Abstract

In an Abelian gauge symmetry, spontaneously broken at a first order phase transition, we investigate the evolution of two bubbles of the broken symmetry phase. The full field equations are evolved and we concentrate in particular on gauge invariant quantities, such as the integral around a closed loop of the phase gradient. An intriguing feature emerges, namely, the geodesic rule, commonly used in numerical simulations to determine the density of defects formed is shown not to hold in a number of circumstances. It appears to be a function of the initial separation of the bubbles, and the coupling strength of the gauge field. The reason for the breakdown is that in the collision region the radial mode can be excited and often oscillates about its symmetry restoring value rather than settling to its broken symmetry value. This can lead to extra windings being induced in these regions, hence extra defects (anti-defects) being formed.

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There is much evidence that the early Universe was characterised by a series of phase transitions in which a high energy ‘old’ symmetry phase was spontaneously broken to a low energy ‘new’ symmetry phase, possibly leading to the formation of topological defects. Such objects are also readily found in condensed matter systems (although of a much lower energy scale) [2], so although direct observational evidence for them in cosmology is still in doubt there exists plenty of evidence for their existence in terrestrial experiments. Given they could exist, we need to know their initial distribution so that we can determine the cosmological implications of the defects.

In this letter we are concerned with the dynamics of Abelian-Higgs fields in a first order phase transition when bubbles of the new phase are nucleated in the background of the old phase. Topological defects form in the regions where the bubbles collide, so it would be useful to investigate the behaviour of the fields in this region. We will be following the work of [1] in determining the evolution of the bubble walls, but will be concerning ourselves with defect formation rather than the rate at which the energy in the bubble walls would be thermalized.

We will be considering a $U(1)$ theory with a complex scalar field $\Phi$, and Lagrangian

$$\mathcal{L} = (D_\mu \Phi)^* D^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\Phi)$$  

(1)

where $D_\mu \Phi = \partial_\mu \Phi - ie A_\mu \Phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $e$ is the gauge coupling constant and the potential $V$ is a first order potential, a function of $|\Phi|^2$ with a local minimum at $|\Phi| = 0$ and a global minimum at $|\Phi| = \eta/\sqrt{2}$. The dynamics of the phase transition proceed as follows. As the Universe expands and cools, tunnelling can occur from the old phase $\Phi = 0$ to the new phase where $\Phi \approx \eta/\sqrt{2} e^{i\theta}$. The symmetry has been spontaneously broken and within each bubble there is a random choice of the phase angle $\theta$. The bubbles, nucleated at random points expand and collide, with their nucleation rate being determined from the bounce solution to the Euclidean action [3].

The first definitive explanation for defect formation was provided in the work of Kibble [4]. Consider the case of cosmic strings. It is assumed that within each bubble the phase $\theta$ is constant, with neighbouring bubbles being uncorrelated. When two bubbles with phases $\theta_1$ and $\theta_2$ meet, any discontinuity between the phases is smoothed out. On energetic grounds, the shorter path between $\theta_1$ and $\theta_2$ is chosen, a result known as the ‘geodesic rule’. For the collision of three bubbles, a string may be trapped in the region between the bubbles. This depends on the net phase change in going sequentially $\theta_1 \rightarrow \theta_2 \rightarrow \theta_3 \rightarrow \theta_1$.

In a series of papers [5], Srivistava demonstrated that the geodesic rule does indeed hold for global theories. Recently though, with Rudaz [6], he questioned the reliability of the rule when gauge fields are present, pointing out that it may not make sense to talk about phase differences between bubbles in a gauge theory. It is possible to gauge transform the phase difference to any value. As a consequence they argued that string formation in such gauge theories may well be strongly suppressed as compared to the global theory case. Hindmarsh et al [7] analysed the bubble collisions using an analytic approach and concluded that the geodesic rule did actually hold. They pointed out that the geodesic rule emerges not from energy considerations but from the equations of motion themselves– the dynamics. Recently Kibble and Vilenkin [8], also using an analytic approach addressed the same issue, concluding that the geodesic rule nearly always held. They went further, including dissipation terms induced by the finite plasma conductivity that the bubbles expand in, they demonstrated
that this can cause the phases to equilibrate on a timescale much smaller than the bubble radii at the time of collision. Such a result seems to vindicate the common assumption that the geodesic rule holds. However, as the authors stressed in [7] and [8], throughout the calculations various assumptions have to be made about the behaviour of the fields. For example in [8] it was assumed that the radial mode of the Higgs field is strongly damped, settling into its equilibrium value on a timescale short compared to the phase equilibrium process. A similar assumption was made in [7] where the variation of the radial mode was dropped inside the bubble. In general though, this may not happen. In this letter we report on a project in which we solved the full field evolution equations for two colliding bubbles numerically, keeping track on the variation of the radial as well as the phase degrees of freedom. The results open up the possibility that the geodesic rule may not be as widespread as first thought. However, we must point out that so far this result is without the presence of an explicit damping term. This is for computational reasons that will become clear.

Following [1] we consider a potential which is a function of $|\Phi|^2$ and which has a local minimum at the origin and a circle of degenerate global minima away from the origin. The simplest such potential is a cubic in $|\Phi|^2$, hence we consider

$$V(\phi) = a(|\phi|^2 + B)(|\phi|^2 - c^2)^2. \quad (2)$$

Two of the three parameters ($a$, $c$) in Eqn. (2) can be set to unity by redefining the fields and coordinates.

The nucleation of a single bubble follows the bounce solution [3]. The gauge fields are taken to vanish and the phase is required to be constant within the bubble. The bounce solution found numerically was seen to compare favourably with a tanh($x$) profile, which was subsequently used in the simulations as the initial condition on the Higgs field. (Our results are robust to this approximation due to the Lorentz contraction of the bubble walls.) In the problem of two bubbles colliding the tunnelling solution has $SO(2,1)$ symmetry. The fields must then evolve independently of the two angles $\theta$ and $\psi$, which greatly simplifies the calculation since the solution becomes a function of two variables, the $x$ coordinate, and the ‘time’ $s$ where $s^2 = t^2 - y^2 - z^2$. [See [1] for details].

There are two useful representations for $\Phi$. Firstly one has $\Phi = \frac{\rho}{\sqrt{2}} e^{i\theta}$. This is easier to visualise but difficult to implement numerically, due to the ambiguity in $\theta$ when the field modulus $\rho$ vanishes. Secondly there is a Cartesian description $\Phi = \frac{1}{\sqrt{2}} (u + iv)$, where $u$, $v$ are real. The choice of the Lorentz gauge ($\nabla_\alpha A^\alpha = 0$) and the independance of $\psi$, $\theta$ leads to a set of evolution equations. Of particular interest is the $\theta$ equation:

$$\frac{\partial^2 \theta}{\partial s^2} = \frac{\partial^2 \theta}{\partial x^2} - \frac{2}{s} \frac{\partial \theta}{\partial s} + \frac{2e}{\rho} \left( A_s \frac{\partial \rho}{\partial s} - A_x \frac{\partial \rho}{\partial x} \right) - \frac{2}{\rho} \left( \frac{\partial \rho}{\partial s} \frac{\partial \theta}{\partial s} + \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial x} \right) \quad (3)$$

Note that there is a term depending on $e$, the gauge fields and the derivatives of $\rho$, which may be interpreted as a forcing term for $\theta$ and is the central difference between taking $\rho$ as constant [8] and considering the global case [5].

For the initial conditions of two bubbles with a phase difference, the description of the field’s modulus is easy to understand. As the bubbles collide, the intersection forms a ring,
the large amount of energy available in the walls at collision allows the $U(1)$ symmetry to be restored in this ring. It is this localised symmetry restoration which allows (but does not require) the phase to have a non-trivial winding about the ring, thus indicating the break down of the geodesic rule. Such a winding occurs in many cases. It is found that the amount of winding and its direction depends on the size of the gauge coupling, the initial separation and initial phase difference of the bubbles. By holding the phase and bubble separation constant we may investigate the $e$ dependance. When $e = 0$, the global case, the winding around the ring is seen to vanish, consistent with the geodesic rule. Increasing $e$ leads to a generation of winding, which can flip sign from being positive to negative and also produce multiple $2\pi$ windings. Subsequent flips of sign occur for values of $e$ which are integer multiples of the value of $e$ for which the flip first occurs.

The dependance on initial separation is probed by holding $e$ and the phase difference constant. It is found that the greater the separation, the greater the winding. As for the dependance on the initial phase separation, we observe that there is a cutoff around $\frac{\pi}{2}$ above which no winding is generated.

The picture describing the evolution of the Higgs field is shown for example in Figure 1. Two regions of true vacuum expand, collide and form a pocket of restored $U(1)$ symmetry which subsequently collapses and forms another pocket. These oscillations continue until the energy has been radiated away [1].

A non-trivial winding is observed to occur over the time period that the first pocket collapses (see Figure 2). This timescale can be estimated [1] by considering the thin wall approximation for a global theory. Let $s_0$ be the time at which the walls collide and $s_2$ when the pocket collapses. By matching the wall velocities before and after collision (consistent with simulations) then it is possible to show:

$$s_2 \simeq \left[ 2 \left( 2 \left[ 1 - \frac{1}{3} \alpha^2 \right] \right)^{\frac{1}{3}} - 1 \right] s_0$$

where $\alpha$ is the phase separation between the bubbles. The limiting case is when $s_2 \simeq s_0$ for which we find a value of $\alpha$ of $\simeq \sqrt{\frac{3}{2}} \simeq \frac{\pi}{2}$, a result that explains why no windings are found above a certain initial phase separation. Shortly we shall see that the amount of winding depends on how long this first pocket can survive. From Eqn. (4) the lifetime varies in proportion to $s_0$ so, since the bubbles reach relativistic velocities rapidly, the pocket’s lifetime will vary in proportion to the initial separation, a result that explains why more windings were found as the separation was increased.

Now we consider the cause of the winding. If we set the phase in the left hand bubble to be zero and in the right hand bubble to be $0 < \alpha < \pi$, the equations of motion require $A_x > 0$ for all $x$ and $A_s < 0$ for $(x > 0)$ just after the collision. As the pocket initially expands then $\dot{\rho} < 0$, $\rho' > 0$ which means the term $2\frac{e}{\rho} (A_s\dot{\rho} - A_x\rho')$ in Eqn. (3) has components which oppose each other and lead to a small forcing term for $\theta$. However, when the pocket starts to collapse then $\dot{\rho} > 0$ and the components combine to create a negative forcing term for $\theta$, driving the field around the vacuum manifold and generating a winding (see Figures 2, 4). Note that the longer the pocket survives, then the longer this forcing can act leading to more winding. An important feature of the winding is that as the bubbles collide and the radial component of the field overshoots the global minima restoring the symmetry, the
field traces a twist in the configuration space which leads to a winding of $+2\pi$ instead of the $-2\pi$ which would occur without the twist (see Figures 3, 4 at $s=31.5$, $s=32$).

The sense of the winding depends upon the sign of the gauge fields. If it is possible to change the sign of the gauge fields by the time the pocket starts to collapse then the winding direction is reversed. In the equations for $A_s$ and $A_x$ we find the harmonic term $-e^2 \rho^2 A$ which can generate a sign change on a timescale of $\pi e \rho$. This scale is only significant for large $\rho$ and should be compared with the timescale of the modulus wall, $\delta$. If the gauge fields $A_s, A_x$ can change sign in the time taken for the modulus to drop to zero, then they will retain their new sign until the pocket collapses. This leads to a change in the sense of the winding at $e \simeq \frac{\pi}{e\rho} n$, where $n$ is an integer. This demonstrates the sign flipping observed in the simulations at regular intervals of $e$ and agrees numerically with the observed periodicity. It should be noted however, that as $e$ becomes very large then the timescale $\frac{\pi}{e\rho}$ becomes sufficiently small even when $\rho$ is near the false vacuum. This complicates the description of when a change in the sense of winding will occur.

Determining the conditions for multiple windings is more complicated. It is clearly a function of the initial separation, but it also depends on the magnitude of the gauge field in the interaction region. The impact of dissipation on these results is an important point that needs to be addressed. In [8] it has been shown that under the assumption that the radial field is fixed in its broken symmetry phase, then the phase difference is damped exponentially fast due to the presence of the background plasma in which the bubbles are expanding. There is little doubt that any plasma will act to damp down fluctuations present in the field. It remains to be determined whether, in the case where the radial field is not pinned to the true vacuum value, there could still be induced windings as the phase rotates through $2\pi$. It is not sufficient to say that the field is damped, because as we have seen the winding is generated within the first period of the bubble wall oscillations. It then becomes a question of timescales as to which will win out. We are currently investigating this issue. The problem we face is that any damping term breaks the SO(2,1) symmetry of the problem making it computationally much more demanding to solve the field equations.

On the face of it the cosmological implications could be very significant. Typically, as mentioned earlier, the phases are set at random in each bubble and the geodesic rule is used to determine the density of defects formed initially. This leads to a distribution of open and closed string. Now we seem to find that depending on the parameters, if the phase difference between two bubbles is less than about $\pi/2$, then it is likely that new windings will be induced in the collision region. We performed a toy simulation by laying phases (continuously from 0 to $2\pi$) down at random on a 3-torus, with a cubic lattice. Between neighbouring correlation domains, where the phase difference was less than $\pi/2$, we created an extra string. The net effect was equivalent to previous cubic-lattice based simulations [9], in that we found the distribution for loops scaling with number density $n(l) \propto l^{-5/2}$ and for string winding around the torus $n(l) \propto l^{-1}$. We have not attempted to determine the ratio of long to short string as the result is sensitive to the lower cut-off present on the lattice. It would appear that the generation of extra defects, anti-defects does not alter the number distribution. However, in the cases where multiple windings are generated the initial string network will contain either strings of higher winding, or multiple $N=1$ strings in close proximity. A technique has recently been developed [10] in which the length distribution of strings can be calculated numerically, without a regular lattice dependance. This method is
particularly suited to modelling a first order phase transition and it would be interesting to extend those results to take into account the effects described above.

There is a precedent for extra vortices being found in the collision regions of the bubble walls. Digal and Srivastava [11] have analysed the behaviour of two bubbles colliding in a global U(1) theory (in 2+1 dimensions) where the global symmetry is broken both spontaneously and explicitly. They find that in the coalesced region of the bubbles, field oscillations result in the production of a number of defects (vortices and anti-vortices). We have been able to confirm this behaviour in the case of 3+1 dimensions and will report on this elsewhere [12].

In conclusion, the results we have obtained suggest that a set of rules exist for predicting the likely outcome of defects after two bubble collisions. The number of windings found depends on the initial separation of the bubbles, the strength of the coupling constant $e$, the magnitude of the gauge field present in the collision region and on the currently unknown strength of the dissipation term. We hope to be able to quantify these rules shortly.

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FIG. 1. field evolution plot for $e = 0.0 \, 0 < s < 60, -30 < x < 30$. The magnitude of the Higgs field is represented by the arrow length and the phase by its direction.

FIG. 2. field evolution plot for $e = 0.5 \, 0 < s < 60, -30 < x < 30$ where a winding of $2\pi$ is generated.
FIG. 3. Sequence in u-v space for e=0.0

FIG. 4. Sequence in u-v space for e=0.5