O(10) grand unification models which do not necessarily have an extra global symmetry are discussed, taking the model with one 10-plet in the Yukawa sector as an example. A strong correlation between mass ratios and CP is found. The mass relation $m_t/m_b = v_u/v_d$ is recovered when $G_W = 0$; and another special relation $m_t/m_b = G_E/G_W$ appears when $v_d = 0$, where $G_E, W$ are Yukawa coupling constants and $v_{u,d}$ are VEVs. To facilitate this discussion, a set of $O(10)$ γ- matrices is offered based on a physical representation of the spinors and that of the vector of the $SO(10)$ group. Flavor changing neutral currents in such models are also discussed.
I. Motivations

It is well-known that in order to somehow explain the fermion masses and mixing, multi-Higgs doublets are generally needed\cite{1}. In addition, flavor changing neutral currents (FCNC) due to multi-Higgs doublets generally exist. The FCNC situation in an $O(10)$ model\cite{2} is somewhat subtle. Taking the model with one 10-plet mass provider (minimal O-ten models, MOTM) as an example, the 10-plet is decomposed into $SU(5)$ representations as

$$10 = 5_1 + 5^*_2$$

The concerned two doublets are in $5_1$ and $5^*_2$ respectively. Because the two 5-d representations may not be mutually conjugated, therefore a subscript is added to distinguish them. The fermions 16 and $16^*$ are decomposed as

$$16 = 10 + 5^* + 1, \quad 16^* = 10^* + 5 + 1$$

Therefore there are two sets of possible mass terms:

$$5_1 \, 10^* \, 5, \quad 5^*_2 \, 10^* \, 10^*;$$

and

$$5_1 \, 10 \, 10, \quad 5^*_2 \, 10 \, 5^*.$$

There will be no FCNC at low energies for MOTM with three generations of fermions, if only one of the two sets contributes, or only one of the 5-plets contributes. However neither of these two scenarios is obviously obliged within the group of $O(10)$. It becomes clear that only in some very special cases the above scenarios without FCNC appear in MOTM.

Indeed, one will see that the top-bottom mass ratio can be written as

$$\frac{m_t}{m_b} = \left( \frac{\left[ Re\left(g_e v_1 + g_o v_6\right) \right]^2 + \left[ Im\left(g_e v_6 + g_o v_1\right) \right]^2}{\left[ Re\left(g_e v_1 - g_o v_6\right) \right]^2 + \left[ Im\left(g_e v_6 - g_o v_1\right) \right]^2} \right)^{\frac{1}{2}}. \quad (1)$$

Here,

$$v_1 = (v_u + v_d)/2, \quad v_6 = (v_u - v_d)/2 \quad (2)$$

where $v_u$ and $v_d$ are the VEVs of $5_1$ and $5^*_2$ respectively of the 10-plet Higgs. $g_e$ and $g_o$ are respectively coupling constants of $O(10)$ parity even and odd terms. The 10-d space
reflection transformation is not an element of the $SO(10)$ group. This is the reason why $O(10)$ is of interest. Indeed in general there can be two different $SO(10)$ invariant couplings of the Higgs 10-plet and fermion bilinears: one $g_e$ term which is parity even and the other $g_o$, odd. One can see from this formula that in order to have $|m_t/m_b| \neq 1$, not only one needs both vacua $v_1$ and $v_6$ but also both couplings $g_e$ and $g_o$. When there is a maximal CP mixing, $g_o = \pm ig_e$, the mass ratios will be adversely affected in the MOTM.

Two special cases are worth noting: 1) $v_1 = v_6$: This means that there is only one nonzero VEV, $v_u \neq 0$, $v_d = 0$, which is typical for a vector (the 10-plet) to develop VEV. In this case, one obtains $m_t/m_b = (g_e + g_o)/(g_e - g_o)$. For a further insight, one can read the paragraph with Eq (38). 2) The Yukawa couplings satisfy the condition for being self-dual, $g_o = g_e$. The definition of dual (denoted by E for convenience) and anti-dual (denoted by W) in $O(10)$ is similar to that of left-handed (L) and right-handed (R) in the Lorentz group, however with some new characters. In this case one obtains a two Higgs doublet model with a global $U(1)$ symmetry[3] at low energies.

In general, the counterpart of a MOTM at low energies is a general two Higgs doublet model[4] with FCNC and a complicated relation between $m_t/m_b$ and $v_u/v_d$. For a further insight, one can read Eq (41). The second special case, particularly when it is resulted from supersymmetry, is widely applied for discussions of $O(10)$ mass relations[5]. In this note we will analyze the $O(10)$ model without any constraints.

The above properties of MOTM are crucially related to the $O(10)$ group structure. All gamma matrices can only be established on a $32 = 16 + 16^*$ reducible spinor representation basis. In terms of $16^2$ blocks, one can make these matrices look like

$$
\begin{pmatrix}
RL & RR \\
LL & LR
\end{pmatrix},
$$

(3)

if 16 is assigned to be left-handed ($16^*$ will then be right-handed). All the 10 $\gamma$-matrices are block-off-diagonal, therefore all odd rank products of $\gamma$-matrices are block-off-diagonal and the even ones are block-diagonal. Furthermore, among the 10 Hermitian $\gamma$-matrices, 5 are L-R symmetric and 5 are L-R anti-symmetric. Two $\gamma$-matrices, one symmetric and one anti-symmetric, can be used as mass operators. They correspond exactly to two components of 10-plet which can develop VEVs without a contradiction with electric charge conservation.
Therefore a one 10-plet Higgs O(10) model (MOTM) may in general correspond to a two Higgs doublet model with FCNC. If more Higgs multiplets are involved in the Yukawa sector, then there will be more FCNCs. In either case, the Yukawa coupling constants can be complex, which may cause explicit CP violation. In addition to this, spontaneous CP violation due to a relative phase of the VEVs may appear in all the electro-weak interactions. In general, one may need eight $3 \times 3$ U-matrices in order to diagonalize the mass matrices of the up-, down-, neutrino- and lepton- mass matrices:

$$U_L^U, U_R^U; U_L^D, U_R^D; \ U_L^\nu, U_R^\nu; \text{ and } U_L^l, U_R^l.$$  \hspace{1cm} (4)

All of them can be physically relevant; in other words, the CKM matrix

$$V_{CKM} = U_L^U U_L^{D\dagger}$$  \hspace{1cm} (5)

appears in the left-handed charged current gauge interactions; the matrix

$$V' = U_R^U U_R^{D\dagger}$$  \hspace{1cm} (6)

appears in the right-handed charged current gauge interactions; the matrix

$$U_L^U G_Y^U U_R^U$$  \hspace{1cm} (7)

leads to the scalar mediated FCNC interactions among up-type quarks, where $G_Y^U$ is the matrix of Yukawa couplings of this scalar; and finally, the matrix

$$U_L^D G_Y^D U_R^D$$  \hspace{1cm} (8)

leads to the scalar mediated FCNC among the down-type quarks. Unless $G_Y^\nu$ of a Higgs doublet is proportional to the corresponding mass matrix, FCNC mediated by this Higgs field is in general nontrivial. CP violation can in principle appear in any of the four interactions. Similarly, there are four combinations in the leptonic sector also, such as

$$U_L^l G_Y^l U_R^l.$$  \hspace{1cm} (9)

Eqs (6) to (9) represent physics beyond the CKM matrix[4, 6]. The detection of these interactions will help to provide information on the mass matrix elements themselves and the relevant new particles, the scalars.
This work is devoted to the general relations among mass ratios, FCNC, and CP violation in an arbitrary $O(10)$ model. The MOTM will be taken as an explicit example. However, the results can be applied to any non-minimal $O(10)$ models. Before such a discussion, a set of reliable and explicit $\gamma$-matrices will be provided in Section II. The properties of mass operators will be discussed in detail in Section III. A formula similar to Eq(1) will be given in terms of the dual (E) and anti-dual (W) coupling constants. In Section IV, symmetries beyond $O(10)$ are discussed which may help to forbid the W term or the E term.

II. The $O(10)$ Gamma Matrices and Mass Operators

There have been discussions on the group of $O(10)$, since it was recognized as a potential candidate group for grand unification theories (GUT)[2, 7]. A different approach will be taken in this work. The components of the fundamental spinor and vector representations will be given first, in terms of the familiar quantum numbers, such as color, flavor, and $B - L$. Then the $\gamma$-matrices will be built up on this specific basis. Therefore $\sum_{N=1}^{10} \gamma_N H_N$ is guaranteed an $O(10)$ 0-rank tensor matrix, because the numbering of the 10 $\gamma$-matrices properly matches the numbering of the 10 components in the 10-plet $H_N$. Tensor matrices and mass operators will also be given. There seem to be no contradictions between the results of this work and those of the previous rigorous studies of the group, except that results here are more specific.

It is important to make sure that the symbols for Weyl fields used here are clear. A fermion field can be seen as a sum of the left-handed and the right-handed parts

$$\psi = \psi_L + \psi_R, \quad \psi_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\psi.$$  \hfill (10)

A generic mass term is

$$\bar{\psi}_L \psi_R = \psi_L^T C \psi_R = \psi_R^T C \psi_L.$$  \hfill (11)

where $\psi^c = C\psi^T$, and $C = i\gamma_2\gamma_5$ is the C-matrix of the Lorentz group, $C^T = C^{-1} = -C$. Since $\psi_L$ and $\psi_R$ are in one irreducible representation in the $O(10)$ models, one often meet both orders. The left and right fields are essentially two component fields, in particular, in the Majorana representation, where $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the right- and left- spinors occupy the first two and last two components respectively of a given four component field $\psi$. From
here on, all Lorentz group matrices will be underlined, in order to distinguish them from the 
$O(10)$ matrices.

The present task is to find ten $32 \times 32$ matrices which satisfy the Clifford algebra:

$$\{\gamma_M, \gamma_N\} = \gamma_M \gamma_N + \gamma_N \gamma_M = 2\delta_{MN} I^{32}. \quad (12)$$

$\delta_{MN}$ in the above equation is the metric of the 10-d internal space which is unity when 
$M = N$ and vanishes otherwise. Eq (12) is not restrictive enough to render a unique set 
of gamma matrices. Indeed, the anti-commuting relation (12) is invariant under orthogonal 
transformations

$$\gamma'_M = a_{MN} \gamma_N. \quad (13)$$

In addition one has the freedom to choose the components of the $2^5 = 16 + 16^*$ spinors.

The two irreducible spinor representations of $SO(10)$ are represented here by $\psi$ and $\psi^c$. 
They are used to represent respectively Lorentzian left-handed and right-handed Weyl fields 
of one family fermions

$$\psi^T = (u_L, d_L, e^c_R, u^c_R, \nu_L, e_L, e^c_R, \nu^c_R), \quad (14)$$

and the $16^*$-plet is just its charge conjugate. The color indices (from 1 to 3) for the quarks are 
suppressed. It is easy to discover that the arrangement of the components in (14) complies 
with a $SU(3) \times SU(2)_L \times SU(2)_R$ decomposition of the $SO(10)$ spinor$^3$.

$$16 = (3, 2, 1, +1/3) + (3^*, 1, 2, -1/3) + (1, 2, 1, -1) + (1, 1, 2, +1), \quad (15)$$

where the fourth number in each parenthesis is the $B - L$ quantum number. The 32-d 
reduced representation is chosen as $\Psi$,

$$\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}. \quad (16)$$

It is convenient to first define 10 symmetric $16 \times 16$ matrices in two groups:

$$\alpha_i, \beta_p, (i = 1, 2, 3, 4, 5; p = 6, 7, 8, 9, 10) \quad (17)$$

$^3$There is a difference between the subgroup $SU(2)_L \times SU(2)_R$ here as a part of the internal symmetry, 
and that in $SO(3, 1) = SU(2)_L \times SU(2)_R$ of the Lorentz group. The subindices “L, R” in (14) are for the representations of the former.
where $\alpha$s and $\beta$s mutually commute, while each groups make separate Clifford algebras,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}I^{16}, \{\beta_p, \beta_q\} = 2\delta_{pq}I^{16}; \quad [\alpha_i, \beta_p] = 0. \quad (18)$$

The gamma matrices are then simply

$$\gamma_I = \begin{pmatrix} 0 & \alpha_I \\ \alpha_I & 0 \end{pmatrix}, \quad (I = 1, \ldots 5) \quad (19)$$

$$\gamma_P = \begin{pmatrix} 0 & -i\beta_P \\ i\beta_P & 0 \end{pmatrix}, \quad (P = 6, \ldots 10)$$

Note that the RR part is the Hermitian conjugate of the LL part for all the $\gamma$-matrices.

When choosing the basis components for 10-plet as the following \(^4\)

$$H = \frac{1}{2} \left( \nu^e_{1L}\nu^c_{2R} + \nu^c_{2L}\nu^e_{1R}, \epsilon^e_{1L}\nu^c_{2R} + \epsilon^c_{2L}\nu^e_{1R}, d^{1e}_{1R}\nu^c_{2R} + d^{2e}_{2R}\nu^c_{1R}, d^{1c}_{1R}\nu^c_{2R} + d^{2c}_{2R}\nu^c_{1R}; \right) \quad (20)$$

\(4\)The fermion symbols are used for their $SO(10)$ quantum numbers. The subscripts are attached in order to have complex components of the 10-plet. Readers who do not prefer this basis for 10, which mixes components with opposite quantum numbers, may read the next section for other representations in which no Clifford algebra can be found though.
the $\alpha$- and $\beta$-matrices in (19) on the basis (14) are

\[
\alpha_1 = \begin{pmatrix} I^3 & I^3 \\ I^3 & \tau_i \end{pmatrix}, \quad \beta_6 = \begin{pmatrix} I^3 & -I^3 \\ -I^3 & -\tau'_2 \end{pmatrix};
\]

\[
\alpha_2 = \begin{pmatrix} I^3 & -I^3 \\ -I^3 & \tau_3 \end{pmatrix}, \quad \beta_7 = \begin{pmatrix} -I^3 & -I^3 \\ -I^3 & -I^2 \end{pmatrix};
\]

\[
\alpha_3 = \begin{pmatrix} -\phi^3 & \phi^3 \\ -\phi^3 & \phi^3 \end{pmatrix}, \quad \beta_8 = \begin{pmatrix} -\phi^3 & \phi^3 \\ -\phi^3 & \phi^3 \end{pmatrix};
\]

\[
\alpha_4 = \begin{pmatrix} -h^3 & h^3 \\ -h^3 & h^3 \end{pmatrix}, \quad \beta_9 = \begin{pmatrix} -h^3 & h^3 \\ -h^3 & h^3 \end{pmatrix};
\]

\[
\alpha_5 = \begin{pmatrix} -\bar{\phi}^3 & \bar{\phi}^3 \\ -\bar{\phi}^3 & \bar{\phi}^3 \end{pmatrix}, \quad \beta_{10} = \begin{pmatrix} -\bar{\phi}^3 & \bar{\phi}^3 \\ -\bar{\phi}^3 & \bar{\phi}^3 \end{pmatrix},
\]

where

\[
P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \phi^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \bar{\phi}^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad (22)
\]

and

\[
P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau'_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad (23)
\]
finally

\[ B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}; \]

\[ C_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}. \]

(24)

In all of these matrices, empty fields correspond to zeros. One can see that the non-zero elements in each matrices have the corresponding quantum numbers as those of corresponding components in (20).

There are some additional matrices which will be useful for the discussion of discrete symmetries. First, \( \gamma_{11} \) is defined as the product of all the ten gamma matrices,

\[ \gamma_{11} = -i \gamma_1 \gamma_2 \cdots \gamma_{10} = \text{diag} \left( I^{16}, -I^{16} \right). \]

(25)

This matrix is similar to \( \gamma_4 \) in the Lorentz group. Secondly the C-matrix is the product of the first five \( \gamma \)-matrices

\[ C = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 = \begin{pmatrix} 0 & I^{16} \\ I^{16} & 0 \end{pmatrix}. \]

(26)

\( \gamma_{11} \) anti-commutes with all the ten \( \gamma \)-matrices. It therefore can be used to construct derived \( \gamma \)-matrices

\[ \tilde{\gamma}_N = \gamma_{11} \gamma_N, \]

(27)

which satisfy the same conditions for a Clifford algebra, except for a sign difference in normalization. The relation between \( i \tilde{\gamma}_N \) and \( \gamma_N \) is called mutually dual, because one set can also be transformed to the other set by the use of the ten index anti-symmetric tensor \( \varepsilon \).

To find out tensor representations decomposed from the product of two 32-spinor representations \( \Psi_1 \) and \( \Psi_2 \), let us first define

\[ \tilde{\Psi} = \Psi^{\sigma T} = \Psi^{T} C \]

(28)

The \( a \)-th order tensor \( \gamma \)-matrices are

\[ \Gamma^{(a)}_{N_1 \ldots N_a} = \frac{1}{a!} \gamma_{[N_1} \cdots \gamma_{N_a]} \gamma_{a}, \quad (a = 0, \ldots, 2n). \]

(29)
The bracket \([\cdots]\) for the subindices means anti-symmetrization. Obviously \(\Gamma^{(10)} = i\gamma_{11}\). The \(O(10)\) anti-symmetric tensor representations are simply

\[\tilde{\Psi}_1 \Gamma^{(a)} \Psi_2,\]

where a \(C\) of the Lorentz group is implied as is shown in Eq(11).

The second order \(\gamma\)-matrices \(\Gamma^{(2)}\) are worth special attention because they are the \(SO(10)\) infinitesimal group operators in the 32 reduced representation. They are all anti-Hermitian and block diagonalized, therefore one has\(^5\)

\[e^{\Gamma^{(2)}_{MN} \gamma_{a} MN} \Gamma^{[0]}_e e^{\Gamma^{(2)}_{PQ} \gamma_{a} PQ} = C.\]  

5 among the 45 operators can be simultaneously diagonalized. They are

\[D_I = i \Gamma_{t, t+5} = i \gamma_{t} \gamma_{t+5}.\]  

Note that

\[
\begin{align*}
\sqrt{\frac{1}{2}} (D_2 + D_4) &= \sqrt{8} T_{3R}, \\
\sqrt{\frac{1}{2}} (D_2 - D_4) &= \sqrt{8} T_{3L}, \\
\sqrt{\frac{1}{3}} (D_3 + D_4 + D_5) &= -\sqrt{3} (B - L), \\
\sqrt{\frac{1}{2}} (D_3 - D_4) &= -\sqrt{8} I_3^{\text{color}}, \\
\sqrt{\frac{1}{6}} (D_3 + D_4 - 2D_5) &= -\sqrt{8} Y^{\text{color}}.
\end{align*}
\]

Here, \(T_{3L}=\text{diag}[1/2, -1/2]\) for each left-handed doublet; \(I_3^{\text{color}} = \text{diag}[1/2, -1/2, 0]\) and \(Y^{\text{color}} = \sqrt{1/12} \text{diag}[1, 1, -2]\) for each color triplet.

It is interesting to find operators which are block off-diagonal, color-singlet, and electrically neutral. They therefore can be used to write mass terms. It is easily seen that

\[\gamma_{1} \quad \text{and} \quad \gamma_{6}\]  

\(^5\)However, when one of the ten dimensions (say, the 10th) is time-like, part of \(\Gamma_{MN}\) are Hermitian due to the following definition: \(\gamma_6 = i \gamma_{10}\) which is anti-Hermitian. One then has

\[e^{\Gamma^{(2)}_{MN} \gamma_{a} MN} C \gamma_6 e^{\Gamma^{(2)}_{PQ} \gamma_{a} PQ} = C' \gamma_6.\]

This is the main difference between \(SO(9, 1)\) and \(SO(10)\) groups.
in 10 can contribute to fermion masses if a 10-plet Higgs $H_N$ develops VEV in the first and the sixth positions\(^6\). There are four mass operators in 120. They are

$$\gamma_{i,\sigma}(i)\gamma_7 = \gamma_{i,\sigma}D_2; \quad \gamma_{i,\sigma}(B - L).$$  \hspace{1cm} (34)

All non-zero elements of these operator matrices in (33) and (34) have quantum numbers

$$(SU(3), T_{3L}, T_{3R}, B - L) = (1, \pm 1/2, \mp 1/2, 0).$$  \hspace{1cm} (35)

An operator which includes $B - L$ as a factor will give $3m_q = \pm m_t$. Otherwise, $m_q = \pm m_t$.

Note that the off-diagonal block of operators in 120-plet is made of $16 \times 16$ anti-symmetric matrices. There are four mass operators in 126 also. Two of them are

$$\gamma_{i,\sigma}(B - L)D_2,$$  \hspace{1cm} (36)

which enjoy the same property as described in (35). The other two, from its quantum number analysis, are found all to have zero elements except those with quantum numbers

$$(1, 0, \pm 1, \mp 2), \text{ or } (1, \pm 1, 0, \mp 2)$$

and the first one is normally used to give right-handed neutrinos huge Majorana masses in order that a “see-saw” mechanism may take place to render a tiny left-handed neutrino mass\([9]\). These operators have only two non-zero elements in each $16 \times 16$ off-diagonal block, therefore the magnitude of each non-zero element is $\sqrt{3}$.

A linear combination of the operators in (33) and (34) or (36) can provide flexibility to produce desired quark-lepton Dirac mass relations. However, the relation $m_d/m_e = m_u/m_\nu$ is valid for all of these operators. The up-down mass relation in the $O(10)$ models is quite arbitrary, see the next section.

The method used here to produce all the necessary matrices on a physical basis can be used for other $O$ groups.

III. Masses and CP Violation

The most general Yukawa term involving one 10 is

$$L_Y = \frac{1}{2} \left( g_e \bar{\psi}_i \gamma_N H_N \psi_j + g_\nu \bar{\psi}_i \gamma_\nu H_N \psi_j \right) + h.c. \hspace{1cm} (37)$$

\(^6\)The naturalness of developing VEVs at two specific positions of a 10-plet is a question subject to study[8].
Note that the expressions $\psi_j \gamma_N \psi_j + \bar{\psi}_j \gamma_N \bar{\psi}_j$ and $\psi_j i \gamma_i \gamma_N \psi_j + \bar{\psi}_j i \gamma_i \gamma_N \bar{\psi}_j$ are real. $g^{ij}_{e,o}$ are two sets of Yukawa coupling constants which give rise to parity even and odd terms. Both terms in (37) can be CP even if $\text{Im} g^e_a = 0$.

As an illustration, we write down the explicit form for the third family

\[
L^Y = \text{Re}(g_e H_1 + g_o H'_o) \bar{u}_3 u_3 + \text{Im}(g_e H'_o + g_o H_1) \bar{u}_3 i \bar{u}_5 u_3 \\
+ \text{Re}(g_e H_1 - g_o H'_o) \bar{d}_3 d_3 - \text{Im}(g_e H'_o - g_o H_1) \bar{d}_3 i \bar{u}_5 d_3 \\
+ \text{charged current inter.} + \text{leptonic counterpart,}
\]

with $H'_o = -i H_o$ and $g_{e,o} = g_{e,o}^{33}$. It is easy to check that when $g_e$ and $g_o$ are real, the Yukawa term is indeed CP even if $\text{Re} H_1$ and $\text{Re} H'_o$ are assigned CP even while $\text{Im} H_1$ and $\text{Im} H'_o$ are CP odd. In obtaining Eq (38) one sees that $\gamma_i$ creates a current $(\psi^\dagger \alpha_1 \psi^* - \psi^T \alpha_1 \psi)$; while $\gamma_o$ creates $(\psi^\dagger \alpha_1 \psi^* + \psi^T \alpha_1 \psi)$. Consequently the first current is scalar and the second, psuedo-scalar. When there is a phase difference between $g_e$ and $g_o$, the Lagrangian has explicit CP odd terms. The mass relation in (1) is obtained, with a substitution of

\[
\langle H_1 \rangle = v_1, \quad \langle H'_o \rangle = \langle -i H_o \rangle = v_o.
\]

To discuss the special cases, it is more convenient to use the dual representation. This is done by introducing the following two sets of reduced $\gamma$-matrices, which do not belong to any Clifford algebra. To avoid confusion, they are called E-type and W-type matrices,

\[
\gamma_N^E = \frac{1}{2} (1 + \gamma_i) \gamma_N, \quad \gamma_N^W = \frac{1}{2} (1 - \gamma_i) \gamma_N, \quad \gamma^E = \gamma^W.
\]

The E and W project operators $(1 \pm \gamma_i)/2$ are $SO(10)$ invariant. They separate 16 from 16* in a 32 reduced representation. The Yukawa terms expressed in the E-W basis is

\[
L^Y = \frac{1}{2} \left( G^{ij}_E \bar{\psi}_i \gamma_N^E H_N \psi_j + G^{ij}_W \bar{\psi}_j \gamma_N^W H_N \psi_i \right) + h.c.
\]

\[
= G^{33}_E \left( H_u \bar{u}_3 L u_3 R + H_d \bar{d}_3 L d_3 R \right) + G^{33}_W \left( H_d \bar{u}_3 L u_3 R + H_u \bar{d}_3 L d_3 R \right) + h.c.
\]

\[
+ \text{leptonic and charged} + \cdots,
\]

where

\[
H_{u,d} = H_1 \pm i H_o, \quad G_{E,W} = g_e \pm g_o.
\]
$H_u$ and $H_d$ are respectively in 5 and $5^*$ of $SU(5)$ which are respectively up and down components in a certain representation of 10 (please compare with Eq (20)).

Let us now return to the special cases discussed in Section I.

**Case 1),** $v_d = \langle H_d \rangle = 0$: One obtains

$$m_t/m_b = (g_e + g_o)/(g_e - g_o) = G_E/G_W. \quad (43)$$

It can give the phenomenological mass ratio with only one VEV. But if $\text{Re} g_e g_o^* = 0$ (which corresponds to a maximal CP phase when $g_e g_o^* \neq 0$) one is forced to have the trivial $O(10)$ mass relation $|m_t| = |m_b|$.

The mass matrix for the Up-type quarks and Down-type quarks are, in the case of three families of quarks and leptons,

$$M^U = G^{ij}_E v, \quad M^D = G^{ij}_W v, \quad (i, j = 1, 2, 3) \quad (44)$$

while the Yukawa couplings for the $H_d$ field, which does not develop VEV, are

$$Y^U = G^{ij}_W, \quad Y^D = G^{ij}_E. \quad (45)$$

There does not necessarily exist a proportionality relation between the masses and the Yukawa coupling constants unless $g_o = 0$ or $g_e = 0$. Therefore, the $H_d$ mediated FCNC exist, as does new CP violation in the Yukawa sector, if $\text{Im} g_e g_o^* \neq 0$. When $g_o = 0$, half of the Higgs degrees of freedom decouple from the Yukawa sector.

**Case 2),** $G_W = 0$: $H_u$ only provides up-type masses; and $H_d$ only provides down-type masses. This case has attracted the most attention in the literature. But it raises the question of how one can forbid the $G_W$ term by certain symmetries. The only candidate within the $O(10)$ group seems to be the $O(10)$ dual transformation

$$\gamma_N \rightarrow \gamma_1; \gamma_N. \quad (46)$$

The E-type term is self-dual and the W-type term is anti-self-dual. An extra $U(1)$ global symmetry appears along with self-duality.

Another representation can separate $H_{\pm}$ at the beginning, where

$$H_{I\pm} = H_I \pm i H_{I+5} \quad (I = 1 \text{ to } 5) \quad (47)$$
\[ H_{I^+} \text{ and } H_{I^-} \text{ are in } 5_1 \text{ and } 5^*_2 \text{ respectively of } SU(5) \text{ and } 10 = (5_1, 5^*_2) \text{ in this representation.} \]
The associated reduced \( \gamma \)-matrices are
\[
\gamma_{I\pm} = (\gamma_I \pm i\gamma_{I+b})/2. \quad (I = 1 \text{ to } 5)
\]
The even \((G_E = G_W)\) Yukawa interaction is then
\[
\bar{\Psi} \sum_{I=1}^{5}(\gamma_{I^-} H_{I^+} + \gamma_{I^+} H_{I^-})\Psi. \quad (49)
\]
This representation is convenient for discussions of charged currents and gauge interactions.

IV. Discussion and Speculation

It has been explicitly shown that a general \(O(10)\) grand unification model with one 10-plet Higgs provides a natural motivation for the most general two Higgs doublet model (2HDM) at low energies as discussed in detail in Ref. 4. It turns out that the behavior of \(m_t/m_b = v_u/v_d\) is valid only for a special 2HDM which is, from the \(O(10)\) grand unification point of view, just a special case of the MOTM. This case appears when a self dual (or anti-dual) condition is imposed upon the Yukawa terms. Furthermore, in a general MOTM, there must be FCNC, if trivial up-down \(O(10)\) mass relations is to be avoided. It is easy to see that this behaviour also appears when one Higgs 120 or 126 contributes to Dirac fermion masses, except that 120 contributes only inter-family masses.

Since a global symmetry comes along with self-duality, one can also rule out the W-current-H coupling by the use of: a) an extra \(U(1)\) quantum number; b) a discrete symmetry of an order higher than five; c) a complex nonabelian group. If the previous successful mass relations and top mass predictions are significant, then the \(O(10)\) grand unification theory requires one of these symmetries to be fundamental. One or other of these symmetries have been used in models to explain specific patterns of the mass matrices[5].

Actually, certain amount of FCNC is tolerable within the accuracy of the present experimental data, as discussed in Ref. 4. While self duality can make \(m_b \neq m_t\), the most general condition for \(m_b \neq m_t\) is
\[
\text{Re}(g_s g_{s^*}) \neq 0, \quad \text{Re}(v_i v_{i^*}) \neq 0,
\]
\[
or \quad |G_W| \neq |G_E|, \quad |v_u| \neq |v_d|. \]
When FCNC is allowed there are possibilities to realize the desired up-down mass ratio by a combination of two VEVs and two coupling constants.

It is very interesting that within the realm of the explicit CP invariant MOTM, one can adjust the up-down mass ratio in one family by adjusting $g_o/g_e$ and $v_u/v_d$. Therefore it is possible to find an $O(10)$ model with a CP invariant Lagrangian. In such a model, CP violation all will be spontaneous. Nevertheless, the phenomenology of the MOTM with explicit CP violation is also interesting. As expected, the minimal $O(10)$ model in the most general context provides a low energy physics which coincides to every point with the most general two Higgs doublet model.

In conclusion, there is a correlation between explicit CP violation and fermion mass relation in an $O(10)$ grand unification model. The $O(10)$ Higgs multiplets which may contribute to fermion masses have two or more doublets whose CP transformation properties are different. The introduction of self duality, or symmetries beyond the $O(10)$ group, is crucial for getting rid of possible flavor changed neutral currents. For the robustness of this discussion, the explicit $O(10)$ gamma matrices are given, based on physical representations of spinors and vectors.

Except for the $O(10)$ group, other $O(2n)$ ($n \geq 2$) and $E_6$ models may also have similar correlation between CP violation and mass.

One of the authors (DD) sincerely thanks H.J. He, for very useful discussions. The CERN theory group and the DESY theory group are acknowledged for their hospitality during DD’s visit. The work of DD is supported in part by the U.S. Department of Energy under contract DE-FG03-95 ER40914/A00. The work of YL is supported in part by the U.S. Department of Energy under contract DOE/ER/01545-675.
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