Supersymmetry algebra in $N = 1$ chiral supergravity

Motomu Tsuda and Takeshi Shirafuji

Physics Department, Saitama University
Urawa, Saitama 338, Japan

Abstract

We consider the supersymmetry (SUSY) transformations in the chiral Lagrangian for $N = 1$ supergravity (SUGRA) with the complex tetrad following the method used in the usual $N = 1$ SUGRA, and present the explicit form of the SUSY transformations in the first-order form. The SUSY transformations are generated by two independent Majorana spinor parameters, which are apparently different from the constrained parameters employed in the method of the 2-form gravity. We also calculate the commutator algebra of the SUSY transformations on-shell.
Ashtekar's canonical formulation of general relativity was extended to \( N = 1 \) supergravity (SUGRA) introducing the right- and left-handed supersymmetry (SUSY) transformations [1, 2]. The first-order formulation and its extension to \( N = 2 \) SUGRA were made using the 2-form gravity [3, 4]. In this formulation, however, the SUSY transformation parameters are constrained. The purpose of this brief report is to reconsider the SUSY transformations in the \( N = 1 \) chiral SUGRA, following as closely as possible the method originally employed in the usual \( N = 1 \) SUGRA [5, 6].

The \( N = 1 \) chiral SUGRA has characteristic features to be contrasted with the usual \( N = 1 \) SUGRA: Firstly a complex tetrad field is introduced, and secondly spin-3/2 fields \( \psi_\mu \) and \( \overline{\psi}_\mu \) are assumed to be independent of each other. The motivations of taking such chiral Lagrangian as analytic in complex field variables are (a) to evade a consistency problem for matter field equations when more than two spin-3/2 fields are coupled, \(^1\) and (b) to construct the SUSY transformations compatible with the complex tetrad. We present the explicit form of the SUSY transformations in the first-order form. The present formulation has the merit that the SUSY transformation parameters are not constrained at all in contrast with the method of the 2-form gravity. We also calculate the commutator algebra of the SUSY transformations on-shell.

We start with the chiral Lagrangian density for \( N = 1 \) SUGRA,

\[
\mathcal{L}^{(+)} = \mathcal{L}_G^{(+)} + \mathcal{L}_R.
\]

The independent variables in \( \mathcal{L}^{(+)} \) are a complex tetrad \( e_\mu^i \), a self-dual connection \( A_{ij\mu}^{(+)} = A_{ij\mu}^{(+)} \) which satisfies \( (1/2)\epsilon^{ijkl}A_{kl\mu}^{(+)} = iA_{ij\mu}^{(+)} \), and two independent (Majorana) Rarita-Schwinger fields \( \psi_{R\mu} = (1/2)(1 + \gamma_5)\psi_\mu \) and \( \overline{\psi}_{R\mu} \). \(^2\) The chiral gravitational Lagrangian density, \( \mathcal{L}_G^{(+)} \), constructed from the complex tetrad and the self-dual connection is

\[
\mathcal{L}_G^{(+)} = -\frac{i}{2} e^{\mu\nu\sigma} e_\mu^i e_\nu^j R_{ij\mu\nu\sigma}.
\]

\(^1\) If the tetrad is real and the self-dual connection satisfies its equation of motion, the chiral Lagrangian including more than two spin-3/2 fields becomes complex, and its imaginary part gives an additional equation for spin-3/2 fields which gives rise to inconsistency \([7]\).

\(^2\) Greek letters \( \mu, \nu, \ldots \) are space-time indices, and Latin letters \( i, j, \ldots \) are local Lorentz indices. We denote the Minkowski metric by \( \eta_{ij} = \text{diag}(-1, +1, +1, +1) \). The totally antisymmetric tensor \( \epsilon_{ijkl} \) is normalized as \( \epsilon_{0123} = +1 \). The antisymmetrization of a tensor with respect to \( i \) and \( j \) is denoted by \( A_{[i\cdots j]} := (1/2)(A_{i\cdots j} - A_{j\cdots i}) \).
where the unit with $8\pi G = c = 1$ is used, $e$ denotes $\det(e^i_\mu)$ and the curvature of self-dual connection $R^{(+)}_{ij\mu\nu}$ is

$$R^{(+)}_{ij\mu\nu} := 2(\partial_{[\mu} A^{(+)}_{ij\nu]} + A^{(+)}_{k[j}\lambda \gamma \mu} A^{(+)}_{i]k\lambda}).$$  

(3)

The chiral Lagrangian density of (Majorana) Rarita-Schwinger fields, $\mathcal{L}_{RS}^{(+)}$, is

$$\mathcal{L}_{RS}^{(+)} = -e \epsilon^{\mu\nu\rho\sigma} \bar{\psi} \gamma_\mu D^{(+)}_\sigma \psi R_{\nu\rho},$$  

(4)

where $D^{(+)}_{\mu}$ denotes the covariant derivative with respect to $A^{(+)}_{ij\mu}$:

$$D^{(+)}_{\mu} := \partial_\mu + \frac{i}{2} A^{(+)}_{ij\mu} S^{ij}$$  

(5)

with $S^{ij}$ being the Lorentz generator.  

The field equations derived from the $\mathcal{L}^{(+)}$ of (1) are slightly different from the usual $N = 1$ SUGRA. Varying $\mathcal{L}^{(+)}$ with respect to $A^{(+)}_{ij\mu}$ and solving the equation for $A^{(+)}_{ij\mu}$ yield

$$A^{(+)}_{ij\mu} = A^{(+)}_{ij\mu}(e) + K^{(+)}_{ij\mu},$$  

(6)

where $A^{(+)}_{ij\mu}(e)$ is the self-dual part of the Ricci rotation coefficients $A_{ij\mu}(e)$, while $K^{(+)}_{ij\mu}$ is that of $K_{ij\mu}$ given by

$$K_{ij\mu} := \frac{i}{2}(e^i_\mu e^j_\rho \psi R_{[\rho} \gamma_{[i} \psi R_{\sigma]} + e^i_\rho \psi R_{[\rho} \gamma_{[i} \psi R_{j]} - e^i_\rho \psi R_{[\rho} \gamma_{[i} \psi R_{j]}).$$  

(7)

From (7) we obtain

$$T^{i}_{[\mu\nu]} := 2 K^{i}_{[\mu\nu]} = -i \bar{\psi} R_{[\mu} \gamma^i \psi R_{\nu]}.$$  

(8)

Varying $\mathcal{L}^{(+)}$ with respect to $e^i_\mu$, $\bar{\psi} R_{\mu}$, and $\psi R_{\mu}$ yields

$$\epsilon^{\mu\nu\rho\sigma}(e^j_\rho R^{(+)}_{ij\rho\sigma} + i \bar{\psi} R_\rho \gamma_i D^{(+)}_\sigma \psi R_\nu) = 0,$$  

(9)

$$\epsilon^{\mu\nu\rho\sigma} \gamma_\rho D^{(+)}_\sigma \psi R_\nu = 0,$$  

(10)

$$\epsilon^{\mu\nu\rho\sigma} D^{(+)}_\sigma (\gamma_\rho \psi R_\nu) = 0,$$  

(11)

$\text{a}^{3}$ In our convention $S_{ij} = i[\gamma_i, \gamma_j]$ and $\{\gamma_i, \gamma_j\} = -2\eta_{ij}$.
respectively. If the tetrad is real together with $\bar{\psi}_{R\mu} = \overline{\psi}_{R\mu}$ and if the self-dual connection satisfies its equation of motion, the field equations of (9) to (11) are equivalent to those of the usual $N = 1$ SUGRA.

It is possible to establish the right- and left-handed SUSY transformations in the $\mathcal{L}^{(+)}$ of (1) as in the case of the real tetrad [1, 2]. Since $\psi_{R\mu}$ and $\bar{\psi}_{R\mu}$ in (4) are independent of each other, we need two anticommuting Majorana spinor parameters $\alpha$ and $\bar{\alpha}$ which generate the SUSY transformations. The chiral Lagrangian density $\mathcal{L}^{(+)}$ is invariant under the right-handed SUSY transformations generated by $\alpha$,

$$\delta_R \psi_{R\mu} = 2D^{(+)}_{\mu} \alpha_R \quad \delta_R \bar{\psi}_{L\mu} = 0,$$

$$\delta_R e^i = -i \bar{\psi}_{R\mu} \gamma^i \alpha_R,$$

$$\delta_R A^{(+)}_{ij\mu} = 0,$$

and also under the left-handed SUSY transformations generated by $\bar{\alpha}$,

$$\delta_L \psi_{R\mu} = 0 \quad \delta_L \bar{\psi}_{L\mu} = 2D^{(-)}_{\mu} \bar{\alpha}_L,$$

$$\delta_L e^i = -i \bar{\psi}_{L\mu} \gamma^i \bar{\alpha}_L,$$

$$\delta_L A^{(-)}_{ij\mu} = -\frac{1}{2} \left( (\bar{B}_{(R)\mu} e_{\mu}^i - e_{\mu}^i \bar{B}_{(R)\mu}^m | m | | j \rangle) - \frac{i}{2} e_{ij}^{kl} (\bar{B}_{(R)\mu} e_{\mu}^k \bar{B}_{(R)\mu}^l m l) \right)$$

with

$$\bar{B}_{(R)}^{\lambda \mu \nu} := \epsilon^{\lambda \mu \nu \rho} \bar{\alpha}_R R_{\rho}^\lambda D^{(+)}_{\rho} \psi_{R\sigma}.$$

Here we put a tilde on $B_{(R)}^{\lambda \mu \nu}$ because the parameter $\bar{\alpha}$ is used. In (15) of the left-handed SUSY transformations, $D^{(-)}_{\mu}$ denotes the covariant derivative with respect to antself-dual connection, $A^{(-)}_{ij\mu}$, and we assume that $A^{(-)}_{ij\mu}$ is the solution derived from the “unphysical” Lagrangian density, $\mathcal{L}^{(-)}$:

$$\mathcal{L}^{(-)} = \frac{i}{2} e^{\mu \nu \rho} e^i_{\mu} e^j_{\nu} R^{(-)}_{ij\rho \sigma} + e^{\mu \nu \rho} \bar{\psi}_{L\mu} \gamma_{\rho} D^{(-)}_{\sigma} \psi_{L\nu},$$

where $R^{(-)}_{ij\mu \nu}$ is the curvature of antself-dual connection. If the tetrad is real and $\bar{\psi}_{R\mu} = \overline{\psi}_{R\mu}$, the $\mathcal{L}^{(-)}$ of (19) becomes just the complex conjugate of the $\mathcal{L}^{(+)}$ of (1).

In the SUSY transformations in $\mathcal{L}^{(+)}$, the self-dual connection $A^{(+)}_{ij\mu}$ is one of the independent variables, while the antself-dual connection $A^{(-)}_{ij\mu}$ is the solution derived from $\mathcal{L}^{(-)}$. If we consider the SUSY transformations in $\mathcal{L}^{(-)}$, however, the role of
$A_{ij\mu}^{(+)}$ and $A_{ij\mu}^{(-)}$ is exchanged each other. Indeed, the $\mathcal{L}^{(-)}$ of (19) is invariant under the SUSY transformations which has the same form as in $\mathcal{L}^{(+)}$, if we take

$$\delta_{R}A_{ij\mu}^{(-)} = \frac{1}{2} \left\{ (B_{(L)}^{\mu i j} - e_{\mu}[i B_{(L)}^{m}[i]m[j]) + \frac{i}{2} e_{ij}^{kl}(B_{(L)}^{\mu i k l} - e_{i j k l} B_{(L)}^{m}[i]m[j]) \right\} \tag{20}$$

with

$$B_{(L)}^{\lambda \mu} := e^{\mu \rho \sigma} \bar{\alpha}_{L} \gamma^{\lambda} D_{\rho}^{(-)} \bar{\psi}_{L \sigma}, \tag{21}$$

for the right-handed SUSY transformations, and

$$\delta_{L}A_{ij\mu}^{(-)} = 0 \tag{22}$$

for the left-handed SUSY transformations. But in this case, we assume that the self-dual connection $A_{ij\mu}^{(+)}$ is the solution derived from the chiral Lagrangian $\mathcal{L}^{(+)}$.

If the tetrad is real and $\bar{\psi}_{R \mu} = \bar{\psi}_{R \mu}$, the left-handed SUSY transformation of the self-dual connection, (17), can be written as

$$\delta_{L}A_{ij\mu}^{(+)}(1st-order) = \text{self-dual part of} \quad \frac{1}{2} \left\{ \delta_{A_{ij\mu}}(1st-order) \right|_{N=1 \text{ SUGRA}}$$

$$- (e_{ij}^{\rho \sigma} \bar{\alpha}_{\mu} D_{\rho} \psi_{\sigma} + e_{[i}^{\lambda} e_{j]}^{\rho \sigma} \bar{\alpha}_{\gamma} \gamma_{\lambda} D_{\rho} \psi_{\sigma}). \quad \tag{23}$$

This form of (23) does not agree with the self-dual part of the first-order transformation of the connection in the usual $N = 1$ SUGRA. In the second-order formulation, however, the situation is changed. Indeed, if we use the equation for the self-dual connection, the spin-3/2 field equation $e^{\mu \rho \sigma} \gamma_{\mu} D_{\rho} \psi_{\nu} = 0$ and its variant forms [8]

$$D_{[\mu} \psi_{\nu]} + \frac{i}{2} e_{\mu}[i D_{\rho} \psi_{\sigma} = 0, \tag{24}$$

$$\gamma_{\mu} D_{[\nu} \psi_{\lambda]} + \gamma_{\nu} D_{[\lambda} \psi_{\mu]} + \gamma_{\lambda} D_{[\mu} \psi_{\nu]} = 0, \tag{25}$$

then we can show that (23) becomes

$$\delta_{L}A_{ij\mu}^{(+)}(2nd-order) = \text{self-dual part of} \quad \delta_{A_{ij\mu}}(2nd-order) \right|_{N=1 \text{ SUGRA}}. \tag{26}$$

On the other hand, the transformations of (14) and (17) seem to be quite different from the SUSY transformations based on the 2-form gravity [3, 4]: In fact, in the
2-form gravity, the SUSY transformation parameters with constraint are used, and the transformations of the connection do not include the covariant derivative.

For the purpose of calculating the SUSY algebra, let us write the other equivalent forms of the spin-3/2 field equations (10) and (11) explicitly. Since we restrict ourselves to the algebra on-shell, we substitute the solution $A_{ij\mu}^{(+)\,\nu}$ of (6) into (10) and (11). Then we obtain

$$e^{\mu\nu\rho\sigma}\gamma_{\rho}D_{\sigma}\bar{\psi}_{R\nu} = 0,$$

$$e^{\mu\nu\rho\sigma}\gamma_{\rho}D_{\sigma}\bar{\psi}_{L\nu} - M^\mu = 0,$$

where

$$M^\mu := -\frac{1}{2}e^{\mu\nu\rho\sigma}T_{\rho\sigma}^{i}\gamma_{i}\bar{\psi}_{L\nu}.$$  \hspace{1cm} (29)

The $M^\mu$ of (29) vanishes for $N = 1$ SUGRA because of a Fierz transformation. After the contraction of (27) and (28) with $\gamma_{\mu}$, we have

$$S^{\mu\nu}D_{\mu}\bar{\psi}_{R\nu} = 0,$$

$$S^{\mu\nu}D_{\mu}\bar{\psi}_{L\nu} + \frac{1}{4}\gamma_{\mu}M^\mu = 0.$$  \hspace{1cm} (31)

Further using the relation $2\gamma^{\lambda}S^{\mu\nu} = -i(g^{\mu\nu}\gamma^{\lambda} - g^{\lambda\nu}\gamma^{\mu}) + e^{\lambda\mu\nu\rho}\gamma_{\rho}\gamma_{\nu}$, (27) and (28) become

$$i\gamma^{\rho}(D_{\rho}\bar{\psi}_{R\lambda} - D_{\lambda}\bar{\psi}_{R\rho}) = 0,$$

$$i\gamma^{\rho}(D_{\rho}\bar{\psi}_{L\lambda} - D_{\lambda}\bar{\psi}_{L\rho}) + M_{\lambda} + \frac{1}{2}\gamma_{\lambda}(\gamma \cdot M = 0.$$  \hspace{1cm} (33)

The commutator algebra of the SUSY transformations in $\mathcal{L}^{(+)}$ on the complex tetrad is easily calculated as

$$[\delta_{R1}, \delta_{R2}]e^{i}_{\mu} = 0 = [\delta_{L1}, \delta_{L2}]e^{i}_{\mu},$$

$$[\delta_{R1}, \delta_{L2}]e^{i}_{\mu} = 2iD_{\mu}(\bar{\alpha}_{2\nu}^{R}y^{\nu}\alpha_{1R}),$$

where the equation for the self-dual connection is used in (35). Therefore, we have

$$[\delta_{1}, \delta_{2}]e^{i}_{\mu} = D_{\mu}(\xi^{i} + \eta^{i}),$$

where $\delta := \delta_{R} + \delta_{L}$ and

$$\xi^{i} := 2i\bar{\alpha}_{2\nu}^{R}y^{\nu}\alpha_{1R},$$

$$\eta^{i} := 2i\bar{\alpha}_{2\nu}^{L}y^{\nu}\alpha_{1L}.$$  \hspace{1cm} (38)
Next we calculate the commutator algebra of the SUSY transformations in $\mathcal{L}^{(+)}$ on $\psi_{R\mu}$ and $\tilde{\psi}_{L\mu}$. For $\psi_{R\mu}$, we have

\begin{align}
[\delta_{R1}, \delta_{R2}]\psi_{R\mu} &= 0 = [\delta_{L1}, \delta_{L2}]\psi_{R\mu}, \\
[\delta_{R1}, \delta_{L2}]\psi_{R\mu} &= -i(\delta_{L2}A_{ij\mu}^{(+)}) \varepsilon_{ij} \alpha_{1R}.
\end{align}

Further we rewrite (40) as follows. Using the identity

\[ \eta_{ij} \varepsilon_{klmn} = \eta_{ik} \varepsilon_{jlmn} + \eta_{il} \varepsilon_{kjm} + \eta_{im} \varepsilon_{kljn} + \eta_{ln} \varepsilon_{kilmj} \]

stated in [5], we can rewrite (17) as

\[ \delta_{L} A_{ij\mu}^{(+) = \text{self–dual part of}} \{ -\varepsilon_{ij} \lambda_{\rho} \alpha_{1R} \gamma_{\lambda} (D_{\mu}^{(+)}) \psi_{R\rho} - D_{\rho}^{(+)}) \psi_{R\mu} + \varepsilon_{ij} \lambda_{\rho} \alpha_{1R} \gamma_{\lambda} D_{\mu}^{(+)}) \psi_{R\rho} \}. \]

The last term in (42) vanishes by means of the field equation (10), and hence it can be omitted. Substituting the first term of (42) into (40), and using the solution $A_{ij\mu}^{(+) = \text{self–dual part of}}$ of (6), we get

\[ [\delta_{R1}, \delta_{L2}]\psi_{R\mu} = \xi^{\rho}(D_{\rho}^{(+)}) \psi_{R\mu}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho} + \frac{1}{4}(\xi \cdot \gamma) \gamma^{\rho}(D_{\rho}^{(+)}) \psi_{R\mu}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho} \} \]

after a Fierz transformation. The last term in (43) may be dropped because of the field equation of (32). Finally, we obtain

\[ [\delta_{1}, \delta_{2}]\psi_{R\mu}^{(+) = (\xi^{\rho} + \eta^{\rho})(D_{\rho}^{(+)}) \psi_{R\mu}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}^{(+) - D_{\mu}^{(+)}) \psi_{R\rho}} \]

discarding those terms which vanish due to the matter field equation.

In the same way, we have the following algebra on $\tilde{\psi}_{L\mu}$ by using (20), (22) and the field equation of (28):

\begin{align}
[\delta_{R1}, \delta_{R2}]\tilde{\psi}_{L\mu} &= 0 = [\delta_{L1}, \delta_{L2}]\tilde{\psi}_{L\mu}, \\
[\delta_{R1}, \delta_{L2}]\tilde{\psi}_{L\mu} &= \xi^{\rho}(D_{\rho}^{(+)}) \tilde{\psi}_{L\mu}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho} + \frac{1}{4}(\xi \cdot \gamma) \gamma^{\rho}(D_{\rho}^{(+)}) \tilde{\psi}_{L\mu}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}} \] \]

The last term in (46) may be dropped due to the field equation of (33) and we find

\[ [\delta_{1}, \delta_{2}]\tilde{\psi}_{L\mu}^{(+) = (\xi^{\rho} + \eta^{\rho})(D_{\rho}^{(+)}) \tilde{\psi}_{L\mu}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}^{(+) - D_{\mu}^{(+)}) \tilde{\psi}_{L\rho}} \]
If the tetrad is real and \( \overline{\psi}_{R\mu} = \overline{\psi}_{R\mu} \), the algebra of (36), (44) and (47) coincides with that of the usual \( N = 1 \) SUGRA [5].

The commutator algebra of (36), (44) and (47) can be interpreted as the complex extension of the usual \( N = 1 \) SUGRA. Indeed, the infinitesimal general coordinate transformations of the fields can be written as [5]

\[
\delta_G e^i_\mu = D_\mu e^i - e^p A^i_{j\rho} e^j_\mu + e^p T^i_{\mu\rho},
\]

(48)

\[
\delta_G \psi_{R\mu} = e^p (D_\rho \psi_{R\mu} - D_\mu \psi_{R\rho}) - \frac{i}{2} e^p A_{ij\rho} S^{ij} \psi_{R\mu} + D_\mu (\epsilon \cdot \psi_R),
\]

(49)

\[
\delta_G \tilde{\psi}_{L\mu} = e^p (D_\rho \tilde{\psi}_{L\mu} - D_\mu \tilde{\psi}_{L\rho}) - \frac{i}{2} e^p A_{ij\rho} S^{ij} \tilde{\psi}_{L\mu} + D_\mu (\epsilon \cdot \tilde{\psi}_L),
\]

(50)

with \( e^\mu \) being the infinitesimal transformation parameter: \( x'^\mu = x^\mu - e^\mu \). If we take \( e^\mu \) as

\[
e^\mu := \xi^\mu + \eta^\mu,
\]

(51)

then the commutator algebra of (36), (44) and (47) can be rewritten as follows:

\[
[\delta_1, \delta_2] e^i_\mu = \delta_G e^i_\mu + e^p A^i_{j\rho} e^j_\mu - e^p T^i_{\mu\rho},
\]

(52)

\[
[\delta_1, \delta_2] \psi_{R\mu} = \delta_G \psi_{R\mu} + \frac{i}{2} e^p A_{ij\rho} S^{ij} \psi_{R\mu} - D_\mu (\epsilon \cdot \psi_R),
\]

(53)

\[
[\delta_1, \delta_2] \tilde{\psi}_{L\mu} = \delta_G \tilde{\psi}_{L\mu} + \frac{i}{2} e^p A_{ij\rho} S^{ij} \tilde{\psi}_{L\mu} - D_\mu (\epsilon \cdot \tilde{\psi}_L),
\]

(54)

The first and second terms of (52) to (54) describe the complex general coordinate transformations and the field-dependent complex local Lorentz transformations, respectively. The third terms in (53) and (54) are just the right- and left-handed SUSY transformations generated by the field-dependent parameters \( \alpha'_R := -(1/2)\epsilon \cdot \psi_R \) and \( \alpha'_L := -(1/2)\epsilon \cdot \tilde{\psi}_L \). Similarly, the third term in (52) can be interpreted as the sum of the right- and left-handed SUSY transformations by using (8):

\[
- e^p T^i_{\mu\rho} = i e^p \overline{\psi}_{R\mu} \gamma^i \psi_{R\rho} = -i (\overline{\psi}_{R\mu} \gamma^i \alpha'_R + \overline{\psi}_{L\mu} \gamma^i \alpha'_L).
\]

(55)

Note that the structure constants defined by these results are field-dependent, i.e., structure functions, as in the case of the usual \( N = 1 \) SUGRA.

So far we have considered the SUSY transformations in \( \mathcal{L}^+ \) of (1) with the complex tetrad. In the usual local field theory, the spinor fields \( \psi \) and \( \tilde{\psi} \) are not independent of each other. However, since the independent \( \psi_{R\mu} \) and \( \tilde{\psi}_{R\mu} \) have been
used in $\mathcal{L}^{(+)}$, it is difficult to see the correspondence of $\mathcal{L}^{(+)}$ to the usual local field theory. This correspondence can be seen if we add the complex conjugate of the chiral Lagrangian density, $\overline{\mathcal{L}}^{(+)}$, to the Lagrangian density $\mathcal{L}^{(+)}$. For example, in special relativistic limit, the total Lagrangian density of (Majorana) Rarita-Schwinger fields, $\mathcal{L}_r^{\text{tot}} := \mathcal{L}_r^{(+)} + \mathcal{L}^{(+)}$, becomes

$$L_r^{\text{tot}} = \epsilon^\mu_{\nu\rho\sigma} \overline{\psi}_\mu \gamma_\rho \partial_\sigma \psi_\nu,$$  \hspace{1cm} (56)

which can be diagonalized as

$$L_r^{\text{tot}} = \epsilon^\mu_{\nu\rho\sigma} \left( \overline{\psi}_\mu^1 \gamma_\rho \partial_\sigma \psi_\nu^1 - \overline{\psi}_\mu^2 \gamma_\rho \partial_\sigma \psi_\nu^2 \right),$$  \hspace{1cm} (57)

with $\psi_\mu^1 := (1/2)(\psi_\mu + \overline{\psi}_\mu)$ and $\psi_\mu^2 := (1/2)(\psi_\mu - \overline{\psi}_\mu)$. The minus sign in (57) means the appearance of negative energy states. The SUSY transformations in the total Lagrangian density, $\mathcal{L}^{\text{tot}} := \mathcal{L}^{(+)} + \overline{\mathcal{L}}^{(+)}$, can be constructed as in $\mathcal{L}^{(+)}$.

We have also seen that the SUSY algebra for $N = 1$ chiral SUGRA with the complex tetrad closes only on-shell, because the field equation terms appear in the SUSY algebra on (Majorana) Rarita-Schwinger fields. In the usual $N = 1$ SUGRA, additional auxilialy fields are introduced in order that the SUSY algebra closes off-shell [9]. We are now trying to construct such a formulation for $N = 1$ chiral SUGRA.

We would like to thank the members of Physics Department at Saitama University for discussions and encouragement.
References


