We argue that the statistical entropy relevant for the thermal interactions of a black hole with its surroundings is (the logarithm of) the number of quantum microstates of the hole which are distinguishable from the hole's exterior, and which correspond to a given hole's macroscopic configuration. We compute this number explicitly from first principles, for a Schwarzschild black hole, using nonperturbative quantum gravity in the loop representation. We obtain a black hole entropy proportional to the area, as in the Bekenstein-Hawking formula.

In this letter, we present a derivation of the Bekenstein-Hawking expression for the entropy of a Schwarzschild black hole of surface area $A$ [1]

$$S = c \frac{k}{\hbar G} A$$

(1)

(where $c$ is a constant of the order of unity, $G$ the Newton constant, $k$ the Boltzmann constant, and we have put the speed of light equal to 1) via a statistical mechanical computation from a full theory of quantum gravity [2]. We use the loop representation of quantum gravity [3], and, in particular, we make use of the spectrum of the area operator recently computed in loop quantum gravity [4]. This work is strongly influenced by (but conceptually different from) a number of ideas presented by Krasnov in [5].

Consider a system containing, among other components, a non-rotating and non-charged black hole of mass $M$ (and therefore surface area $A = 16\pi G^2 M^3$). We are interested in the (statistical) thermodynamics of such system. The macroscopic state of the black hole is determined by the single parameter $M$. However, there may be a large number of microstates corresponding to the same macroscopic configuration. The number of such microstates determines the entropy to be associated to the black hole in analyzing its thermal interactions with the surroundings – in the same way in which the number of microstates of any given subsystem determines the entropy of the subsystem at any given macroscopic equilibrium state [6]. Taking quantum theory into account, such an entropy will be determined by the number of (linearly independent) quantum states $|\psi_i\rangle$ that correspond to the given macroscopic configuration of the hole.

The precise specification of these states is crucial. First, macroscopical spherical symmetry does not imply that individual statistical, or quantum, fluctuations be spherically symmetric as well; therefore the quantum microscopic configurations we have to consider do not need to be individually spherically symmetric. Second, only the configurations of the hole itself, and not the configurations of the surrounding geometry, affect the hole entropy. Indeed, the surrounding geometry will have many microscopic configurations corresponding to the same "macroscopic metric", but this multiplicity will contribute to an eventual entropy of the gravitational field –of the gravitational radiation– not to the entropy to be associated to the hole. Thus, we must focus on quantum states of the hole. Finally, what we are considering is the thermodynamical behavior of a system containing the hole. This behavior cannot be affected by the hole's interior. Indeed, the black hole interior may be in an infinite number of states. For instance, the black hole interior may (in principle) be given, say, by a Kruskal extension, so that on the other side of the hole there is another huge universe (maybe spatially compact, if not for the hole) possibly with millions of galaxies. The number of those internal states cannot affect the interaction of the hole with its surroundings. In other words, we are only interested in configurations of the hole that are distinguishable from the exterior of the hole. From the exterior, the hole is completely determined by the properties of its surface. Thus, the entropy relevant for the thermodynamical description of the thermal interaction of the hole with its surroundings is determined by the states of the quantum gravitational field (of the quantum geometry) on the black hole surface.

An important point to take into account (missed in an earlier stage of this work [7]) is that for an external observer different regions on the black hole surface are distinguishable from each other. This is indeed easy to see: consider initial data for the Einstein equations given in an asymptotically flat space containing a horizon. Consider data
Therefore, the quantum geometry on the surface is determined by the corresponding to a non-spherical deformation of a spherically symmetric event horizon. Imagine that this deformation is located in a certain region of the horizon. Then the future evolution of the field— for instance the radiation that reaches future infinity— depends on the location of the deformation on the event horizon. Thus we are interested in the quantum states of the geometry on a surface \( \Sigma \) of area \( A \), where different regions of \( \Sigma \) are distinguishable from each other. At this point the problem is well defined, and can be translated into a direct computation, provided that a quantum theory of geometry is given [8].

In loop quantum gravity, the quantum states of the gravitational field are represented by s-knots [9]. An s-knot is an equivalence class under diffeomorphisms of graphs immersed in space, carrying colors on their links (corresponding to irreducible representations of \( SU(2) \)), and colors on their vertices (corresponding to invariant couplings between such representations). The relation between s-knots and classical geometries was explored in [10]. If a surface \( \Sigma \) is given, the geometry of \( \Sigma \) is determined by the intersections of the s-knot with the surface. Given a quantum state and a surface, let \( i = 1 \ldots n \) label such intersections, and \( p_i \) be the color of the link through \( i \). Generically, no node of the graph will be on the surface; here, we disregard the "degenerate" cases in which a node falls on the surface. Thus, the quantum geometry of the surface is characterized by an n-tuple of \( n \) colors \( \vec{p} = (p_1, \ldots, p_n) \), where \( n \) is arbitrary. In particular, it was shown in [4] that the total area of the surface \( \Sigma \) is

\[
A = \sum_{i=1,n} 8\pi\hbar G \sqrt{p_i(p_i + 2)}. \tag{2}
\]

Recall that we are assuming that the points of \( \Sigma \) are distinguishable. Thus a quantity observable from outside the hole is, say, the area associated to a region containing only the intersections \( i = 1, \ldots, l < n \). Its value is

\[
A = \sum_{i=1,l} 8\pi\hbar G \sqrt{p_i(p_i + 2)}. \tag{3}
\]

Therefore, the quantum geometry on the surface is determined by the ordered n-tuples of integers \( \vec{p} = (p_1, \ldots, p_n) \). States labeled by different orderings of the same unordered n-tuple are distinguishable for an external observer.

We are thus interested in the number of ordered n-tuples of integers \( \vec{p} \) such that the macroscopic geometry of the surface is the geometry of the surface of the black hole. The geometry of the surface of the Schwarzschild black hole is characterized by the total area \( A \), and by the uniformity of the distribution of the area over the surface. Uniformity is irrelevant in a count of microscopic configurations, because the number of configurations and the number of uniform configurations are virtually the same for large area (for the vast majority of random configurations of air molecules in a room, the macroscopic air density is uniform). Thus, our task of counting microscopic configurations is reduced to the task of counting the ordered n-tuples of integers \( \vec{p} \) such that (2) holds. More precisely, we are interested in the number of microstates (n-tuples \( \vec{p} \)) such that the l.h.s of (2) is between \( A \) and \( A + dA \), where \( A >> \hbar G \) and \( dA \) is much smaller than \( A \), but still macroscopic.

Let

\[
M = \frac{A}{8\pi\hbar G}, \tag{4}
\]

and let \( N(M) \) be the number of ordered n-tuples \( \vec{p} \), with arbitrary \( n \), such that

\[
\sum_{i=1,n} \sqrt{p_i(p_i + 2)} = M. \tag{5}
\]

First, we over-estimate \( M(N) \) by approximating the l.h.s. of (5) dropping the +2 term under the square root. Thus, we want to compute the number \( N_+(M) \) of ordered n-tuples such that

\[
\sum_{i=1,n} p_i = M. \tag{6}
\]

The problem is a simple exercise in combinatorics. It can be solved, for instance, by noticing that if \( (p_1, \ldots, p_n) \) is a partition of \( M \) (that is, it solves (6)), then \( (p_1, \ldots, p_n, 1) \) and \( (p_1, \ldots, p_n + 1) \) are partitions of \( M + 1 \). Since all partitions of \( M + 1 \) can be obtained in this manner, we have

\[
N_+(M + 1) = 2N_+(M). \tag{7}
\]

Therefore
Where \( C \) is a constant. In the limit of large \( M \) we have
\[
\ln N_+(M) = (\ln 2) M. \tag{9}
\]
Next, we under-estimate \( M(N) \) by approximating (5) as
\[
\sqrt{p_i(p_i + 2)} = \sqrt{(p_i + 1)^2 - 1} \approx (p_i + 1). \tag{10}
\]
Thus, we wish to compute the number \( N_-(M) \) of ordered n-tuples such that
\[
\sum_{i=1,n} (p_i + 1) = M. \tag{11}
\]
Namely, we have to count the partitions of \( M \) in parts with 2 or more elements. This problem can be solved by noticing that if \((p_1, ..., p_n)\) is one such partition of \( M \) and \((q_1, ..., q_m)\) is one such partition of \( M - 1 \), then \((p_1, ..., p_n + 1)\) and \((q_1, ..., q_m, 2)\) are partitions of \( M + 1 \). All partitions of \( M + 1 \) in parts with 2 or more elements can be obtained in this manner, therefore
\[
N_-(M + 1) = N_-(M) + N_-(M - 1). \tag{12}
\]
It follows that
\[
N_-(M) = Da_+^M + Ea_-^M \tag{13}
\]
where \( D \) and \( E \) are constants and \( a_\pm \) (obtained by inserting (13) in (12)) are the two roots of the equation
\[
a_\pm^2 = a_\pm + 1. \tag{14}
\]
In the limit of large \( M \) the term with the highest root dominates, and we have
\[
\ln N_-(M) = (\ln a_+) M = \ln \frac{1 + \sqrt{5}}{2} M. \tag{15}
\]
By combining the information from the two estimates, we conclude that
\[
\ln N(M) = d M. \tag{16}
\]
where
\[
\ln \frac{1 + \sqrt{5}}{2} < d < \ln 2 \tag{17}
\]
or
\[
0.48 < d < 0.69. \tag{18}
\]
Since the integers \( M \) are equally spaced, our computation yields immediately the density of microstates. Using (4), the number \( N(A) \) of microstates with area \( A \) grows for large \( A \) as
\[
\ln N(A) = \frac{A}{8\pi\hbar G} \tag{19}
\]
We have argued above that the statistical entropy that controls the thermal properties of the hole in its interactions with its surroundings is determined by the number of microstates of the quantum geometry of the hole surface distinguishable from the exterior of the hole, namely
\[
S(A) = k \ln N(A) \tag{20}
\]
Equations (19) and (20) yield
\[
S(A) = c \frac{k}{\hbar G} A. \tag{21}
\]
which is the Bekenstein-Hawking formula. The constant of proportionality that we have obtained is

\[ c = \frac{d}{8\pi}, \]

which is roughly \(4\pi\) times smaller than Hawking's value \(c = \frac{1}{4}\).

Notice that the dynamics (the Hamiltonian) does not enter our derivation directly. However, it does enter indirectly by singling out the states with given area \(A\) as the ones with the same energy \(M\). This is the usual role of the Hamiltonian in the microcanonical framework. In particular, in a gravitational theory different from GR the relation between the hole's energy \(M\) and its area \(A\) may be altered or even lost. Thus, the relation between number of states and area is purely kinematical, by the relation between this number and the entropy (which is the number of states with the same energy) is theory dependent [11].

We leave a number of issues open (which may affect the proportionality factor). We have disregarded the degenerate states in which a node falls over the surface. Also, we have worked in the simplified setting of a black hole interacting with an external system with given geometry, instead of working with a fully generally covariant statistical mechanics [12].

Finally, we comment on the relation of our result with Ref. [5]. We learned the idea of associating entropy to classical configurations of the geometry—seen as macroscopical states—from Krasnov [13]. An earlier attempt to realize this idea along the lines described here failed, yielding an entropy proportional to the square root of the area [7]. A crucial breakthrough in [5] was the intuition that intersections are distinguishable. In [5], however, the setting of the problem is substantially different than ours: internal configurations of the black hole are considered, the Bekenstein entropy-bound conjecture and the holographic conjecture are invoked in order to justify the counting considered. Here, we do not need those hypotheses. Furthermore, the entropy-area proportionality is derived in [5] by means of an elegant but complicated argument involving a phase transition in a fictitious auxiliary statistical system, while here the combinatorial computation is performed explicitly.

In summary: we have argued that the black hole entropy relevant for the hole's thermodynamical interaction with its surroundings is the number of the quantum microstates of the hole which are distinguishable from the exterior of the hole; we have counted such microstates using loop quantum gravity. We have obtained that the entropy is proportional to the area, as in the Bekenstein-Hawking formula, but with a different numerical proportionality factor.

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[6] The canonical partition function of a system containing a black hole is notoriously tricky, due to the negative specific heat of the hole, but the problem can be circumvented. See for instance J York, Phys Rev D33 (1986) 2092


[8] For an overview of current ideas on quantum gravity and quantum geometry, see C. Isham: “Structural Issues in Quantum Gravity”, lecture at the GR14 meeting, Florence 1995, gr-qc/9501063; and Quantum Geometry and Diffeomorphism Invariant Quantum Field Theory, Special Issue of the Journal of Mathematical Physics, eds. C Rovelli, L Smolin, November 1995


[11] I thank Ranjeet Tate for clarifying this point.

[12] C Rovelli: Class and Quantum Grav 10, 1549 (1993), Class and Quantum Grav 10, 1567 (1993); A Connes, C Rovelli: Class and Quantum Grav 11, 2899 (1994)