Quantizing Regge Calculus.

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Abstract

A discretized version of canonical gravity in (3+1)-d introduced in a previous paper is further developed, introducing the Liouville form and the Poisson brackets, and studying them in detail in an explicit parametrization that shows the nature of the variables when the second class constraints are imposed. It is then shown that, even leaving aside the difficult question of imposing the first class constraints on the states, it is impossible to quantize the model directly, using complex variables and leaving the second class constraints to fix the metric of the quantum Hilbert, because one cannot find a metric which makes the area variables hermitean.

1. Introduction.

In a previous paper [1] I discussed a possible way of discretizing canonical gravity in 3+1 dimensions, dividing space in tetrahedra, and associating pairs of variables to the triangles that separate them: the oriented area of the triangle, and the Lorentz transformation that takes us from one tetrahedron to the next through the triangle. This scheme is not very different from others recently proposed [2] [3](these authors use a cubical lattice, but that perhaps is not very important), but draws rather from the well known work of T. Regge [4] to adapt to a discretized theory the new variables of A. Ashtekar [5]. If one could quantize this particular form of ”Regge calculus” one would have a quantum gravity which is regularized, yet is faithful to the geometrical spirit of Einstein’s theory; in particular, unlike other regularizations, does not require an arbitrary background metric.

With this aim in mind, this program is continued in this paper specifying and analysing in detail the Liouville form, and therefore the Poisson brackets between the basic variables.

From this one should go on to eliminate second class constraints (perhaps introducing Dirac brackets), check the algebra between first class constraints, finally proceeding to the quantization of the theory. However, it is central to the quantization program based on the Ashtekar variables the idea that one does not solve the second class constraints (which are reality conditions on the complex basic variables), but uses them to fix the metric in the Hilbert
space of the quantized theory. In this spirit I have ignored the difficult problem of checking whether the algebra of the constraints closes, but rather attempted a straight conversion of Poisson brackets into commutators, and checked whether one can find an acceptable scalar product in the Hilbert space. The result is that no, there cannot be a scalar product which makes the basic area variables hermitian. Thus the procedure which was shown to work fine for linearized gravity \cite{6} fails in this case, and so does the attempt to produce a discretized version of the "net states" of C. Rovelli and L. Smolin \cite{7} or of their volume and area operators. I am not able to say whether this should be taken as evidence against the viability of their program, or of the quantization program in general, but find little reason for optimism. And, in a way, one should not be too surprised that a relatively naïve approach like this fails, considering how tricky is the construction of the scalar product in the simpler case of $2+1$–d gravity \cite{8}.

The alternative, real $SU(2)$ connection introduced by J. F. Barbero \cite{9} is not a Levi–Civita connection, and therefore does not have an obvious geometric interpretation in this discretized context. Nor can one hope to start with an $SU(2)$ connection and recover the full theory by analytic continuation, unless one relaxes the constraints that come from the geometric interpretation. It is plausible to conjecture that the imaginary part of the Liouville form vanishes when the Gauss law and the reality conditions are imposed, and therefore if the group were restricted to $SU(2)$ there would be no local degrees of freedom, and nothing to analytically continue.

I have tried (but failed) to prove this conjecture using an explicit real parametrization of the basic canonical variables that makes clear the meaning of the reality conditions; numerically, on a simple 5–tetrahedra model, the conjecture appears to be true. The Liouville form, if its imaginary part is indeed zero depends only on the areas of the triangles and the rapidities of the Lorentz transformations, very much like the one proposed by t’Hooft \cite{10} for his discretized 2+1 dimensional theory.

The discretization scheme is briefly summarized in \S2, while in \S3 the Liouville form and the Poisson brackets are introduced. The Liouville form is further discussed in \S4, using an explicit, real parametrization, and the restriction imposed by the reality conditions are analysed in detail. Finally in \S5 I discuss the quantization of the theory and the reasons why it is not possible to construct a scalar product that make the area variables hermitean.

2. Outline of the model.

Following the basic ideas of Regge calculus, 3–space is divided in tetrahedra, each with a flat inside. We may therefore choose an inertial frame for each tetrahedron, and expect that a Lorentz transformation will be needed to go from one tetrahedron to the next, and that because of curvature one will end up in a different Lorentz frame going round an edge.

In terms of Ashtekar variables, this intuitive picture means that we have in each tetrahedron some constant $\tilde{E}^{i\alpha}$, and a complex connection $A^i_\alpha$ with support on the triangles
separating tetrahedra. So, if tetrahedron A ((1234) in fig.1) shares the triangle (123) with its neighbour B (1235), we can associate to the triangle variables \((S, g)\) given in local coordinates by:

\[
S_A := \tau_i S^i_A := \tau_i \tilde{E}_{A}^{i\ell} \ abc S^{bc} = -g(AB) S_B g(AB)^{-1}; \quad g(AB) := P \exp \int_{A}^{B} A \cdot \tau dl
\]  

where \(\tau_i = \frac{g_{ij}}{2}\), \(g(AB)\) is the \(SL(2,\mathbb{C})\) element that takes from the B to the A frame across the triangle, \(S^i_A\) (or the traceless \(2 \times 2\) matrix \(S_A\)) is the (oriented) area of the triangle in the frame of A; if the vertices of the triangle are \((x_1^a, x_2^a, x_3^a)\), \(S^{ab} = \frac{1}{2}(x_1 x_2 + x_2 x_3 + x_3 x_1)^{[ab]}\). The areas are oriented outwards in each frame, which explains the minus sign in (1). Curvature is found going round an edge, e.g.:

\[
R_{(12)A} := g(AB)g(BC)g(CA) := \exp(F_{(12)A})
\]  

Gauge transformations are Lorentz \((SL(2, \mathbb{C}))\) transformations in each tetrahedron:

\[
S_A \to g_A S_A g_A^{-1}, \quad S_B \to g_B S_A g_B^{-1}, \quad g(AB) \to g_A g(AB) g_B^{-1}
\]  

From this equations we see that the 3-vectors \(S^i\) transform according to the self-dual (or \((1, 0)\)) representation of the Lorentz group. It is the use of this representation that assures the nice match between the formalism and the geometric picture. The price to pay is that the variables are complex, subject to "reality conditions" mutated from the continuum theory [5]. The \(S^i\) variables are in general complex, but a first set of conditions is that in a tetrahedron all
$S^2$ must be real positive and all scalar products real; this insures that we can choose a frame such that all the $S^i$ are real vectors ("time gauge"). To express the second set of conditions, which involve the $g$ variables, we have to define for each pair of triangles belonging to a tetrahedron (say (123) and (214) of A in fig.1) a variable $\tilde{l}$, associated to their common edge by:

$$[S_{(123)A}, S_{(214)A}] = \frac{3}{4} V_A [\tau_i, \tau_j] \xi_{abc} \overline{E}_A^{ia} \overline{E}_A^{jb} (x^2 - x^1)^c := \frac{3}{2} V_A \tilde{l}_{(12)A}$$

where $V_A = \frac{1}{6} \xi_{abc}(x_1 x_2 x_3 + x_2 x_1 x_4 + x_3 x_4 x_1 + x_4 x_3 x_2)^{abc}$ is the volume of A in local coordinates. In the same way the triangles (123), (215) give a variable $\tilde{l}_{(12)B}$. If $g(AB)$ were just 1, the three vectors $S_{(123)}, S_{(214)}, S_{(215)}$ would be coplanar; this basic fact is generalized to:

$$[\tilde{l}_{(12)A}, g(AB) \tilde{l}_{(12)B} g(AB)^{-1}] = ic S_{(123)A}$$

for some real constant $c$, and this is the second set of reality conditions.

Apart from these conditions, which are second class constraints, we have first class constraints, again mutated from the continuum theory. First of all, the tetrahedra must close, so, for tetrahedron A with $I = (123), (231), (341), (432)$:

$$\sum_I S_{I,A} = 0$$

which is our "Gauss law". Further, at a more tentative level, we have analogues of the vector and the scalar constraints; for each tetrahedron:

$$\sum_I \epsilon_{acb} F^c_{I} \tilde{l}_I = 0; \quad \sum_I F^a_{I} \tilde{l}_I = 0$$

these expressions were shown to give the Ashtekar constraints in the continuum limit, and to be respectively pure imaginary and real using the reality conditions (5).

This is the setting on which I propose to build a quantum theory.

3. Poisson brackets.

To put some dynamics in the kinematic setting, we need first of all to specify a Liouville form, so that the Poisson brackets between the variables can be calculated.

The form I propose is, labeling triangles and tetrahedra with indices $I$ and $T$ respectively:

$$\Theta = i \sum_T \sum_{I \in T} \text{Tr}(S_{I,T} dg_I g_I^{-1})$$

This is similar to what other authors have proposed [2,3], and looks very much like the Liouville form for a set of rotating tops (but notice the factor $i$, and that the $g_I$ are $SL(2, \mathbb{C})$ matrices, not $SU(2)$). One can show that (8) reproduces the continuum Liouville form in a
naïve continuum limit. In fact, a given triangle contributes twice to the sum, but the two contributions are equal; explicitly for say (123) of fig.1:

\[ i \text{Tr}(S_A dg(AB) g(AB)^{-1}) + i \text{Tr}(S_B dg(BA) g(BA)^{-1}) = 2i \text{Tr}(S_A dg(AB) g(AB)^{-1}) \]  

(9)

We may split the displacement from the middle of A to the middle of B in two parts, so that

\[ g(AB) \approx 1 + \tau_i A_a^i (\delta_a^A + \delta_a^B) \]  

(10)

from this approximate form and (1) we get for the contribution of a tetrahedron to \( \Theta \):

\[ \Theta_T = i \sum_{t \in T} \text{Tr}(\tau_i \tilde{E}^{ia}_\xi_{abc} S^{bc}_t \tau_j dA^d_t \delta^d_{i T}) = -i \frac{1}{2} \tilde{E}^{ia}_d dA^i_d \xi_{abc} \sum_{t \in T} S^{bc}_t \delta^d_{i T} = -i \tilde{E}^{ia}_d dA^i_d V_T \]  

(11)

which agrees with the continuum theory [5]; in the last step we have used an identity (eq.31 of [1]) for the volume \( V_T \) of \( T \).

The other merit of the proposed form is that under gauge transformations:

\[ \text{Tr}(S_A dg(AB) g(AB)^{-1}) \rightarrow \text{Tr}(S_A dg(AB) g(AB)^{-1}) + \text{Tr}(S_A dg_A g_A^{-1}) + \text{Tr}(S_B dg_B g_B^{-1}) \]  

(12)

summing over all triangles and all tetrahedra we see that if the Gauss law (6) is satisfied, the Liouville form is invariant, just like in the continuum.

The sympletic form is \( \Omega = d\Theta \), and the algebra to calculate the Poisson brackets can be borrowed from the theory of the rotating top. The result is (I use the notation \( g^{(1)} = g \otimes 1 \), \( g^{(2)} = 1 \otimes g \)):

\[ \{ S^i_A, S^j_A \} = -i \epsilon_{ijk} S^k_A ; \quad \{ g(AB), S^i_A \} = -i \tau_i \cdot g(AB) ; \quad \{ g(AB)^{(1)}, g(AB)^{(2)} \} = 0 \]  

(13)

We see that the \( S \) variables (summed for each tetrahedron) are the generators of gauge transformations. The factor \( i \) may appear surprising, and it will prove fatal to the naïve quantization scheme of §5, but our insisting that the Liouville form has the correct continuum limit implies that it must be inexorably there. In the continuum, when the Gauss law and the reality conditions are satisfied and in time gauge, the Ashtekar connection becomes the Sen connection:

\[ A^i_a = -\frac{1}{2} \epsilon_{ijk} \Omega^j_a + ie^{ib} K_{ab} \]  

(14)

from this we see that the \( i \) that appears in the last member of (11) is indeed correct, because the dynamics is in the imaginary part of the Ashtekar connection, the extrinsic curvature \( K_{ab} \) (\( \Omega^j_a \) is the Levi–Civita connection).
4. Geometrodynamics.

Given that the presentation of the variables and of the model has been rather formal, one may wonder whether the details are so important: could one change the Poisson brackets removing the offending factor $i$? or solve the reality conditions and use real variables, like in geometrodynamics? Removing the factor $i$ would mean to start with a model in which the group is $SU(2)$, from which the full theory could be reached by analytic continuation. I think this unlikely: as we shall see, taking an explicit real parametrization for the canonical variables, one finds that it is unlikely that a geometrically sensible $SU(2)$ theory exists, because its Liouville form would most likely vanish, and there would be no local degrees of freedom left.

Choosing the time gauge for tetrahedra A and B, i.e. frames in which all the $S^i$ are real 3–vectors, we can introduce the following parametrization for the basic variables:

$$S_A := u_A \cdot \tau_3 s \cdot u_A^{-1} ; \quad S_B := u_B \cdot \tau_3 s \cdot u_B^{-1} ; \quad \text{with: } u_A = e^{\alpha \tau_3} e^{\beta \tau_2} ; \quad u_B = e^{\gamma \tau_3} e^{\delta \tau_2}$$

(15)

where $\alpha \ldots \delta$ are the polar angles of $S^i_A$, $S^i_B$ and $s > 0$. Then by eq.(1) $g(AB)$ must be of the form:

$$g(AB) := u_A e^{(\Phi + i \zeta) \tau_3} 2 \tau_2 u_B^{-1}$$

(16)

Here $\zeta$ is the rapidity of the Lorentz transformation; the matrix $2 \tau_2$ is there because the $S$ have opposite orientation in the two frames, and $\Phi$ determines the axis around which the rotation by $\pi$ takes place. Notice that this is essentially the parametrization used in [11].

Expressed in terms of these variables the contribution of a triangle to the the Liouville form becomes:

$$2i \text{Tr}(S_A d g(AB) g(AB)^{-1}) = s d \zeta - i s d \Phi + 2is \text{Tr}(\tau_3 u_A^{-1} du_A) + 2is \text{Tr}(\tau_3 u_B^{-1} du_B) =$$

$$= s d \zeta - i s d \Phi - i s \cos \beta d \alpha - i s \cos \delta d \gamma$$

(17)

The real part of this form is just what was assumed in [10] in the $(2 + 1)$–d case, which is promising. As for the imaginary part: in the continuum, when the Gauss law and reality condition are imposed, the imaginary part of the Liouville form involves the Levi–Civita connection (see eq.(14)), and it integrates to zero. I therefore conjecture that, under the same hypothesis,

$$Im \Theta = \sum_t s_t (d \Phi_t + \cos \beta_t d \alpha_t + \cos \delta_t d \gamma_t) = 0 \ ?$$

(18)

This is a crucial test of consistency for the whole scheme, but I have not been able to prove that indeed it is so. It is however very plausible that when all the $\zeta$ are set to zero we are left with the Levi–Civita connection. And indeed, replacing eqs.(15)(16) in the reality conditions
(5), one can determine the angle $\Phi$ in terms of the others. To keeps things simple, define, with reference to the edge (12) of fig.1:

$$u_{(214)}^{-1} A u_{(123)} A := \exp(\tau_3 \phi_{(12)} A) \exp(\tau_2 \theta_{(12)} A) \exp(-\tau_3 \psi_{(12)} A)$$

$$u_{(213)}^{-1} B u_{(125)} B := \exp(\tau_3 \phi_{(12)} B) \exp(\tau_2 \theta_{(12)} B) \exp(-\tau_3 \psi_{(12)} B)$$

where $\theta_{(12)} A$, $\theta_{(12)} B$ are the angles between the faces (123) and (214), and between (213) and (125). After some algebra, we find that eq.(5) for the edge (12) and tetrahedra A and B becomes:

$$\cosh \zeta \sin \theta_{(12)} A \sin \theta_{(12)} B \sin(\phi_{(12)} B + \psi_{(12)} A - \Phi_{(123)}) + i \sinh \zeta (\ldots) = i c$$

for some real c, so that:

$$\Phi_{(123)} = \phi_{(12)} B + \psi_{(12)} A \mod \pi$$

This simple relation implies that the angles $\Phi$ neatly cancel from the expression for the curvature, which becomes:

$$R_A = u_{3A} e^{(\phi_3 + i \zeta_3) \tau_3} 2 \tau_2 u_{3B}^{-1} u_{5B} e^{(\phi_5 + i \zeta_5) \tau_5} 2 \tau_2 u_{5C}^{-1} u_{4C} e^{(\phi_4 + i \zeta_4) \tau_4} 2 \tau_2 u_{4A}^{-1} =$$

$$= u_{4A} e^{\phi_A \tau_3} e^{\theta_A \tau_2} e^{i \zeta_3 \tau_3} 2 \tau_2 e^{\theta_B \tau_2} e^{i \zeta_5 \tau_3} 2 \tau_2 e^{\theta_C \tau_2} e^{i \zeta_4 \tau_3} 2 \tau_2 e^{-\phi_A \tau_3} u_{4A}^{-1}$$

(we have dropped subscripts (12)), in agreement with the expression found in [1].

The reality conditions eq.(5) also give more relations between the angles, because we should obtain the same $\Phi_{(123)}$ repeating the argument that leads to eq.(21) for the edges (23) and (31); therefore:

$$\phi_{(12)} B + \psi_{(12)} A = \phi_{(23)} B + \psi_{(23)} A \mod \pi$$

These relations cannot be all independent, although I have found the detailed trigonometry too complicated to disentangle. One can however argue as follows: suppose the Regge lattice has $N_0$ vertices, $N_1$ edges, $N_2$ triangles and $N_3$ tetrahedra. If it is the discretization of a closed 3-manifold we know that [12]:

$$2 N_3 = N_2 \quad ; \quad N_0 - N_1 + N_2 - N_3 = 0$$

The metric and the orientation of each frame requires that we give the lengths of the $N_1$ edges and three angle for each of the $N_3$ tetrahedra. Alternatively, we can give the $N_2$ areas and $8 N_3$ angles, subject to the $3 N_3$ condition that each tetrahedron closes (eq.(6)) and to $X$ conditions coming from the reality conditions eq.(23). Therefore:

$$N_1 + 3 N_3 = N_2 + 8 N_3 - 3 N_3 - X \quad \Rightarrow \quad X = 2 N_2 - N_1$$
Hence, between the $2N_2$ relations eq.(23) we see that only $2N_2 - N_1$ are independent. This is plausible if we remember that these relations basically state the coplanarity between the triangles that share an edge. But just because of this geometric motivation, it is difficult to imagine a theory in which the $\Phi$ angles would be independent variables, and the angular contribution to the Liouville form, i.e. its imaginary part, is most likely to vanish as conjectured above, eq.(18).

Having failed to prove eq.(18), I have tried numerically to check whether it is true at least in the simplest model: $S^3$ divided in five tetrahedra, i.e. the boundary of a 4–simplex [13]. One may assign random lengths to the 10 edges, choose arbitrarily a frame in each tetrahedron and calculate all the areas and angles; then vary slightly the lengths, recalculate everything and in this way estimate $\text{Im} \Theta$ when the Gauss law and the reality conditions are imposed. However carefully one repeats the calculation, the extent to which it can be trusted is debatable, but it does confirm the conjecture; more precisely one finds that eq.(18) holds, numerically, more or less like the Regge theorem (see [4], app.1) that the contribution of the angle variation to the variation of his action is zero, $\sum l_i d\Theta_i = 0$. However, while the Regge theorem holds for each tetrahedron, it does not appear that, if one uses eq.s(21)(23) to reorganize the sum in eq.(18) as a sum over tetahedra, the individual contributions vanish.

I would like to conclude this section with one further question: when all constraints are taken into account, how many degrees of freedom does the theory have? This is an important, and non trivial question, first put by Regge in [4], but I do not know of an articulate answer. We would like to know how many of the $4N_3$ constraints eq.(7) are independent, and by how much do they reduce the number of independent variables. From areas and angles, taking the relevant constraints into account, we get $N_1$ independent variables , to which one should add the $N_2$ rapidities $\zeta$. As a first guess, I would expect $N_1$ relations between these to come from the constraints eq.(7), leaving us with $N_2 = 2N_3$ independent variables, simply because I see no other way of getting an even number. The analysis of the constraints is certainly not going to be easy.

5. Quantization.

Given the Poisson brackets, one should go on to check the algebra of the constraint, a very difficult task, that I have not attempted. On the other hand, one may simply hope for the best, and try to quantize the theory by blindly turning Poisson brackets into commutators

$$[\hat{S}^i, \hat{S}^j] = -\epsilon_{ijk} \hat{S}^k; \quad [\hat{S}^i, \hat{g}] = \tau_i \cdot \hat{g}; \quad [\hat{g}^{(1)}, \hat{g}^{(2)}] = 0$$

The form of these commutation relations makes the connection representation appear "natural": we take wave functions to be holomorphic, gauge invariant functions of the $g$–s. On this wave functions the $S^i$-s act like (minus) the generators of the left–regular representation
of \( SL(2, \mathbb{C}) \) \(^{14}\) \(^{15}\):

\[
- \hat{S}^i \Psi(\ldots, g, \ldots) = \Psi(\ldots, -\tau^i g, \ldots)
\]  

(27)

A basis for the wavefunctions is provided by the "net states"\(^7\): given a net in the dual lattice, (the lattice made of links that cross our triangles), choose a representation \((j_l, 0), \ j_l = 0, \frac{1}{2}, 1, \ldots \) of \( SL(2, \mathbb{C}) \) for every link \( l \) in the net, and an appropriate invariant tensor \( C(\{j\}) \) for every vertex. Then a gauge invariant, holomorphic wavefunction will be given by:

\[
\Psi = \sum C(\{j\}) \prod_l \mathcal{D}^{j_l}(g_l)
\]  

(28)

The scheme is however ruined because it cannot have an acceptable scalar product, i.e. one such that the \( \hat{S}^i \) are hermitian, and/or \( S^2 \) is positive.

Suppose one takes as scalar product, starting from the Haar measure \( d\mu_H(g) \) on \( SL(2, \mathbb{C}) \):

\[
< \Psi | \Phi > = \int f(g) \overline{\Psi(g)} \Phi(g) \ d\mu_H(g)
\]  

(29)

with some \( f(g) \) to be determined. Then since \( \int \hat{S}^i(\ldots) d\mu_H(g) = 0 \), and \( \hat{S}^i \overline{\Psi}(g) = 0 \) (see the previous footnote):

\[
\int f(g) \overline{\Psi(g)} \hat{S}^i \Phi(g) \ d\mu_H(g) = \int f(g) \hat{S}^i \Psi(g) \Phi(g) \ d\mu_H(g) \implies \hat{S}^i f(g) = \overline{\hat{S}^i f(g)}
\]  

(30)

but this cannot be, because using the commutation relations:

\[
\hat{S}^i f = -\epsilon_{ijk} \hat{S}^j \hat{S}^k f = -\epsilon_{ijk} \hat{S}^j \hat{S}^k \hat{S}^j f = -\epsilon_{ijk} \hat{S}^k \hat{S}^j f = -\epsilon_{ijk} \hat{S}^k \hat{S}^j f = -\overline{\hat{S}^i f}
\]  

(31)

So the search for a measure fails. A similar argument shows that it fails also with the more general ansatz for the scalar product (suggested by Carlo Rovelli)

\[
< \Psi | \Phi > = \int f(g_1, g_2) \overline{\Psi(g_1)} \Phi(g_2) \ d\mu_H(g_1) d\mu_H(g_2)
\]  

(32)

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\(^1\) Explicitly: If \( g \in SU(2) \), it is represented in the left–regular representation by an operator \( U_g \) which acts on functions on \( SU(2) \) itself like: \( (U_g F)(g') = F(g^{-1} g') \). If we choose the Euler angles \( \{\phi^a\} \) as coordinates on the group, the generators of this representation will be some linear differential operators \( \hat{T}_{Li} = E^a_i(\phi) \frac{\partial}{\partial \phi^a} \), which give on irreducible representations \( \hat{T}_{Li} \mathcal{D}^j(g) = -T^j_i \mathcal{D}^j(g) \). Analytically continuing to \( SL(2, \mathbb{C}) \), \( \phi^a \to \varphi^a + i\eta^a \), \( E^a_i \to e^a_i + ih^a_i \); the continued generators can be grouped into:

\[
\hat{T}_{Li}^c = \frac{1}{2}(e^a_i + ih^a_i)(\frac{\partial}{\partial \varphi^a} - i \frac{\partial}{\partial \eta^a}); \quad \hat{T}_{Li}^c = \frac{1}{2}(e^a_i - ih^a_i)(\frac{\partial}{\partial \varphi^a} + i \frac{\partial}{\partial \eta^a})
\]

which satisfy separately the algebra and commute with each other. Continuing the irreducible representations to \( SL(2, \mathbb{C}) \), one has \( \hat{T}_{Li}^c \mathcal{D}^j(g) = -T^j_i \mathcal{D}^j(g) \), \( \hat{T}_{Li} \overline{\mathcal{D}}^j(g) = 0 \). What I am saying is that \( \hat{S}^i = -\hat{T}_{Li}^c \).
where we would require $\hat{S}_2^i f = \hat{S}_1^i f$.

On the contrary, the same argument was successful with linearized gravity, and for the $(q, z = q - ip)$ oscillator [6][5]. In this latter case one sets:

$$\hat{z}\psi(z) = z\psi(z) , \quad \hat{q}\psi(z) = \frac{\partial \psi}{\partial z} ; \quad \hat{q}^\dagger = \hat{q} , \quad \hat{z}^\dagger = -\hat{z} + 2\hat{q}$$

(33)

and finds that these (abelian) reality conditions do have a solution:

$$<\psi|\phi> = c \int d^2 z \exp \left( -\frac{1}{4}(z + z^*)^2 \right) \psi(z)^* \phi(z)$$

(34)

In our case the $i$ factor spoils the whole scheme.

This failure is somewhat disheartening, and would seem to block further progress in this direction.

5. Conclusions.

I have tried to develop a discretized canonical theory of gravity in the spirit of Regge calculus, and that means keeping as much as possible of the underlying geometric structure of the theory. Inevitably many of the original difficulties remain, plus new ones which may be artefacts of the discretization. The discussion of §4 makes perhaps clear that the "geometric-dynamic" alternative is far too complicated to be practicable. In §5 it has been shown that a direct use of the Ashtekar-like variables does not lead to a sensible quantum theory. Some suitable middle ground should be found, perhaps restricting the configuration space with a careful analysis of the other constraints.

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References

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