terminates after finite iterations. Thus we get the formula. \( \Box \)

Remark 2. In order to get the representations of the correlation functions in terms of self-avoiding random walk, we used only Fourier transformations of single spin distributions and Laplace expansions of determinants. Then the \( n \)-point functions of various lattice spin...
systems with various boundary conditions have similar representations. However, we may not apply the method to get similar formula for lattice gauge systems.

III. INTEGRATION ON $\Gamma_{\Lambda}^{[A]}$

In this section, we prepare some properties of the integration with respect to the complex variables $\{a_j\}_{j \in \Lambda}$ on $\Gamma_{\Lambda}^{[A]}$. We give them for a certain class of functions specified below for later convenience.

For an arbitrary but fixed $\delta > 0$, let $C_\delta$ be the set of multiple sequences of non-negative numbers defined by

$$C_\delta = \{ c : \tilde{N}^A \to [0, \infty) \mid \sum_{\alpha \in \tilde{N}^A} c_\alpha \delta^{\mid \alpha \mid} < \infty \},$$

where $\tilde{N} = \{0, 1, 2, \cdots \}$ and $\mid \alpha \mid = \sum_{j \in \Lambda} \alpha_j$. We then define the set of analytic functions of complex variables $\{ z_j \}_{j \in L}$ whose coefficients belong to $C_\delta$:

$$F_\delta = \left\{ f(z) = \sum_{\alpha \in \tilde{N}^A} c_\alpha z^\alpha \mid \{ c_\alpha \}_{\alpha \in \tilde{N}^A} \in C_\delta \right\},$$

where $z^\alpha = \prod_{j \in \Lambda} z_j^{\alpha_j}$. We will need another class of analytic functions defined by

$$E^s = \left\{ h(z) = C z^{(s)+\alpha} \exp(\sum_{j \in \Lambda} c_j z_j) \mid C > 0, \alpha \in \tilde{N}^A, c_j \geq 0 (\forall j \in \Lambda) \right\}$$

for an arbitrary but fixed $s > 0$. Here, $z^{(s)+\alpha} = \prod_{j \in \Lambda} z_j^{s+\alpha_j}$. The following properties are obvious.

**Proposition 1**

(i) $F_\delta$ contains all polynomials with positive coefficients.

(ii) $f, g \in F_\delta \Rightarrow c^f, f + g, fg \in F_\delta$

Let us introduce an integration of functions of the form $fh$ ($f \in F_\delta, h \in E^s$). We put
\[
[fh] = \int_{\Gamma_\lambda} f\left(\frac{1}{2ia}\right) h\left(\frac{1}{2ia}\right) \prod_{j \in \Lambda} \frac{e^{ia_j}}{2\pi} \quad (3.4)
\]

for \( f \in \mathcal{F}_\delta \) and \( \lambda \in (1/2\delta, \infty) \). Since \( f \) and \( h \) are bounded and \( e^{ia}da \) is a finite (complex valued) measure on \( \Gamma_\lambda \), the integral is well-defined. Note also that the expectation value \( \langle fh \rangle \) does not depend on the choice of \( \lambda > 1/2\delta \) because of Cauchy's integral theorem.

**Proposition 2** For \( \alpha \in \mathbb{N}^\Lambda \), \( f, g \in \mathcal{F}_\delta \) and \( h \in \mathcal{E}^* \), the following relations hold:

(i) \( \| z^{(\alpha)} \| = \prod_{j \in \Lambda} 2^{-\ell(s+\alpha_j)} \Gamma(s+\alpha_j)^{-1} \)

(ii) \( \| fh \| \geq 0 \), \( \| fh \| = 0 \iff f = 0 \)

(iii) \( \| fgh \| \| h \| \leq \| fh \| \| gh \| \) \hspace{1cm} (3.5)

**Proof.** To prove the first relation, it is enough to show

\[
\int_{\Gamma_\lambda} \frac{e^{ia}}{(2ia)^u} \frac{da}{2\pi} = \frac{1}{2^u \Gamma(u)} \hspace{1cm} (3.6)
\]

for any \( u > 0 \).

\[
\text{l.h.s. of (3.6)} = \lim_{c \to 0} \int_{\mathbb{R}^{-i\lambda}} \frac{da}{2\pi(2ia)^u} \exp \left[ ia - \frac{c^2 a^2}{2} \right]
\]

\[
= \lim_{c \to 0} \int_{\mathbb{R}^{-i\lambda}} \frac{da}{2\pi \Gamma(u)} \int_0^\infty \exp \left[ ia(1-2t) - \frac{c^2 a^2}{2} \right] t^{u-1} dt
\]

\[
= \lim_{c \to 0} \frac{1}{\Gamma(u)} \int_0^\infty \frac{t^{u-1}}{\sqrt{2\pi c^2}} \exp \left[ -\frac{(2t-1)^2}{2c^2} \right] dt
\]

\[
= \frac{1}{2^u \Gamma(u)}
\]

The case that \( f \) is a monomial in (ii) is an obvious consequence of (i). The dominated convergence theorem leads the general case because \( f \in \mathcal{F}_\delta \) has non-negative coefficients.

For the third relation, it is enough to see

\[
\mathcal{I}_{s+2}(c) \mathcal{I}_s(c) \leq \mathcal{I}_{s+1}(c)^2 \hspace{1cm} (3.7)
\]

where
\[ I_s(c) \equiv \int_{\Gamma_A} \exp(ia + \frac{c}{2ia}) \frac{da}{(2ia)^s2\pi} = \frac{I_{s-1}(\sqrt{2c})}{2(\sqrt{2c})^{s-1}}, \] (3.8)

where \( s \) and \( c \) are non-negative constants and \( I_s \) is the \( s \)-th modified Bessel function. In fact, using (3.7) repeatedly, we get

\[ I_{s+n+m}(c)I_s(c) \leq I_{s+m}(c)I_{s+n}(c) \] (3.9)

for \( n, m \in \mathbb{N} \). The case where \( f \) and \( g \) are monomials is the multiplication of those inequalities with appropriate numbers \( n, m \) and \( c \). The dominated convergence theorem and (3.9) lead the general case. For the proof of (3.7), we refer to [14]. (See also [9].)

Let us apply this formulation to the \( O(N) \) Heisenberg models. We choose \( \lambda \) and \( \delta^{-1} \) so large that

\[ \lambda > 1/2\delta > 3\nu\beta \] (3.10)

holds. Let \( 2iA - J \) be the operator on \( C^\Lambda \) defined in §2, and \( Q \) the orthonormal projection onto the subspace spanned by \( \{ e_j \}_{j \in \Delta} \) defined similarly as (2.12), where \( \Delta \) is an arbitrary subset of \( \Lambda \). Then we have

**Proposition 3** As functions of complex variables \( z_j = (2ia_j)^{-1} \) (\( j \in \Lambda \)),

\[ \det^{-N/2}(2iA) \in E^{N/2}, \]

and the following functions belong to \( F_3 \):

\[ (2iA - J)^{-1}, (Q(2iA - J)Q)^{-1}, \det^{N/2}(2iA)\det^{-N/2}(2iA - J), \]

\[ \det^{N/2}(Q2iAQ)\det^{-N/2}(Q(2iA - J)Q), \]

where the determinants and the inverses of the operators \( Q(2iA - J)Q \) and \( Q2iAQ \) are considered as those of the corresponding matrices with the index set \( \Delta \). (See Remark 1.)

**Proof.** This proposition directly follows from the condition (3.10) and the random walk representations of determinants and inverses of matrices (see [4]). \( \square \)
IV. ESTIMATES OF $Z(\omega)/Z$

In this section, we estimate $Z(\omega)/Z$ using the formulation of the preceding section. The result is summarized in

**Theorem 2** For every self-avoiding walk $\omega$ in the lattice $\Lambda$,

$$0 < Z(\omega)/Z \leq \frac{1}{N\beta^{k-1}} \left( \frac{I_{N/2}(\beta)}{I_{(N-2)/2}(\beta)} \right)^{k-1}. \quad (4.1)$$

We prove the theorem in three steps: block diagonalization of $2iA - J$, shift of the integral variables $\{a_j\}_{j \in \Lambda}$ and applications of the inequalities of §3. Let $B, C, K$ and $K^T$ denote the operators

$$B = P_\omega(2iA - J)P_\omega, \quad C = Q_\omega(2iA - J)Q_\omega, \quad K = P_\omega J Q_\omega \quad (4.2)$$

and the transpose of $K$, $K^T = Q_\omega J P_\omega$.

**Lemma 2** The representations

$$Z(\omega) = \left[ \det^{-N/2} B \det^{-(N+2)/2} (C - K^T B^{-1} K) \right], \quad (4.3)$$

$$Z = \left[ \det^{-N/2} B \det^{-N/2} (C - K^T B^{-1} K) \right] \quad (4.4)$$

hold, where $\det B$ and $\det (C - K^T B^{-1} K)$ denote the determinants of $B$ and $C - K^T B^{-1} K$ in the sense of Remark 1.

**Proof.** Operating $P_\omega + Q_\omega = I_d$ to $2iA - J$ from both sides, we obtain

$$2iA - J = B + C - K - K^T$$

$$= (I_d - K^T B^{-1})(B + C - K^T B^{-1} K)(I_d - B^{-1} K). \quad (4.5)$$

By $B^{-1}$, we mean the inverse of $B$ in the sense of Remark 1. It is given by the expansion

$$B^{-1} = (2iA)^{-1} P_\omega \sum_{n=0}^{\infty} (J(2iA)^{-1} P_\omega)^n,$$

which converges because of the condition (3.10) and $a_j \in \Gamma_\lambda (j \in \Lambda)$. In fact, $\|B^{-1}\| = \sup_{j \in \Lambda} \sum_{k \in \Lambda} |\langle e_j, B^{-1} e_k \rangle| \leq \delta/(1 - 2b\nu\beta)$. Since the determinants of the first and the third factors of (4.5) is 1, we have
\[ \det(2iA - J) = \det B \det(C - K^TB^{-1}K). \]

Remark 3. It also follows from (3.10) that \( \det(2iA) \det^{-1}B \det^{-1}(C - K^TB^{-1}K) \in \mathcal{F}_S. \)

Next, we diagonalize \( C - K^TB^{-1}K \) by triangular matrices. For \( n = 0, 1, \cdots, \|\omega\|, Q_n \) denotes the orthonormal projection to the one dimensional subspace \( C\omega_{(n)} \) of \( C^A \). Let the operator \( C_n \) and the function \( V_n \) be given inductively by

\[
V_n = (\omega_{(n)}, J(B^{-1} + (I_d + B^{-1}J)C^{-1}_{n+1}(I_d + JB^{-1}))J\omega_{(n)}),
\]

\[
C^{-1}_n = C^{-1}_{n+1} + \frac{(I_d + C^{-1}_{n+1} J(I_d + B^{-1}J))Q_n(I_d + (I_d + JB^{-1}J)C^{-1}_{n+1})}{2ia_{(n)} - V_n},
\]

\[
C^{-1}_{[\|\omega\|] + 1} = 0.
\]

Then we have the following lemma.

Lemma 3

\[
\det(C - K^TB^{-1}K) = \prod_{n=0}^{[\|\omega\|]} (2ia_{(n)} - V_n)
\]

Proof. Put \( R_1 = Q_\omega - Q_0 \), which is the orthonormal projection to the subspace spanned by \( \{ \omega_{(1)}, \cdots, \omega_{([\|\omega\|])} \} \). Then we have

\[
C_0 = C - K^TB^{-1}K = Q_\omega (2iA - J - JB^{-1}J)Q_\omega
\]

\[
= (2ia_{(0)} - (\omega_{(0)}, JB^{-1}J\omega_{(0)}))Q_0 - K_1 - K_1^T + C_1,
\]

where \( C_1 = R_1(2iA - J - JB^{-1}J)R_1, K_1 = R_1(J + JB^{-1}J)Q_0 \) and its transpose \( K_1^T = Q_0(J + JB^{-1}J)R_1 \). Let us perform the block diagonalization of \( C_0 \) by the triangular matrices \( I_d - K_1^TC_1^{-1} \) and \( I_d - C_1^{-1}K_1 \):

\[
C_0 = (I_d - K_1^TC_1^{-1})(C_1 + Q_0(2ia_{(0)} - V_0))(I_d - C_1^{-1}K_1),
\]

where \( V_0 = (\omega_{(0)}, J(B^{-1} + (I_d + B^{-1}J)C_1^{-1}(I_d + JB^{-1}))J\omega_{(0)}) \) and \( C_1^{-1} \) denotes the inverse operator of \( C_1 \) in the sense of Remark 1. It is given by the expansion \( C_1^{-1} = (2iA)^{-1}R_1 \sum_{n=0}^{\infty} ((J + JB^{-1}J)(2iA)^{-1}R_1)^n \), which converges because of the condition (3.10).
In fact, \( \|C_1^{-1}\| \leq \delta(1 - 2\delta \nu \beta)/(1 - 4\delta \nu \beta) \). And then \( 2i\alpha_{\omega(0)} - V_0 \) does not take the value zero. Thus we can invert (4.12), and get
\[
C_0^{-1} = (I_d + C_1^{-1}K_1) \left( C_1^{-1} + \frac{Q_0}{2i\alpha_{\omega(0)} - V_0} \right) (I_d + K_1^T C_1^{-1}) \tag{4.13}
\]
\[
= C_1^{-1} + \frac{(I_d + C_1^{-1} J(I_d + B^{-1}J))Q_0(I_d + (I_d + JB^{-1})JC_1^{-1})}{2i\alpha_{\omega(0)} - V_0}. \tag{4.14}
\]

From (4.12), we also have
\[
det C_0 = (2i\alpha_{\omega(0)} - V_0) \det C_1. \tag{4.15}
\]

We make a similar procedure with \( \omega(1) \) instead of \( \omega(0) \), and so on. In general, we put \( R_n = R_{n-1} - Q_{n-1} (n = 1, 2, \cdots) \), which is the orthonormal projection to the subspace spanned by \( \{ e_{\omega(n)}, \cdots, e_{\omega(|\omega|)} \} \). Then we have
\[
C_{n-1} \equiv R_{n-1}(2iA - J - JB^{-1}J)R_{n-1}
\]
\[
= (2i\alpha_{\omega(n-1)} - (e_{\omega(n-1)}, JB^{-1}Je_{\omega(n-1)}))Q_{n-1} - K_n - K_n^T + C_n, \tag{4.16}
\]
where \( C_n = R_n(2iA - J - JB^{-1}J)R_n, \ K_n = R_n(J + JB^{-1}J)Q_{n-1} \) and its transpose \( K_n^T = Q_{n-1}(J + JB^{-1}J)R_n \). We again perform the block diagonalization of \( C_{n-1} \) by the triangular matrices:
\[
C_{n-1} = (I_d - K_n^T C_n^{-1})(C_n + Q_{n-1}(2i\alpha_{\omega(n-1)} - V_{n-1}))(I_d - C_n^{-1} K_n). \tag{4.17}
\]

Then we get (4.6), (4.7) and
\[
det C_{n-1} = (2i\alpha_{\omega(n-1)} - V_{n-1}) \det C_n. \tag{4.18}
\]
This completes the proof of the lemma. \( \square \)

Let the operators \( \hat{C}_n \) and the functions \( \hat{V}_n \) be defined inductively by
\[
\hat{V}_n = \left( e_{\omega(n)}, J(B^{-1} + (I_d + B^{-1}J)\hat{C}_{n+1}^{-1}(I_d + JB^{-1}))Je_{\omega(n)} \right), \tag{4.19}
\]
\[
\hat{C}_n^{-1} = \hat{C}_{n+1}^{-1} + \frac{(I_d + \hat{C}_{n+1}^{-1} J(I_d + B^{-1}J))Q_n(I_d + (I_d + JB^{-1})J\hat{C}_{n+1}^{-1})}{2i\alpha_{\omega(n)}}, \tag{4.20}
\]
\[
\hat{C}_{|\omega|+1}^{-1} = 0, \tag{4.21}
\]
where \( n = 0, 1, \cdots, |\omega| \). Then we have the following lemma.
Lemma 4

\[ Z(\omega) / Z = \frac{\det^{-N/2} B \exp\left(\sum_{n=0}^{\infty} \frac{\chi}{2} V_n / 2\right) \prod_{n=0}^{\infty} \left(2ia_{\omega(n)}\right)^{-N+2}/2}{\det^{-N/2} B \exp\left(\sum_{n=0}^{\infty} \frac{\chi}{2} V_n / 2\right) \prod_{n=0}^{\infty} \left(2ia_{\omega(n)}\right)^{-N}/2} \]  
\[ (4.22) \]

Proof. We obtain the lemma from Lemma 3 by changing the integral variables. From (4.6), (4.7), and (4.8), it is obvious that \( B_{\alpha+1} \) and \( V_n \) do not depend on the complex variables \( \{a_{\omega(0)}, \cdots, a_{\omega(n)}\} \). Let us consider the integration with respect to \( a_{\omega(0)} \) for fixed \( \{a_j\}_{j \in \alpha - \{\omega(0)\}} \in \Gamma_\lambda^{\alpha-1} \). We shift the integral variable \( a_{\omega(0)} = a_{\omega(0)} - V_0 / 2i \), and then deform the contour of integration with respect to \( a_{\omega(0)} \) from \( \Gamma_\lambda - V_0 / 2i \) to \( \Gamma_\lambda \). Note that the deformation can be made avoiding the singularity \( a_{\omega(0)} = 0 \) because of (3.10). It follows from Cauchy's integral theorem that

\[ Z(\omega) = \left[ \det^{-N/2} B \exp(V_0 / 2) \left(2ia_{\omega(0)}\right)^{-(N+2)/2} \prod_{n=1}^{\infty} \left(2ia_{\omega(n)} - V_n\right)^{-(N+2)/2} \right], \]
\[ (4.23) \]

where we put the notation \( a_{\omega(0)} \) back to \( a_{\omega(0)} \). Next, using Fubini's theorem, we consider the integration with respect to \( a_{\omega(1)} \) for fixed \( \{a_j\}_{j \in \alpha - \{\omega(1)\}} \in \Gamma_\lambda^{\alpha-1} \). We perform the shift \( a_{\omega(1)} \rightarrow a_{\omega(1)} + V_1 / 2i \), followed by the deformation of the contour of integration. Note that \( V_0 \) is changed by this shift. After performing these operations on variables \( \{a_{\omega(0)}, a_{\omega(1)} \cdots, a_{\omega(\infty)}\} \), we get the representation for \( Z(\omega) \). The same procedure also yields the denominator. \( \square \)

To finish the proof of the theorem, let us apply the inequality (3.5) to self-avoiding random walk representation (4.22) of the correlation functions of the \( O(N) \) Heisenberg models. It is seen from (4.19), (4.20), (4.21) that \( V_n \in F_\delta \) and \( V_n \) contain the term \( \beta^2 / 2ia_{\omega(n+1)} \). Because of Prop. 3, we can decompose

\[ \det^{-N/2} B \exp\left(\sum_{n=0}^{\infty} \frac{\chi}{2} V_n / 2\right) \prod_{n=0}^{\infty} \left(2ia_{\omega(n)}\right)^{-N/2} = hf \]
\[ (4.24) \]

where \( f \in F_\delta \) and

\[ h = \exp\left(\sum_{n=1}^{\infty} \beta^2 / 4ia_{\omega(n)}\right) \prod_{j \in \alpha} \left(2ia_j\right)^{-N/2} \in \mathbb{C}^{N/2}. \]
\[ (4.25) \]

Using (3.5) for the above \( f, h \) and
\[ g = \prod_{n=0}^{\infty} (2i\omega(n))^{-1} \in \mathcal{F}_5, \quad (4.26) \]

we get

\[ Z(\omega) / Z = \frac{[h f g]}{[h f]} \leq \frac{[h g]}{[h]} \quad (4.27) \]

\[ = \frac{1}{N} \left( \frac{I_{(N+2)/2}(\beta^2/2)}{I_{N/2}(\beta^2/2)} \right)^{\text{dim}} = \frac{1}{N\beta^{\text{dim}}} \left( \frac{I_{N/2}(\beta)}{I_{(N-2)/2}(\beta)} \right)^{\text{dim}}. \quad (4.28) \]

\[
\Box
\]

V. LOWER BOUNDS OF $\beta_C$

In this section, we discuss lower bounds of the inverse critical temperatures of the $O(N)$ symmetric Heisenberg models. From Theorem 1 and 2, we get

\[ 0 < \langle S^z I S^z \rangle \leq \sum_{\omega=1}^{m} \frac{1}{N} \left( \frac{I_{N/2}(\beta)}{I_{(N-2)/2}(\beta)} \right)^{\text{dim}}. \quad (5.1) \]

If the summation over $\omega$ is taken over all self-avoiding walk in $\mathbb{Z}^\nu$, the bound is uniform in $\Lambda$. Then the above inequality also holds for the thermodynamic limit taken under the free boundary condition. Let $\mu_\nu$ be the connectivity constant in $\nu$ dimensional lattice defined by $\log \mu_\nu = \lim_{l \to \infty} l^{-1} \log s^\nu_l$, where $s^\nu_l$ is the total number of self-avoiding nearest neighbor walks in $\mathbb{Z}^\nu$ of length $l$ starting from the origin (see e.g. [13]). Since the critical inverse temperature $\beta_c$ is defined as the maximum number of those $\beta$ below which the correlation function exhibits exponential decay, we have:

**Corollary 1** For the $\nu$ dimensional $O(N)$ symmetric Heisenberg model,

\[ \beta_c \geq \inf \{ \beta > 0 \mid \mu_\nu / I_{N/2}(\beta)/I_{(N-2)/2}(\beta) \geq 1 \}. \quad (5.2) \]

Let us apply the corollary to one dimensional case. The connectivity constant $\mu_1$ is 1. The inequality $I_{N/2}(\beta) < I_{(N-2)/2}(\beta)$ holds for every $\beta > 0$ and $N \in \mathbb{N}$. So we recover the fact $\beta_c = \infty$ from Corollary 1.
For the cases $\nu \geq 2$, the precise values of the connectivity constants have not been known, yet. But it is rigorously known that $\mu_2 \leq \tilde{\mu}_2^{(12)} = 2.712, \mu_3 \leq \tilde{\mu}_3^{(4)} = 4.863, \mu_4 \leq \mu_4^{(4)} = 6.820 [5]$, and it is expected that $\mu_2 = 2.638, \mu_3 = 4.683, \mu_4 = 6.775 [15]$. Here, $\mu_\nu^{(k)}$ is defined by $\log \mu_\nu^{(k)} = \lim_{l \to \infty} l^{-1} \log \epsilon_{(k)}^{(l)}$, where $\epsilon_{(k)}^{(l)}$ is the total number of nearest neighbor $l$-step walks in $\mathbb{Z}^\nu$ starting at the origin, in which $k$-th order and lower order self-intersections are forbidden. The numerical values using Corollary 1 and the above upper bound and expected values of $\mu_\nu$ are in Table, and they are in good agreement with experimental results.

The following properties of the modified Bessel functions can be obtained readily:

(i) $I_s(x)/I_{s-1}(x) \leq x/2s$

(ii) $I_s(x)/I_{s-1}(x) \leq \frac{x}{s-1 + \sqrt{s^2 + x^2}}$.

Summarizing these argument, we have the following bounds.

**Corollary 2**

(i) $\beta_c \geq \frac{N}{\mu_\nu}$ for all $N$

(ii) $\beta_c \geq \mu_\nu N/(\mu_\nu^2 - 1) + O(1)$ \quad $N \to \infty$

(iii) $\beta_c = \infty$ \quad for $\nu = 1$

Unfortunately, we could not prove our long standing conjecture $\beta_c(\nu = 2, N \geq 3) = \infty$ in the present framework. We are trying to find a deviation in the bound of $\beta_c(N)$ for $\nu = 2$, as $N$ increases, by numerical calculation to take the effects of $\tilde{V}_n$ into consideration. This will be reported in another publication.
TABLE I. Comparison of our results with MC Simulations

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\( \beta_0 \): the lower bounds obtained in [4].

\( \beta_1 \): the lower bounds obtained by Corollary 1 and the upper bounds of connectivity constants \( \mu_2 \leq 2.712, \mu_3 \leq 4.863 \) and \( \mu_4 \leq 6.820 \) [5] are used.

\( \beta_2 \): the lower bounds obtained by Corollary 1 and the expected values of connectivity constants \( \mu_2 = 2.638, \mu_3 = 4.683 \) and \( \mu_4 = 6.775 \) [15] are used.

\( \beta_c \): data obtained by Monte Carlo simulations except for that of the 2 dimensional Ising model which is exactly soluble. Data are taken from [1,2,6,7,11,12,17].
REFERENCES


