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**SNOWFLAKE HARMONICS AND COMPUTER GRAPHICS :
NUMERICAL COMPUTATION OF SPECTRA ON FRACTAL DRUMS**

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Abstract

In this work, we study the steady-states vibrations of the “Koch snowflake drum”, both numerically and by means of computer graphics. In particular, we approximate the smallest 50 eigenvalues (or “frequencies” of the drum), along with the corresponding eigenfunctions (called “snowflake harmonics”) of the Dirichlet Laplacian on the Koch snowflake domain. We describe the numerical methods used in the computations, and we display graphical representations of a selected set of the eigenfunctions (as well as of their gradients). In the case of the first harmonic, the graphical results agree with mathematically derived results (by Lapidus and Pang) concerning gradient behavior (“blow up” or “infinite stress” of the membrane) on the boundary and suggest further conjectures regarding the higher eigenfunctions.

According to earlier work by the physicist Sapoval and his collaborators, this research may help better understand the formation and “stabilization” of fractal structures (e.g., coastlines, trees and blood vessels) in nature.

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1 Introduction

Objects with irregular geometry can be found everywhere in nature and their vibrational properties are of general interest. How trees behave in the wind and how coastlines are formed are natural questions. The emergence of fractal geometry provides a framework within which to address these questions and has prompted several recent works on this subject by physicists and mathematicians.

Physical experiments on the mechanical vibrations of fractal drums have been carried out by the physicist B. Sapoval and his collaborators, Th. Gobron and A. Margolina, [Sapoval, 1989, 1990; Sapoval & Gobron, 1993; Sapoval *et al.*, 1991]. The “fractal drum” in their experiments consisted of a stainless steel sheet with the boundary laser etched in the pattern of a square snowflake. A “soap bubble” is deposited on this object. A loudspeaker situated above the fractal drum excites low frequency vibrations in the bubble which are then visually observed. Their results have suggested, in particular, that wave motion is “localized” and severely damped in media with fractal boundary. Other work by the physicist S. Sridhar and his collaborators [Kudrolli & Sridhar, 1994] investigate aspects of quantum chaos by looking at the patterns created when microwaves move about inside thin metal enclosures (microwave cavities), and more recently, the vibrations of certain drums with fractal boundaries [Sridhar *et al.*, 1993] (using severely squashed cavities).

Much of the mathematical work so far—motivated in part by the study of the scattering of waves off fractal or random surfaces as well as the study of porous media [Berry, 1979 & 1980]—has concentrated on the asymptotic distribution of the eigenvalues on regions with fractal boundary. See, e.g., [Brossard & Carmona, 1986; Lapidus, 1991; Evans & Harris, 1993; He & Lapidus, 1995; Lapidus & Maier, 1995; Lapidus & Pomerance, 1993 & 1996] as well as the survey article [Lapidus, 1993] and the references therein. Recently, however, M. L. Lapidus and M. Pang [1995] have studied rigorously the boundary behavior of the Dirichlet Laplacian eigenfunctions and their gradients on a class of planar domains with fractal boundary, including the (triadic and square) Koch snowflake domains, as well as their polygonal approximations. In particular, one of their main results, specialized to the Koch snowflake domain, implies that the (magnitude of the) gradient of the first eigenfunction (or “first harmonic”) “blows up” at infinitely many boundary points, namely, those with an obtuse angle corner as well as those in

a Cantor-type set of boundary points. Consequently, in the terminology of Sapoval [1989], the membrane of the Koch snowflake drum exhibits “infinite stress” near such points.

A long term goal of the present research is to try to understand the formation and self-stabilization of fractal (i.e., extremely irregular) structures in nature. This is of great scientific interest, not only in mathematics and physics, but also in geology, ecology and biology. For example, some of the questions it may help answer are the following [Sapoval, 1989 & 1990]: 1.) Why are waves damped much more efficiently by fractal rather than by smooth coastlines? 2.) Why does the fractal foliage of trees provide such strong resistance to the wind? 3.) How can the extreme fractality of our circulatory system dampen the hammer blow generated by the heart pump?

In this paper we will investigate the eigenfunctions of fractal drums (i.e., “drums with fractal boundary” or “surface fractals”) by means of computer graphics and numerical experiments. We will compare our results with some of the aforementioned experimental results obtained by Sapoval *et al.* as well as theoretically derived results by Lapidus and Pang. (Of course, in contrast to the mathematical work in Lapidus & Pang [1995], the “fractal” boundary in the present numerical work and in that of Sapoval *et al.* is only a *finite approximation* to the limiting fractal shape.) Moreover, we will use the results of our computer simulations to propose conjectures regarding the boundary behavior of the higher eigenfunctions and their gradients.

More specifically, we wish to approximate the first N_e eigenvalues, along with the corresponding eigenfunctions, of the negative Dirichlet Laplacian $-\Delta$ on a planar domain Ω having a fractal boundary $\partial\Omega$:

$$\begin{cases} -\Delta u &= -(u_{xx} + u_{yy}) = \lambda u, & \text{for } (x, y) \in \Omega, \\ u &= 0, & \text{for } (x, y) \in \partial\Omega. \end{cases} \quad (1)$$

(Mathematically, this boundary value problem must be interpreted in the variational sense; see, for example, Sec. 2.1 in Lapidus [1991] .) We focus here on a prototypical model, namely the (triadic) *Koch snowflake drum* corresponding to the Koch snowflake domain Ω (see Fig. 1). The associated eigenfunctions will also be called *snowflake harmonics*.

Recall that Eq. (1) can be considered as a mathematical model for the (steady-state) vibrations of a drum. The eigenvalues (or rather, their square-roots) correspond to the natural frequencies (or fundamental modes) of the

Figure 1: The Koch Snowflake Drum Ω

drum. The first eigenvalue corresponds to the fundamental tone (with associated eigenfunction called the first harmonic) and the higher eigenvalues correspond to the overtones (with associated eigenfunctions referred to as the higher harmonics) of the drum.

Further, recall that the spectrum of the Dirichlet Laplacian on a bounded (planar) domain is discrete and consists of an infinite sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ written in increasing order according to their (finite) multiplicity:

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \text{ with } \lambda_n \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (2)$$

Moreover, let φ_n be the eigenfunction corresponding to λ_n (and normalized by $\|\varphi_n\|_{L^2(\Omega)}=1$). Then φ_n is smooth (i.e., infinitely differentiable) in Ω , but not necessarily up to the boundary $\partial\Omega$. Finally, the first eigenvalue λ_1 is *simple* and the first harmonic can be chosen so that $\varphi_1 > 0$.

Some of our present joint results were announced in Sec. 3.2 of Lapidus [1995].

We close this introduction by explaining how the remainder of the paper will be organized: Section 2 will focus on the computer graphics. We present our graphics of the eigenfunctions (the “snowflake harmonics”) and their gradients (Secs. 2.1-2.4). We also discuss their applications and explain how they help formulate conjectures regarding the boundary behavior of the higher eigenfunctions. Further, we display the nodal lines and the contour plots of the eigenfunctions (Secs. 2.5 and 2.6). Moreover, we comment on the symmetries of the eigenfunctions and the multiplicities of the associated eigenvalues (Sec. 2.7). Finally, in Sec. 2.8, we briefly discuss our method for generating the computer graphics. Section 3 will concentrate on the numerical methods used in the present work. In Sec. 3.1, we give the method used to approximate the Koch (triadic) snowflake domain by “discretizing” our domain into triangular grids. In Sec. 3.2, we discuss the process of generating a discretized Laplacian associated to this triangulation. In Sec. 3.3, we use the SOR method (successive over-relaxation) to solve our sequence of linear systems involving the discretized Laplacian. In Sec. 3.4, we discuss the inverse power method and apply it to our solution of the linear systems. In Sec. 3.5, we describe the storage requirements of the computer algorithm. Section 4 focuses on the implementation of the computer algorithm. First, in Sec. 4.1, we provide an outline of the main steps involved in the algorithm. In Sec. 4.2, we then list the first fifty computed eigenvalues. Finally, in Sec.

5, we conclude this paper with some comments, conjectures and problems regarding future work.

2 The Snowflake Harmonics

We display here the graphical representations of the eigenfunctions of the Koch snowflake drum and also provide comments on these pictures. Unless otherwise specified, the graphics will be presented at fractal level 4. (Please see Fig. 2 in Sec. 3.1 below for the definition of fractal levels and refinements.)

2.1 A closer look at selected eigenfunctions

We first present some of the more interesting snowflake harmonics, beginning with the first one (Plates 1 and 2). Note that in agreement with the classical results, the first eigenfunction φ_1 is positive and the associated eigenvalue λ_1 is simple. Plate 2 depicts a magnified view of the left side of Plate 1. It is both illuminating and deceiving. Indeed, caution must be used in interpreting Plate 2 since one may get the mistaken impression that there is a “wall” of infinite slope all along the boundary. This is not so, however, as is shown in the work of Lapidus & Pang [1995]. We will elaborate on this point in Sec. 2.3 where we will examine the magnitude of the gradient of the eigenfunctions.

In contrast with the first harmonic, the second eigenfunction φ_2 displayed in Plate 3 is not associated with a simple eigenvalue. In fact, λ_2 has multiplicity two. The eigenfunction φ_3 — which also belongs to the eigenvalue λ_2 — would look exactly like φ_2 except for a rotation (and reflection), as will be further commented on in Sec. 2.7. (Also see Plates 6 and 21.) The thirteenth eigenfunction (Plate 4) is highly symmetrical as will be further evidenced by the contour plots displayed in Sec. 2.6 (Plate 21). The final picture in this section (Plate 5) gives several views of the nineteenth eigenfunction (which has multiplicity two) in order to demonstrate how different an eigenfunction may appear when viewed from various angles.

2.2 A selection of the first fifty eigenfunctions

We now present some of the first fifty “snowflake harmonics”. More specifically, we display the first sixteen eigenfunctions (Plates 6–9) and a few of the higher ones (Plate 10): the 36th, 37th, 38th and 45th. The 45th eigenfunction is a very good example of a higher eigenfunction with no obvious symmetries; on the other hand, the 36th, 37th and 38th are highly symmetrical.

2.3 The magnitude of the gradient of the 1st and 13th eigenfunction

In this and the following section, we display the magnitude of the (numerical approximation to) the gradient of several of the eigenfunctions. Consistent with the theoretically derived results of Lapidus & Pang [1995], the magnitude of the gradient of the first eigenfunction, $|\nabla\varphi_1|$, “blows up” at those boundary points with an obtuse angle corner (Plate 11). Moreover, $|\nabla\varphi_1|$ tends to zero as one approaches a boundary point with an acute angle corner. (All angles are measured from the inside.) This same graphical behavior is also exhibited by the higher eigenfunctions (Plates 12 and 16) which suggests that the results of Lapidus & Pang [1995] may be extended beyond the first harmonic. In addition, the magnitude of the gradient produces some very beautiful graphics which are equally stunning when rotated. In Plates 12 and 13, we display a “top” view and a “bottom” view, respectively, of $|\nabla\varphi_{13}|$. In Plate 14, we have also included a one-sixth slice of $|\nabla\varphi_{13}|$ to demonstrate how the shape changes as one approaches the Koch snowflake boundary through increasing (fractal) levels; note that the peaks are becoming sharper and sharper. Finally, Plate 15 depicts the same slice (as in Plate 14) upsidedown at fractal level 5.

2.4 The magnitude of the gradient of a higher eigenfunction

In Plates 16 and 17, we provide an analog of Plates 12 and 14 for the 45th eigenfunction. Note the extreme roughness of $|\nabla\varphi_{45}|$ and the same boundary behavior as $|\nabla\varphi_{13}|$.

2.5 The nodal lines of the eigenfunctions

The nodal line graphs were created to better view the zeros of the eigenfunctions. The regions colored in red correspond to an ϵ -neighborhood of the true nodal lines with radius $\epsilon = .001$. Regions colored in yellow and blue represent respectively the positive and negative portions of the eigenfunction outside of the ϵ -neighborhood. Here we display the nodal lines of the 2nd and 13th eigenfunctions (Plates 18 and 19) and selected higher eigenfunctions (Plate 20).

2.6 Contour plots of the eigenfunctions

The symmetries of the eigenfunctions can best be seen in the contour plots as well as by checking the numerical values (given in Sec. 4.2 below) of the eigenvalues. We display here (with a fifteen color scale) the contour plots for some of the more interesting eigenfunctions. (See Plates 21–23.)

2.7 Symmetries of the eigenfunctions and multiplicities of the associated eigenvalues

A number of apparent symmetries are revealed by the contour plots. They are based on the fact that Ω (or an approximation to Ω) is invariant under a planar rotation by a multiple of 60 degrees or a reflection about any one of the six lines through the center. Note also that the Laplacian is invariant under a change of variables associated with rotation or reflection.

Among the first 50 eigenvalues, 16 are simple and the others have multiplicity 2. In order to determine the multiplicity of the eigenvalues, we have used the present contour plots of the eigenfunctions as well as the numerical values of the λ_i 's provided in Sec. 4.2 below. (Caution: the multiplicities are not rigorously derived.) The indexes of the simple eigenvalues are 1, 6, 7, 10, 13, 14, 23, 24, 25, 26, 29, 36, 37, 38, 43, and 48.

Half of the eigenfunctions associated with the simple eigenvalues (7, 10, 14, 25, 26, 36, 43, 48) have 60 degree rotational symmetry. The other half, when rotated by a 60 degree rotation R , turn into the negative of themselves; namely, $u = -u \circ R$. In fact, any eigenfunction rotated 60 degrees yields an eigenfunction (due to the symmetry of the region and the rotational invariance of the Laplacian). For example, since λ_2 has multiplicity 2, the reader

may wish to verify that the second eigenfunction when rotated 60 degrees is a linear combination of the second and third eigenfunctions. (See Plate 6.)

All of the eigenfunctions are either invariant or change sign under a 180-degree rotation of the domain. Each member of a pair associated with an eigenvalue of multiplicity 2 behaves the same way as the other member, and the pairs alternate in behavior: pairs 4-5, 11-12, 17-18, 21-22, 30-31, 34-35, 41-42, and 46-47 are invariant, while pairs 2-3, 8-9, 15-16, 19-20, 27-28, 32-33, 39-40, 44-45, and 49-50 change sign under 180-degree rotation.

Two of the pairs (2-3 and 8-9) have members related to each other as follows: each can be obtained by a 120-degree rotation and a reflection applied to the other; for example, 120-degree clockwise rotation followed by a reflection about the y axis (in both cases). Recall that by construction the members of a pair are orthogonal to each other (as well as to the remaining eigenfunctions).

Eigenfunction 30 appears to be invariant under reflections about both the x and y axes. Eigenfunction 31 changes sign under reflections about the axes.

The symmetries observed above hold at all fractal levels and are confirmed by contour plots of the gradient norms (which are not shown here). There are several additional instances that exhibit near symmetry. In these cases it is not clear whether the absence of symmetry in the plots is due to an absence of symmetry in the eigenfunctions, or to poor approximations of the eigenfunctions.

2.8 A note about the creation of the graphics

The first 50 eigenfunctions have been approximated at the University of California on the San Diego Supercomputer Center's Cray Y-MP8/864. (For later computations, the Cray C90 was used.) Cray computer systems were used to better handle the amount of computations involved in the algorithm as well the large output files that are created, and to achieve greater numerical accuracy. The output is a binary file containing node X, Y and Z values and a grid map which describe how the nodes are connected to form the Koch snowflake. The file also contains the number of nodes and the number of triangles in the grid. This is fairly standard information (with minor format changes) for many plotting packages. In our case, the binary files were converted to ASCII with minor format changes to be readable by

a plotting package called geomview on the Silicon Graphics (SGI) workstation. Geomview is a shareware package (for SGI systems only) available via anonymous ftp to geom.umn.edu. A color scale was also added during this reformatting process based on the maximum and the minimum in the eigenfunction.

All graphics were created on a Silicon Graphics Onyx at the Center for Visual Computing of the University of California, Riverside. The Onyx was used because of the size of the files. Each level 4 ASCII eigenfunction “plot file” contains approximately 48,000 triangles with a size of approximately 5 Mbytes. The level 5 approximation is roughly ten times as large, and cannot be reasonably visualized on anything less than an Onyx (which slowed down the system considerably!). The “continuous” color scale is based on using fractions of the colors red, green and blue. The maximum eigenfunction value (along the z-axis) is colored yellow. Values greater than zero vary “continuously” between yellow and red depending on their position relative to the maximum value. The analog is true for values less than zero except that the minimum eigenfunction value is colored blue.

3 Numerical Computation and Justification of the Method

3.1 Approximation of the Koch snowflake domain

The domain Ω is taken to be a Koch snowflake (see Fig. 1 above) — an “equilateral triangle” whose sides are triadic von Koch curves. (See also Fig. 0.2, p. xv in Falconer [1990] or Plate 45, pp. 44–45, in Mandelbrot [1983].) A triangular grid system has been used in place of rectangular grids to fit this geometry. A method for rectangular grids is described in Neuberger & Noid [1987]. We approximate Ω by a sequence of finite difference grids consisting of triangulations containing only equilateral triangles and in which each interior node has six neighbors. For each fractal level beginning with a triangle having unit side lengths (fractal level 0), a sequence of refinements is computed in which each triangle is partitioned into nine equilateral subtriangles by adding two nodes on each edge and one node at the barycenter of each triangle. The level $L + 1$ grid (0-order refinement) is obtained from the first refinement of the level L grid by adding a boundary triangle (and boundary node) to each

side (sequence of three collinear boundary nodes). See Fig. 2.

Note that the sequence of triangulations described above not only defines the Koch snowflake Ω (as the limit of an infinite such sequence), but also serves as a high-level description of an iterative algorithm used to construct the approximations to Ω . The exact ordering of the nodes depends on the details of the algorithm but roughly corresponds to the sequence of triangulations, in the sense that new nodes are appended at each step. Due to the regularity of the grids, the actual nodal coordinates are not needed to compute the eigenvalues and eigenvectors but are computed in order to create graphical depictions of the grids. For this purpose, the first three nodes are taken to be $(0, 0)$, $(1, 0)$, and $(.5, \sqrt{3}/2)$, respectively. It follows that all of the triangulations are contained in a rectangle of width 1 and height $2/\sqrt{3}$. The mesh width is $h = (1/3)^{L+R}$ for fractal level L and refinement level R .

The initial solution estimates for the eigenvectors at refinement level $R+1$ are obtained by interpolation from level R ; i.e., values at the added nodes are computed by triangle-based piecewise linear interpolation. In the following,

$$\begin{aligned} N_b &= N_b(L, R) && \text{Number of boundary nodes,} \\ N_t &= N_t(L, R) && \text{Number of triangles.} \end{aligned}$$

The reader interested in computation may wish to consult Sec. 3.5 below which describes how N_b and N_t relate to the storage requirements.

3.2 Discretized Laplacian

Each interior node (x, y) of a triangulation has six neighbors corresponding to the vertices of a regular hexagon centered at (x, y) . We will obtain a second-order finite difference approximation to $\Delta u(x, y)$ by expanding the value of u at each of these neighbors in a Taylor series about (x, y) . Let $\{u_i\}_{i=1}^6$ denote the counterclockwise-ordered sequence of neighboring values beginning with $u(x + h, y)$ for mesh width h . See Fig. 3. Then

Figure 2: Approximations to the Koch Snowflake, with Fractal Level L and Refinement Level R .

Figure 3: Interior Node u and its Six Neighbors u_i .

$$\begin{aligned}
u_1 &= u(x+h, y) = u(x, y) + u_x(x, y)h + \frac{1}{2}u_{xx}(x, y)h^2 + \\
&\quad \frac{1}{6}u_{xxx}(x, y)h^3 + O(h^4), \\
u_2 &= u(x + \frac{h}{2}, y + \frac{\sqrt{3}}{2}h) = u(x, y) + u_x(x, y)\frac{h}{2} + u_y(x, y)\frac{\sqrt{3}}{2}h + \\
&\quad \frac{1}{2}\left[u_{xx}(x, y)\frac{h^2}{4} + 2u_{xy}(x, y)\frac{\sqrt{3}}{4}h^2 + u_{yy}(x, y)\frac{3}{4}h^2\right] + \\
&\quad \frac{1}{6}\left[u_{xxx}(x, y)\frac{h^3}{8} + 3u_{xxy}(x, y)\frac{\sqrt{3}}{8}h^3 + 3u_{xyy}(x, y)\frac{3}{8}h^3 + \right. \\
&\quad \left. u_{yyy}(x, y)\frac{3\sqrt{3}}{8}h^3\right] + O(h^4), \\
u_3 &= u(x - \frac{h}{2}, y + \frac{\sqrt{3}}{2}h) = u(x, y) - u_x(x, y)\frac{h}{2} + u_y(x, y)\frac{\sqrt{3}}{2}h + \\
&\quad \frac{1}{2}\left[u_{xx}(x, y)\frac{h^2}{4} - 2u_{xy}(x, y)\frac{\sqrt{3}}{4}h^2 + u_{yy}(x, y)\frac{3}{4}h^2\right] + \\
&\quad \frac{1}{6}\left[-u_{xxx}(x, y)\frac{h^3}{8} + 3u_{xxy}(x, y)\frac{\sqrt{3}}{8}h^3 - 3u_{xyy}(x, y)\frac{3}{8}h^3 + \right. \\
&\quad \left. u_{yyy}(x, y)\frac{3\sqrt{3}}{8}h^3\right] + O(h^4), \\
u_4 &= u(x-h, y) = u(x, y) - u_x(x, y)h + \frac{1}{2}u_{xx}(x, y)h^2 - \\
&\quad \frac{1}{6}u_{xxx}(x, y)h^3 + O(h^4), \\
u_5 &= u(x - \frac{h}{2}, y - \frac{\sqrt{3}}{2}h) = u(x, y) - u_x(x, y)\frac{h}{2} - u_y(x, y)\frac{\sqrt{3}}{2}h + \\
&\quad \frac{1}{2}\left[u_{xx}(x, y)\frac{h^2}{4} + 2u_{xy}(x, y)\frac{\sqrt{3}}{4}h^2 + u_{yy}(x, y)\frac{3}{4}h^2\right] + \\
&\quad \frac{1}{6}\left[-u_{xxx}(x, y)\frac{h^3}{8} - 3u_{xxy}(x, y)\frac{\sqrt{3}}{8}h^3 - 3u_{xyy}(x, y)\frac{3}{8}h^3 - \right. \\
&\quad \left. u_{yyy}(x, y)\frac{3\sqrt{3}}{8}h^3\right] + O(h^4), \\
u_6 &= u(x + \frac{h}{2}, y - \frac{\sqrt{3}}{2}h) = u(x, y) + u_x(x, y)\frac{h}{2} - u_y(x, y)\frac{\sqrt{3}}{2}h + \\
&\quad \frac{1}{2}\left[u_{xx}(x, y)\frac{h^2}{4} - 2u_{xy}(x, y)\frac{\sqrt{3}}{4}h^2 + u_{yy}(x, y)\frac{3}{4}h^2\right] + \\
&\quad \frac{1}{6}\left[u_{xxx}(x, y)\frac{h^3}{8} - 3u_{xxy}(x, y)\frac{\sqrt{3}}{8}h^3 + 3u_{xyy}(x, y)\frac{3}{8}h^3 - \right. \\
&\quad \left. u_{yyy}(x, y)\frac{3\sqrt{3}}{8}h^3\right] + O(h^4).
\end{aligned}$$

Hence

$$\sum_{i=1}^6 u_i = 6u(x, y) + \frac{3}{2}h^2u_{xx}(x, y) + \frac{3}{2}h^2u_{yy}(x, y) + O(h^4),$$

and so

$$-\Delta u(x, y) = \left(6u(x, y) - \sum_{i=1}^6 u_i \right) / \left(\frac{3}{2}h^2 \right) + O(h^2). \quad (3)$$

Let $\mathbf{A}$ denote the matrix of order $N - N_b$ representing the discretized negative Laplacian operator, and let u_i now denote the i^{th} nodal eigenfunction value. Then, from Eq. (3), the k^{th} component of the matrix-vector product $\mathbf{A}\mathbf{u}$, for each interior node k , is

$$(\mathbf{A}\mathbf{u})_k = \left(6u_k - \sum_{j \in N_k} u_j \right) / c, \quad (4)$$

where $c = (3/2)h^2$ and N_k is the set of indexes of the neighbors of node k (stored in column k of the adjacency array).

Each row of $\mathbf{A}$ has diagonal entry $6/c$ and at most six nonzero off-diagonal entries, classifying the matrix as sparse. Since $\mathbf{A}_{ij} = -1/c$ if and only if nodes i and j are neighbors, and $\mathbf{A}_{ij} = 0$ otherwise for $i \neq j$, $\mathbf{A}$ is symmetric and therefore has real eigenvalues and a complete set of orthonormal eigenvectors [Golub & Van Loan, 1989, p. 410]. Also, since $\mathbf{A}$ is diagonally semi-dominant with positive diagonal entries, the eigenvalues are nonnegative by the Gershgorin theorem [Golub & Van Loan, 1989, p. 341]. In fact, it can be shown that $\mathbf{A}$ is nonsingular and therefore has positive eigenvalues. The proof that $\mathbf{A}$ is nonsingular is a trivial generalization of the proof, which may be found in Atkinson [1989, pp. 557–559], for the case of a discretization of the Laplacian on a rectangular grid.

3.3 Iterative solution of linear systems

The inverse power method described in the next section requires the solutions to a sequence of linear systems involving the discretized negative Laplacian operator $\mathbf{A}$. We employ an iterative method that eliminates the need to actually store the matrix $\mathbf{A}$. Since $\mathbf{A}$ is positive definite, the Gauss-Seidel method is guaranteed to converge [Golub & Van Loan, 1989, p. 509]. This method applied to the system $\mathbf{A}\mathbf{u} = \mathbf{v}$ consists of repeatedly executing the following loop:

for $k = 1$ to N , the number of nodes

if (k indexes an interior node) **then**

$$u_k = (cv_k + \sum_{j \in N_k} u_j) / 6$$

endif

endfor

Note that step k of the loop consists of solving equation k , $(\mathbf{A}\mathbf{u})_k = v_k$, for a new value of u_k and updating the approximate solution vector $\mathbf{u}$. The only restriction on the initial solution estimate is that the entries associated with boundary nodes must be zeros.

We obtain a better rate of convergence by employing the successive over-relaxation (SOR) method in which the update step is

$$u_k = (1 - \omega) u_k + \omega \left(cv_k + \sum_{j \in N_k} u_j \right) / 6$$

for relaxation parameter ω . The optimal value of ω is computed from an approximation to the smallest eigenvalue of $\mathbf{A}$ [Isaacson & Keller, 1966, pp. 463–470]. Convergence of the method is defined by a bound on the maximum relative change in a solution component between iterations. Since the rate of convergence decreases with smaller mesh width h , requiring a tighter tolerance, the tolerance is heuristically chosen to be 10^{-k} for $k = L + R + 2$, where L (resp., R) is the fractal (resp., refinement) level described in Sec. 3.1.

3.4 Inverse power method

Let $\mathbf{A}$ have eigenvalues $\{\lambda_i\}_{i=1}^n$ for $n = N - N_b$, where

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n,$$

and denote the corresponding sequence of orthonormal eigenvectors by $\{\mathbf{u}_i\}_{i=1}^n$ so that $\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and $\mathbf{u}_i^T \mathbf{u}_j = \delta_{ij}$ for $1 \leq i, j \leq n$. (Of course, the present λ_i 's are not the same as in Eq. (2) above.) Then $\mathbf{A}^{-1}$ has eigenvalues $\{\lambda_i^{-1}\}_{i=1}^n$ with the same eigenvectors. Since the eigenvectors span $\mathbf{R}^n$, an arbitrary vector $\mathbf{x}_0$ may be written $\mathbf{x}_0 = \sum_{i=1}^n \alpha_i \mathbf{u}_i$, and m applications of $\mathbf{A}^{-1}$

to $\mathbf{x}_0$ results in

$$\begin{aligned}\mathbf{A}^{-m}\mathbf{x}_0 &= \sum_{i=1}^n \alpha_i \mathbf{A}^{-m}\mathbf{u}_i = \sum_{i=1}^n \alpha_i \lambda_i^{-m} \mathbf{u}_i \\ &= \lambda_1^{-m} (\alpha_1 \mathbf{u}_1 + \sum_{i=2}^n \alpha_i (\lambda_1/\lambda_i)^m \mathbf{u}_i).\end{aligned}$$

If $\alpha_1 \neq 0$ and $\lambda_1 < \lambda_2$, the above expression (suitably scaled) converges to a multiple of the dominant eigenvector $\mathbf{u}_1$ as $m \rightarrow \infty$.

The power method applied to A^{-1} consists of solving the sequence of linear systems

$$\mathbf{A}\mathbf{x}_{m+1} = \mathbf{x}_m \text{ for } m = 0, 1, \dots, \quad (5)$$

resulting in $\mathbf{x}_m = \mathbf{A}^{-m}\mathbf{x}_0$. Approximations to the dominant eigenvalue λ_1^{-1} are obtained by computing the Rayleigh quotients $r_m = (\mathbf{x}_m^T \mathbf{A}^{-1} \mathbf{x}_m) / (\mathbf{x}_m^T \mathbf{x}_m) = \mathbf{x}_m^T \mathbf{x}_{m+1} / \mathbf{x}_m^T \mathbf{x}_m$. Using orthogonality of the eigenvectors, we have

$$\mathbf{x}_m^T \mathbf{x}_{m+1} = \sum_{i=1}^n \alpha_i^2 \lambda_i^{-2m-1} \text{ and } \mathbf{x}_m^T \mathbf{x}_m = \sum_{i=1}^n \alpha_i^2 \lambda_i^{-2m},$$

from which it follows that

$$r_m = \lambda_1^{-1} \left(\frac{\alpha_1^2 + \sum_{i=2}^n \alpha_i^2 (\lambda_1/\lambda_i)^{2m+1}}{\alpha_1^2 + \sum_{i=2}^n \alpha_i^2 (\lambda_1/\lambda_i)^{2m}} \right). \quad (6)$$

Note that this expression converges to λ_1^{-1} even in the case of a multiple eigenvalue.

In order to prevent $\mathbf{x}_m$ from approaching zero ($\lambda_1 > 1$) or growing in magnitude ($\lambda_1 < 1$), $\mathbf{x}_m$ is normalized to a unit vector at each step. The choice of Euclidean norm $\|\cdot\|_2$ has two advantages over alternatives: it simplifies the computation of r_m , and it facilitates the deflation procedure described below.

The computational procedure represented by (5) may be used to find the k^{th} eigenvalue by simply projecting the initial estimate $\mathbf{x}_0$ onto the orthogonal complement of the first $k-1$ eigenvectors: $\widehat{\mathbf{x}}_0 = \mathbf{x}_0 - \sum_{i=1}^{k-1} (\mathbf{x}_0^T \mathbf{u}_i) \mathbf{u}_i$ so that $\widehat{\mathbf{x}}_0^T \mathbf{u}_j = 0$ for $j < k$. Then, for $\mathbf{x}_0 = \sum_{i=1}^n \alpha_i \mathbf{u}_i$, $\mathbf{x}_0^T \mathbf{u}_i = \alpha_i$ and $\widehat{\mathbf{x}}_0 = \sum_{i=k}^n \alpha_i \mathbf{u}_i$, resulting in

$$\begin{aligned}\mathbf{x}_m &= \mathbf{A}^{-m} \widehat{\mathbf{x}}_0 = \sum_{i=k}^n \alpha_i \mathbf{A}^{-m} \mathbf{u}_i = \sum_{i=k}^n \alpha_i \lambda_i^{-m} \mathbf{u}_i \\ &= \lambda_k^{-m} \left(\alpha_k \mathbf{u}_k + \sum_{i=k+1}^n \alpha_i (\lambda_k/\lambda_i)^m \mathbf{u}_i \right).\end{aligned}$$

Thus, provided $\alpha_k \neq 0$, $\mathbf{x}_m / \|\mathbf{x}_m\|_2$ approaches $\mathbf{u}_k$ (or, in the case of a multiple eigenvalue, it converges to an eigenvector in the same eigenspace). Further, an expression analogous to (6) shows that the Rayleigh quotients r_m converge to λ_k^{-1} as $m \rightarrow \infty$. In practice, due to numerical considerations, orthogonality must be maintained by applying the projections at each step or at least periodically.

Note that the more general method of inverse iteration calls for solving linear systems involving matrices $\mathbf{A} - \mu\mathbf{I}$, where μ is an approximation to an eigenvalue of $\mathbf{A}$. With $\mu > 0$, this requires a direct solution method because $\mathbf{A} - \mu\mathbf{I}$ is not positive definite and is close to singular. The storage requirements associated with direct methods would limit the size of the problems that could be treated.

3.5 Storage requirements

The storage requirements depend on the number of nodes in each triangulation and, along with the amount of available memory, determine the maximum fractal level that can be treated. For fractal level L and refinement level R (nonnegative integers), we define the following functions:

$$\begin{aligned} N_b &= N_b(L, R) = \text{Number of boundary nodes,} \\ N_t &= N_t(L, R) = \text{Number of triangles, and} \\ N &= N(L, R) = \text{Number of nodes (vertices).} \end{aligned}$$

The following expressions can be easily verified by induction on L and R :

$$\begin{aligned} N_b &= 4^L \times 3^{R+1}, \\ N_t &= 9^R \left[9^L + 3(9^L - 4^L) / 5 \right], \text{ and} \\ N &= (N_b + N_t) / 2 + 1. \end{aligned}$$

Let $L_{\max}$ denote the maximum fractal level for which the eigenvalues are to be computed. Since the set of interior nodes (at which the eigenfunctions are nonzero) in the level $L_{\max}$ triangulation is identical to the set of interior nodes associated with the first refinement at fractal level $L_{\max} - 1$, the two triangulations have the same set of associated eigenvalues and eigenvectors (ignoring the zero entries corresponding to the additional boundary nodes

in the larger triangulation). Thus there is no need to actually construct the level $L_{\max}$ triangulation, and the maximum value of N , $N_{\max}$, is obtained by setting L to $L_{\max} - 1$ and R to 1 in the above expressions. Hence we obtain

$$N_{\max} = N(L_{\max} - 1, 1) = (4 \times 9^{L_{\max}} + 9 \times 4^{L_{\max}-1}) / 5 + 1.$$

The data structure used by our Fortran program consists of a pair of type Real arrays X and Y of length $N_{\max}$ for the nodal coordinates and a type Integer adjacency array $IADJ$ dimensioned 6 by $N_{\max}$. Column j of $IADJ$ contains the indexes of the counterclockwise-ordered sequence of neighbors of node j followed by (one to four) zeros if j indexes a boundary node. If j is a boundary node, its first and last neighbors are boundary nodes; otherwise the choice of first neighbor is not defined.

Since it is not possible to overwrite the adjacency array during execution of the refinement procedure, we require an additional adjacency array $IADJ0$ with $N0$ columns, where $N0$ must be large enough to store the level-0 refinement at fractal level $L_{\max} - 1$:

$$N0_{\max} = N(L_{\max} - 1, 0) = (4 \times 9^{L_{\max}-1} + 6 \times 4^{L_{\max}-1}) / 5 + 1.$$

A third adjacency array $IADJ1$ with $N1$ columns is needed to save the first refinement of each triangulation, which is required to obtain the initial triangulation at the next fractal level for $L \leq L_{\max} - 2$:

$$N1_{\max} = N(L_{\max} - 2, 1) = (4 \times 9^{L_{\max}-1} + 9 \times 4^{L_{\max}-2}) / 5 + 1.$$

The set of N_e eigenvectors associated with each triangulation is stored on disk. However, efficiency of the computational procedure requires three arrays of length $N_{\max}$ for temporary storage of eigenvectors. These must have type Double Precision unless the target machine has a long word length as, for example, in the case of a Cray.

In summary, the total array storage requirement is $6(N_{\max} + N0_{\max} + N1_{\max}) = 52.8 \times 9^{L_{\max}-1} + 20.7 \times 4^{L_{\max}-1} + 18$ integers, $2N_{\max}$ single precision reals, and $3N_{\max}$ double precision reals. Assuming 4-byte integers, 4-byte reals, and 8-byte double precision reals, the total in bytes is

$$49.067 \times 9^{L_{\max}} + 35.1 \times 4^{L_{\max}} + 104.$$

Table 1 shows the relationship between the maximum fractal level and the required memory size.

$L_{\max}$	$N_{\max}$	Array Storage
4	5,365	331,016
5	47,701	2,933,384
6	426,997	26,219,912
7	3,833,749	235,259,528

Table 1: Memory Requirements in Bytes

The four rows of Table 1 correspond to the typical memory capacities of a PC, a modest workstation, a powerful workstation, and a supercomputer, respectively. Our test results indicate that the fractal level 6 grid is a sufficiently accurate approximation to the snowflake.

4 Implementation of the Computer Algorithm

4.1 Outline of the main steps (inverse power method)

The following algorithm employs two unformatted disk files for storing vectors. An input file **infile** contains unnormalized initial approximations to the eigenvectors obtained by linear interpolation from the previous refinement level or, at level (L, R) , by appending zeros to the first $N1$ components of the eigenvectors associated with the last refinement of the previous fractal level, $L - 1$, where $N1 = N(L - 1, 1)$. The other file, **outfile**, contains the approximations to the orthonormal sequence of eigenvectors $\{\mathbf{u}_i\}_{i=1}^{N_e}$. The algorithm also produces the sequence of approximate eigenvalues $\{\lambda_i\}_{i=1}^{N_e}$.

for $k = 1$ to $\min(N_e, n)$

$\mathbf{u} = \mathbf{x}_0$ (read from **infile**)

$e = 0$

do

$e_0 = e$

$\mathbf{v} = \mathbf{u} / \|\mathbf{u}\|_2$

$\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$

rewind outfile

```

for  $i = 1$  to  $k - 1$ 
     $\mathbf{w} = \mathbf{u}_i$  (read from outfile)
     $\mathbf{u} = \mathbf{u} - (\mathbf{u}^T \mathbf{w}) \mathbf{w}$ 
endfor
 $e = 1/\mathbf{u}^T \mathbf{v}$ 
while (  $|e - e_0| > e_0 * \text{PTOL}$  )
     $\lambda_k = e$ 
    write  $\mathbf{v}$  to outfile
endfor

```

The statement $\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$ represents a call to the SOR routine with initial estimate $\mathbf{u}_0 = \mathbf{v}/e$ (or $\mathbf{u}_0 = \mathbf{0}$ on the first iteration when $e = 0$). In all cases, the average number of SOR iterations per power method iteration was less than 1.5. The deflation procedure consists of applying the product of orthogonal projections $\mathbf{P}_i = \mathbf{I} - \mathbf{u}_i \mathbf{u}_i^T$, $i = 1, \dots, k - 1$, to $\mathbf{u}$. This product is equal to the projection $\mathbf{P} = \mathbf{I} - \sum_{i=1}^{k-1} \mathbf{u}_i \mathbf{u}_i^T$ that maps $\mathbf{x}_0$ to $\widehat{\mathbf{x}}_0$. Convergence is defined by an upper bound PTOL on the relative change in the approximate eigenvalue e (the reciprocal of the Rayleigh quotient). We refer the reader to Wilkinson [1965, Chap. 9] for background on the methods described above.

4.2 The computed eigenvalues

Table 2 contains the approximations to the smallest eigenvalue of $\mathbf{A}$ computed with $L_{\max} = 6$ and $\text{PTOL} = 10^{-7}$ (as defined in Secs. 3.5 and 4.1). Recall from Secs. 3.2–3.4 that the eigenvalues of $\mathbf{A}$ are the eigenvalues of the discretized Laplacian, and are therefore approximations to the eigenvalues of the Dirichlet Laplacian on the true Koch snowflake domain.

Note that the level $(L, 0)$ values agree with the level $(L - 1, 1)$ values as expected. The top row ($L = 0$) corresponds to an equilateral triangle for which the eigenvalues are known explicitly. (See Pinsky [1980] and, in a more general situation, Bérard [1980].) The best approximation on the triangle (0,6) agrees with the true value in all 5 digits. More generally, for refinement level 6 approximations on the triangular domain, the maximum relative error among the smallest 50 eigenvalues is $.733 \times 10^{-3}$.

	R = 0	R = 1	R = 2	R = 3	R = 4	R = 5	R = 6
L = 0		36.000	50.534	52.401	52.612	52.635	52.638
L = 1	36.000	40.671	40.410	40.259	40.219	40.211	
L = 2	40.671	40.120	39.880	39.822	39.810		
L = 3	40.120	39.657	39.542	39.519			
L = 4	39.657	39.463	39.419				
L = 5	39.463	39.390					
L = 6	39.390						

Table 2: The Smallest Eigenvalue λ_1

Note that as the fractal level increases, the domain size increases and, as expected, the eigenvalue decreases. The two best estimates of the smallest eigenvalue on the Koch snowflake, levels (5,1) and (4,2), agree to three digits, and this pattern holds for the first 50 values, indicating that $L_{\max} = 6$ is sufficient to give three correct digits.

In Table 3, we give the numerical values of the first 50 eigenvalues (of **A**) at fractal level $L=4$ and refinement level $R=1$. As mentioned in Sec. 2.7, among the first 50 eigenvalues, 16 are simple and the other 34 have multiplicity 2.

Finally, in Table 4, we display the numerical values of the first 20 eigenvalues for (fractal) levels varying from 0 to 5 with refinement levels varying from 6 to 1. This further shows “experimentally” the rapid convergence of the numerical approximation to the eigenvalues of the Koch snowflake drum. (See also Sec. 5.1 below.)

5 Closing Comments

We close this paper by some concluding remarks and comments concerning future developments.

5.1 Numerical analysis

The numerical method presented here appears to be rather accurate as mentioned previously. However, it would be interesting to conduct a detailed study of the numerical errors made in calculating the eigenfunctions and

EVAL	L,R = 4,1	EVAL	L,R = 4,1
1	39.463	26	697.488
2	97.720	27	740.144
3	97.720	28	740.144
4	165.898	29	768.585
5	165.897	30	783.187
6	190.938	31	783.367
7	209.274	32	822.682
8	273.246	33	822.682
9	273.246	34	877.939
10	313.138	35	879.418
11	315.368	36	912.233
12	315.368	37	963.586
13	362.540	38	963.213
14	426.501	39	989.297
15	444.626	40	989.297
16	444.625	41	1052.858
17	459.795	42	1052.859
18	459.795	43	1060.007
19	561.785	44	1084.722
20	561.785	45	1084.722
21	567.999	46	1124.569
22	567.928	47	1124.569
23	596.383	48	1227.393
24	619.434	49	1231.422
25	678.580	50	1231.423

Table 3: The First 50 Eigenvalues (at Fractal Level L=4 and Refinement Level R=1).

EVAL	L,R = 0,6	L,R = 1,5	L,R = 2,4	L,R = 3,3	L,R = 4,2	L,R = 5,1
1	52.638	40.211	39.810	39.519	39.419	39.390
2	122.906	100.269	98.623	97.863	97.609	97.537
3	122.906	100.269	98.623	97.863	97.609	97.537
4	210.575	172.269	167.572	166.204	165.751	165.622
5	228.085	172.268	167.572	166.204	165.751	165.622
6	228.085	199.129	192.851	191.242	194.760	190.571
7	333.261	224.087	211.405	209.627	209.011	208.837
8	333.261	294.344	275.412	273.737	272.980	272.755
9	368.488	294.344	275.412	273.737	272.980	272.755
10	368.488	316.489	315.851	313.713	312.864	312.645
11	491.276	340.330	317.691	315.658	315.121	314.784
12	491.276	340.330	317.691	315.658	315.121	314.784
13	473.728	398.670	362.511	361.364	360.166	359.879
14	543.924	476.683	428.875	426.129	426.050	425.726
15	543.924	476.682	447.314	445.406	444.151	443.830
16	649.117	478.067	447.314	445.406	444.151	443.830
17	649.117	478.068	463.841	460.624	459.436	459.099
18	683.836	473.774	463.993	460.473	459.436	459.099
19	683.837	593.985	568.158	563.088	561.427	560.913
20	754.398	593.985	568.158	563.088	561.341	560.913

Table 4: The First 20 Eigenvalues at Fractal Levels L=0 through 5 and Refinement Levels R=6 through 1.

their gradients.

5.2 Conjecture for the higher eigenfunctions on the true snowflake domain

As was seen in the graphics presented in Sec. 2.3 (Plate 11), the magnitude of the gradient of the first eigenfunction, $|\nabla\varphi_1|$, “blows up” at those boundary points with an obtuse angle corner. This behavior was even more pronounced for the higher eigenfunctions (Plates 12 and 16). We know that this is true for all polygons; moreover, our computer graphics (Plates 12–17) suggest that this is so for the true Koch snowflake domain as well. Hence, we conjecture that the theoretically derived results of Lapidus & Pang [1995] concerning the boundary behavior of $|\nabla\varphi_1|$ may be extended to the higher eigenfunctions. More specifically, we expect that, for any $n \geq 2$, $|\nabla\varphi_n|$ “blows up” as one approaches (from within the snowflake domain) a boundary point with an obtuse angle corner.

5.3 Graphics for irrational boundary points?

Naturally, our graphical representation of the eigenfunctions at the boundary is limited to the “rational points” (i.e., boundary points which are vertices of a polygonal approximation to the Koch snowflake curve). However, as was mentioned earlier, the results of Lapidus & Pang [1995] imply that $|\nabla\varphi_1|$ “blows up” not only at the rational boundary points with an obtuse angle corner, but also at all “irrational points” in an uncountably infinite Cantor set (of Hausdorff dimension $\ln 2/\ln 3$) contained in the true snowflake curve. A both technically and intellectually challenging problem is to try to visualize this phenomenon on a computer.

5.4 Understanding symmetries

We have obtained in this work a wealth of information concerning the symmetries of the eigenfunctions and the multiplicities of the associated eigenvalues. (See especially Secs. 2.6 and 2.7.) Can we establish these properties rigorously (for instance, by group-theoretical methods)? Further, can we predict the symmetries for other domains with self-similar fractal boundaries?

5.5 Absence of localization

In contrast to the square snowflake drum studied experimentally (as well as numerically) by Sapoval, *et al.* [1989, 1991 & 1993], no “localization” is observed for some of the eigenfunctions in the present case of the triangular snowflake drum. (See also Gerling [1994].) This is intuitively understandable since long narrow channels are present in the former but not in the latter. We plan in future work to study different kinds of fractal geometries allowing various types of “blow up” and localization phenomena.

5.6 The effects of nonlinearity

Finally — in view, in particular, of heuristic comments in Sapoval [1989] regarding the formation of “cascades” and the onset of turbulence in a related context (that of ocean waves damped by fractal coastlines) — it would be interesting to investigate (both numerically and mathematically) the effects of nonlinearities on the conclusions obtained in, for instance, Lapidus & Pang [1995], and the present paper. Specifically, would singularities still form at certain points of fractal boundaries for the (gradient of the) solution of the nonlinear wave equation (or even the Navier-Stokes equation)?

In the long term, we hope that the further study of these and related questions will help us better understand the formation of fractal structures in nature.

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References

- Atkinson, K. E. [1989] *An Introduction to Numerical Analysis*, 2nd ed., (John Wiley and Sons, New York).
- Bérard, P. [1980] “Spectres et groupes cristallographiques I: Domaines euclidiens,” *Invent. Math.* **58**, 179–199.
- Berry, M. V. [1979] “Distribution of modes in fractal resonators,” in *Structural Stability in Physics*, eds. Güttinger, W. & Eikemeier, H. (Springer-Verlag, Berlin) pp. 51–53.
- Berry, M. V. [1980] “Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes,” in “Geometry of the Laplace Operator”, *Proc. Sympos. Pure Math.* **36**, Amer. Math. Soc., Providence, pp. 13–38.
- Brossard, J. & Carmona, R. [1986] “Can one hear the dimension of a fractal?,” *Commun. Math. Phys.* **104**, 103–122.
- Evans, W. D. & Harris, D. J. [1993] “Fractals, trees and the Neumann Laplacian,” *Math. Ann.* **296**, 493–527.
- Falconer, K. J. [1990] *Fractal Geometry: Mathematical foundations and applications* (Wiley, Chicester).
- Gerling, J. [1994] “Localized states on the square Koch drum,” (to appear).
- Golub, G. H. & Van Loan, C. F. [1989] *Matrix Computations*, 2nd ed., (Johns Hopkins Univ. Press, Baltimore).
- He, C. Q. & Lapidus, M. L. [1995] “Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function,” preprint, IHES M/95/47, Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France. (To be announced in *Mathematical Research Letters*.)
- Isaacson, E. & Keller, H. [1966] *Analysis of Numerical Methods* (John Wiley, New York).
- Kudrolli, A. & Sridhar, S. [1994] “Signatures of chaos in quantum billiards: microwave experiments,” *Phys. Rev. E* **49**, No. 11, R11–R14.
- Lapidus, M. L. [1991] “Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture,” *Trans. Amer. Math. Soc.* **325**, 465–529.

- Lapidus, M. L. [1993] "Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl-Berry conjecture," in *Ordinary and Partial Differential Equations*, vol. IV, eds. Sleeman, B. D. & Jarvis, R. J., Pitman Research Notes in Math. Series **289**, Longman, London, pp. 126–209.
- Lapidus, M. L. [1995] "Fractals and vibrations: Can you hear the shape of a fractal drum?," in "Fractal Geometry and Self-Similar Phenomena", Proc. of a Symposium in Honor of Prof. Benoit Mandelbrot's 70th Birthday, eds. Evertsz, C. J. G., Peitgen, H. -O. & Voss, R., *Fractals* (in press).
- Lapidus, M. L. & Maier, H. [1995] "The Riemann hypothesis and inverse spectral problems for fractal strings," *Journal London Math. Soc.* (2), No. 1, **52**, 15–35.
- Lapidus, M. L. & Pang, M. M. H. [1995] "Eigenfunctions of the Koch snowflake domain," *Commun. Math. Physics*, No. 2, **172**, 359–376.
- Lapidus, M. L. & Pomerance, C. [1993] "The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums," *Proc. London Math. Soc.* (3) **66**, 41–69.
- Lapidus, M. L. & Pomerance, C. [1996] "Counterexamples to the modified Weyl-Berry conjecture on fractal drums," *Math. Proc. Cambridge Philos. Soc.*, No. 1, **119** (in press).
- Mandelbrot, B. B. [1983] *The Fractal Geometry of Nature*, rev. and enl. ed., (Freeman, New York).
- Neuberger, J. W. & Noid, D. [1987] "Numerical calculation of eigenvalues for the Schrödinger equation III," *J. Comp. Chem.* **8**, 459–461.
- Pinsky, M. A. [1980] "The eigenvalues of an equilateral triangle," *SIAM J. Math. Anal.* **11**, 819–827.
- Sapoval, B. [1989] "Experimental observation of local modes in fractal drums," *Physica D* **38**, 296–298.
- Sapoval, B. [1990] *Fractals* (Aditech, Paris).
- Sapoval, B. & Gobron, Th. [1993] "Vibrations of strongly irregular or fractal resonators," *Phys. Rev. E* (3) No. 5, **47**, 3013–3024.
- Sapoval, B., Gobron, Th. & Margolina, A. [1991] "Vibrations of fractal drums," *Phys. Rev. Lett.* **67**, 2974–2977.
- Sridhar S., Hogenboom, D. & Kudrolli, A. [1993] "Experimental eigenvalue spectra of "rough" and multiply-connected billiards," in

Quantum Dynamics of Chaotic Systems, eds. Yuan, J. M., Feng, D. H. & Zaslavsky, G. M. (Gordon and Breach, Amsterdam), pp. 297–304.
Wilkinson, J. H. [1965] *The Algebraic Eigenvalue Problem* (Oxford University Press, Oxford).

Figures, Tables and Color Plates — Captions and Placement

Section 1

Figure 1: The Koch Snowflake Drum Ω

Section 2

Subsection 2.1

Plate 1: The First Snowflake Harmonic

Plate 2: The First Snowflake Harmonic (Zoom on the Boundary)

Plate 3: The Second Snowflake Harmonic

Plate 4: The Thirteenth Eigenfunction

Plate 5: Several Views of the Nineteenth Eigenfunction

Subsection 2.2

Plate 6: Snowflake Harmonics, φ_1 – φ_4

Plate 7: Snowflake Harmonics, φ_5 – φ_8

Plate 8: Snowflake Harmonics, φ_9 – φ_{12}

Plate 9: Snowflake Harmonics, φ_{13} – φ_{16}

Plate 10: Selected Higher Eigenfunctions

Subsection 2.3

Plate 11: The Magnitude of the Gradient of the First Eigenfunction

Plate 12: The Magnitude of the Gradient of the 13th Eigenfunction

Plate 13: The Magnitude of the Gradient of the 13th Eigenfunction
(bottom view)

Plate 14: One Sixth Slice of $|\nabla\varphi_{13}|$ at Various Refinements

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Subsection 2.4

Plate 16: The Magnitude of the Gradient of the 45th Eigenfunction

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Subsection 2.5

Plate 18: The Nodal Lines of the 2nd Eigenfunction

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Subsection 2.6

Plate 21: Contour Plots

Plate 22: Contour Plots

Plate 23: Contour Plots

Section 3

Subsection 3.1

Figure 2: Approximations to the Koch Snowflake, with Fractal Level L and Refinement Level R

Subsection 3.2

Figure 3: Interior Node u and its Six Neighbors u_i

Subsection 3.5

Table 1: Memory Requirements in Bytes

Section 4

Subsection 4.2

Table 2: The Smallest Eigenvalue λ_1

Table 3: The First 50 Eigenvalues (at Fractal Level L=4 and Refinement Level R=1).

Table 4: The First 20 Eigenvalues at Fractal Levels L=0 through 5 and Refinement Levels R=6 through 1.

We regret that for practical reasons the following color plates (1-23) will be presented in greyscale in this preprint.

The First Snowflake Harmonic
Level 4 - First Eigenfunction

Level 4 - First Eigenfunction
(Zoom on Boundary)

Level 4 -Thirteenth Eigenfunction

Plate 5: Several Views of the Nineteenth Eigenfunction

Snowflake Harmonics

1st Eigenfunction

2nd Eigenfunction

4th Eigenfunction

3rd Eigenfunction

Snowflake Harmonics

Snowflake Harmonics

9th Eigenfunction

10th Eigenfunction

11th Eigenfunction

12th Eigenfunction

Min

Max

Snowflake Harmonics

Max

Min

45th Eigenfunction

38th Eigenfunction

38th Eigenfunction

45th Eigenfunction

Plate 11: The Magnitude of the Gradient of the First Eigenfunction

The Magnitude of the Gradient of the First Eigenfunction

Plate 12: The Magnitude of the Gradient of the 13th Eigenfunction

The Magnitude of the Gradient of the 13th Eigenfunction

The Magnitude of Gradient of the 13th Eigenfunction
(bottom view)

Plate 14: One Sixth Slice of $|\nabla\varphi_{13}|$ at Various Refinements

Magnitude of the Gradient for the 13th Eigenfunction

Level 5 - One of the Six Outer Hills

Magnitude of the Gradient for the 45th Eigenfunction
(one-sixth slice)

Nodal Lines of the 2nd Eigenfunction

Nodal Lines of the 13th Eigenfunction

Nodal Lines of the Higher Eigenfunctions

37th Eigenfunction

45th Eigenfunction

36th Eigenfunction

38th Eigenfunction

Contour Plots

1st Eigenfunction

2nd Eigenfunction

3rd Eigenfunction

13th Eigenfunction

Contour Plots

17th Eigenfunction

Max

000

Min

18th Eigenfunction

44th Eigenfunction

45th Eigenfunction

Contour Plots

36th Eigenfunction

37th Eigenfunction

38th Eigenfunction

43th Eigenfunction